Regular subalgebras and nilpotent orbits of real graded Lie algebras

Heiko Dietrich, Paolo Faccin, and Willem A. de Graaf

Abstract. For a semisimple Lie algebra over the complex numbers, Dynkin (1952) developed an algorithm to classify the regular semisimple subalgebras, up to conjugacy by the inner automorphism group. For a graded semisimple Lie algebra over the complex numbers, Vinberg (1979) showed that a classification of a certain type of regular subalgebras (called carrier algebras) yields a classification of the nilpotent orbits in a homogeneous component of that Lie algebra. Here we consider these problems for (graded) semisimple Lie algebras over the real numbers. First, we describe an algorithm to classify the regular semisimple subalgebras of a real semisimple Lie algebra. This also yields an algorithm for listing, up to conjugacy, the carrier algebras in a real graded semisimple real algebra. We then discuss what needs to be done to obtain a classification of the nilpotent orbits from that; such classifications have applications in differential geometry and theoretical physics. Our algorithms are implemented in the language of the computer algebra system GAP, using our package CoReLG; we report on example computations.

1. Introduction

Let \( g^c \) be a complex semisimple Lie algebra with adjoint group \( G^c \). Classifying the semisimple subalgebras of \( g^c \) up to \( G^c \)-conjugacy is an extensively studied problem, partly motivated by applications in theoretical physics; see for example \([12, 19, 20, 41, 42]\). In \([20, 41]\), this classification is split into two parts: the construction of regular semisimple subalgebras, that is, semisimple subalgebras normalised by a Cartan subalgebra of \( g^c \), and the construction of semisimple subalgebras not contained in any regular proper subalgebra. Dynkin \([20]\) presented, among other things, an algorithm to list the regular semisimple subalgebras of \( g^c \), and the construction of semisimple subalgebras not contained in any regular proper subalgebra. One of the main facts underpinning this algorithm is that two semisimple subalgebras, normalised by the same Cartan subalgebra \( h^c \), are \( G^c \)-conjugate if and only if their root systems are conjugate under the Weyl group of \( g^c \) (with respect to \( h^c \)). The situation is more intricate for a real semisimple Lie algebra \( g \). As a consequence, here the aim is usually not to classify the semisimple subalgebras, but to decide whether a given real form \( a \) of a complex subalgebra \( a^c \) of \( g^c = g \otimes_{\mathbb{R}} \mathbb{C} \) is contained in \( g \), see \([10, 11, 21, 22, 24, 37]\). However, for some classes of subalgebras a classification up to \( G \)-conjugacy can be obtained, with \( G \) the adjoint group of \( g \). Examples are the subalgebras isomorphic to \( sl_2(\mathbb{R}) \), whose classification up to \( G \)-conjugacy is equivalent to classifying the nilpotent orbits in \( g \); the latter can be performed using the Kostant-Sekiguchi correspondence, see \([9, 16]\). It is the first aim of this paper to show that also the regular semisimple subalgebras of \( g \) can be classified up to \( G \)-conjugacy; we describe an effective algorithm for this task. The main issue is that, in general, there exist Cartan subalgebras of \( g \) which are not \( G \)-conjugate, and that a given regular semisimple subalgebra can be normalised by several non-conjugate Cartan subalgebras. To get some order in this situation, we introduce the notion of “strong \( h \)-regularity”; we show that two strongly \( h \)-regular subalgebras are \( G \)-conjugate if and only if their root systems are conjugate under the real Weyl group of \( g \) (with respect to \( h \)).

Key words and phrases. real Lie algebras; regular subalgebras; carrier algebras; real nilpotent orbits.

Dietrich was supported by an ARC-DECRA Fellowship, project DE140100088.
The second problem motivating this paper is the determination of the nilpotent orbits in a homogeneous component of a graded semisimple Lie algebra. Over the complex numbers (or, more generally, over an algebraically closed field of characteristic 0), the theory of orbits in a graded semisimple Lie algebra has been developed by Vinberg [48–50]. Let \( g^c = \bigoplus_{i \in \mathbb{Z}} g_i^c \) be a graded semisimple complex Lie algebra, where we write \( Z_m = \mathbb{Z}/m\mathbb{Z} \), and \( Z_m = \mathbb{Z} \) when \( m = \infty \). The component \( g_0^c \) is a reductive subalgebra of \( g^c \), and \( G_0^c \) is defined as the connected subgroup of \( G^c \) with Lie algebra \( g_0^c \). This group acts on the homogeneous component \( g_1^c \), and the question is what its orbits are. It turns out that many constructions regarding the action of \( G^c \) on \( g^c \) can be generalised to this setting. In particular, there exists a Jordan decomposition, so that the orbits of \( G_0^c \) in \( g_1^c \) naturally split into three types: nilpotent, semisimple, and mixed. Of special interest are nilpotent orbits: in contrast to semisimple orbits, there exist only finitely many of them. It is known that every nonzero nilpotent \( e \in g_1^c \) lies in a homogeneous \( sl_2 \)-triple \( (h, e, f) \), with \( h \in g_0^c \) and \( f \in g_{-1}^c \), and that \( G_0^c \)-conjugacy of nilpotent elements in \( g_1^c \) is equivalent to \( G_0^c \)-conjugacy of the corresponding homogeneous \( sl_2 \)-triples. Concerning the classification of the nilpotent orbits, Vinberg [50] introduced a new construction: the support, or carrier algebra, of a nilpotent element. This is a regular \( \mathbb{Z} \)-graded semisimple subalgebra \( c^c \leq g^c \) with certain extra properties; here “\( \mathbb{Z} \)-graded” means that \( c^c = \bigoplus_{i \in \mathbb{Z}} c_i^c \) with \( c_i^c \subseteq g_i^c \mod m \), and “regular” means that \( c \) is normalised by a Cartan subalgebra of \( g_0 \). The main point is that the nilpotent \( G_0^c \)-orbits in \( g_1^c \) are in one-to-one correspondence with the \( G_0^c \)-classes of carrier algebras, so that a classification of the former can be obtained from a classification of the latter. This approach has been used in several instances to classify nilpotent orbits, see for example [4, 23, 25]; it has also served as the basis of several algorithms to classify the nilpotent orbits in \( g_1^c \), see [28, 40].

Here we consider the analogous problem over the real numbers: \( g = \bigoplus_{i \in \mathbb{Z}} g_i \) is a real graded semisimple Lie algebra, \( G_0 \) is the connected Lie subgroup of \( G \) with Lie algebra \( g_0 \), and we want to classify the nilpotent \( G_0 \)-orbits in \( g_1 \). Again, this is much more complicated than the complex case. Nevertheless, several attempts have been made to develop methods for such a classification. For example, Djoković [18] considered a \( \mathbb{Z} \)-grading of the split real Lie algebra of type \( E_8 \), so that \( g_0 \) is isomorphic to \( gl(8, \mathbb{R}) \), and \( g_1 \cong \wedge^3(\mathbb{R}^8) \) as \( g_0 \)-modules; this is the kind of problem that is of interest in differential geometry, see [31]. Djoković classified the corresponding nilpotent orbits (which in this case form all orbits) using an approach based on Galois cohomology; it is not clear whether this can serve as the basis of a more general algorithm. Van Lê [39] devised a general method, whose main idea is to list the possible homogeneous \( sl_2 \)-triples; however, the main step in her approach uses heavy machinery from computational algebraic geometry over the real numbers, so that it is questionable whether it will be possible to implement this method successfully. The problem of classifying real nilpotent orbits is also considered in the physics literature (with applications in supergravity), see for example [7, 35]. In [7], the nilpotent orbits corresponding to a \( \mathbb{Z}_2 \)-grading of the split Lie algebra of type \( F_4 \) are classified using a method based on listing \( sl_2 \)-triples. The main idea is to search for real Cayley triples – and depends on the unproven assumption that each orbit has a representative lying in such a triple. Using elements of \( G_0 \), many such triples are shown to be conjugate, and the remaining ones are proven non-conjugate by using several invariants.

Here we approach the problem of classifying the nilpotent orbits in a real graded semisimple Lie algebra by first listing the carrier algebras. For this reason, we formulate the algorithm for listing the regular subalgebras in the more general context of \( \mathbb{Z} \)-graded regular subalgebras of a graded real semisimple Lie algebra; the algorithm for listing the regular subalgebras is then a straightforward specialisation to the trivial grading. From this, we devise an algorithm for listing the carrier algebras in a real graded semisimple Lie algebra up to conjugacy; this is the second aim of this paper. Unfortunately, it is not immediately straightforward to get a classification of the nilpotent orbits from this list of carrier algebras: unlike in the complex case, a given carrier algebra can correspond to more than one orbit, or to no orbit at all. In order to overcome this difficulty, we present a number of ad hoc techniques, partly similar to the ones used in [7]:
given a carrier algebra, we first reduce the set of nilpotent orbits to which it belongs by using elements of the group $G_0$; the remaining elements are then shown to be non-conjugate by using suitable invariants.

### 1.1. Main results and structure of the paper.

We use the previous notation. In Section 2 we recall relevant notation and concepts for semisimple Lie algebras over the complex and real numbers. As noted above, the real Weyl group plays a fundamental role in our algorithms; in Section 3 we describe an algorithm to construct the real Weyl group of a real semisimple Lie algebra, relative to a given Cartan subalgebra. The subalgebra $g_0$ is not semisimple in general, but reductive. We discuss a number of well-known properties of reductive subalgebras in Section 4; these are needed throughout the paper. Section 5 is devoted to toral subalgebras of a real semisimple Lie algebra; for such a subalgebra, we study the two subsets of elements having only purely imaginary and only real eigenvalues, respectively. In Section 6 we consider real graded semisimple Lie algebras; we recall some well-known properties, and present two classes of examples which include many cases of interest. In Section 7 we discuss our first main algorithm, namely, an algorithm to list all regular $\mathbb{Z}$-graded semisimple subalgebras, up to $G_0$-conjugacy. In Section 8 we consider the specialisation to the trivial grading to obtain an algorithm for constructing the regular subalgebras of $g$ up to $G$-conjugacy; we exemplify this with the real form $E_6$. The last three sections are devoted to the problem of classifying the nilpotent orbits of a real graded semisimple Lie algebra; we follow Vinberg’s approach and use carrier algebras. First, in Section 9, we generalise some of Vinberg’s constructions to the real case; based on our algorithm in Section 7, we obtain an algorithm to list the real carrier algebras in $g$, up to $G_0$-conjugacy. In Section 10 we discuss what needs to be done to get the classification of the nilpotent orbits from that list of carrier algebras. Finally, Section 11 reports on example computations; in three examples we elaborate on how our methods behave in practice: We consider the 3-vectors in dimension 8 (as in [18]), an example from the physics literature (as in [35]), and the real orbits of $\text{Spin}_{14}(\mathbb{R})$ on the 64-dimensional spinor representation (in the complex case this representation has, for instance, been considered by [26]). On some occasions we report on the runtimes of our implementations.

### 1.2. Notation.

We use standard notation and terminology for Lie algebras, which, for instance, can be found in the books of Humphreys [32] and Onishchik [43]. All Lie algebras are denoted by fraktur symbols, for example, $g$, and their multiplication is denoted by a Lie bracket $[-,-]: g \times g \to g$; all considered Lie algebras are finite-dimensional. If $\varphi: g \to gl(V)$ is a representation with $V$ a finite-dimensional vector space, then the associated trace form is $(x,y) = \text{tr}(\varphi(x) \circ \varphi(y))$. The adjoint representation $ad_g$ is defined by $ad_g(x)(y) = [x,y]$; its trace form is the Killing form $\kappa_g(x,y) = \text{tr}(ad_g(x) \circ ad_g(y))$; if the Lie algebra follows from the context, then we simply write $ad$ and $\kappa$.

Let $v \subseteq g$ be a subspace and let $a \leq g$ be a subalgebra. The normaliser and centraliser of $v$ in $a$ are

$$n_a(v) = \{x \in a \mid [x,v] \subseteq v\} \quad \text{and} \quad z_a(v) = \{x \in a \mid [x,v] = 0\},$$

respectively. A real form of a complex Lie algebra $g^c$ is a real subalgebra $g \leq g^c$ with $g^c \cong g \otimes_R \mathbb{C}$, that is, $g^c = g \oplus ig$ as real vector spaces; here $i \in \mathbb{C}$ is the imaginary unit. The real forms of the simple complex Lie algebras are classified; we use the standard notation of [36, §C.3 & C.4], cf. [43, Table 5].

As already done above, we endow symbols denoting algebraic structures over the complex numbers by a superscript $c$. If this superscript is absent, then, unless otherwise noted, the structure is defined over the reals. Similarly, if $v$ is a real vector space, then $v^c = v \otimes_R \mathbb{C}$ is its complexification.

If $g^c$ is a complex semisimple Lie algebra, then its adjoint group $G^c$ is the connected Lie subgroup of the automorphism group $\text{Aut}(g^c)$ with Lie algebra $ad\ g^c$; it is the group of inner automorphisms of $g^c$, generated by all $\exp(ad\ x)$ with $x \in g^c$. Similarly, the adjoint group $G$ of a real semisimple Lie algebra $g$ is the connected Lie subgroup of $\text{Aut}(g)$ with Lie algebra $ad\ g$; it is generated by all $\exp(ad\ x)$ with $x \in g$, respectively.
In this preliminary section we recall some notions concerning semisimple Lie algebras; throughout, see for example [30, p. 126–127]. If \( g^c = g \otimes_{\mathbb{R}} \mathbb{C} \), then \( G = G^c(\mathbb{R}) \) is the normaliser of \( g \) in \( G^c \), that is, \( G = \{ g \in G^c \mid g(g) = g \} \) is the set of real points of \( G^c \).

## 2. Semisimple Lie algebras

In this preliminary section we recall some notions concerning semisimple Lie algebras; throughout, \( g^c \) is a semisimple Lie algebra defined over \( \mathbb{C} \).

### 2.1. Semisimple complex Lie algebras.

Let \( h^c \leq g^c \) be a Cartan subalgebra with corresponding root system \( \Phi \). For a chosen root order denote by \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) the associated basis of simple roots. The root space corresponding to \( \alpha \in \Phi \) is \( g^c_{\alpha} = \{ g \in g^c \mid \forall h \in h^c : [h, g] = \alpha(h)g \} \). For \( \alpha, \beta \in \Phi \) define \( \langle \beta, \alpha^\vee \rangle = r - q \) where \( r \) and \( q \) are the largest integers such that \( \beta - r\alpha \) and \( \beta + q\alpha \) lie in \( \Phi \). By [32, §25.2], there is a Chevalley basis of \( g^c \), that is, a basis \( \{ h_1, \ldots, h_\ell, x_\alpha \mid \alpha \in \Phi \} \) which satisfies \( h_i \in h^c, x_\alpha \in g^c_\alpha \), and

\begin{align}
[h_i, h_j] &= 0, \\
[x_\alpha, x_{-\alpha}] &= h_\alpha, \\
[h_i, x_\alpha] &= \langle \alpha, \alpha^\vee \rangle x_\alpha, \\
x_{\alpha}, x_{\beta} &= N_{\alpha, \beta} x_{\alpha + \beta};
\end{align}

where \( h_\alpha \) is the unique element in \( [g^c_\alpha, g^c_{-\alpha}] \) with \( [h_\alpha, x_\alpha] = 2x_\alpha \), and \( N_{\alpha, \beta} = \pm(r + 1) \) where \( r \) is the largest integer with \( \alpha - r\beta \in \Phi \). Note that \( h_{\alpha_i} = h_i \) for \( 1 \leq i \leq \ell \), and \( x_\gamma = 0 \) for \( \gamma \notin \Phi \).

A generating set \( \{ g_i, x_i, y_i \mid i = 1, \ldots, \ell \} \) of \( g^c \) is a canonical generating set if it satisfies

\begin{align}
\{g_i, g_j\} &= 0, \\
\{g_i, x_j\} &= \langle \alpha_j, \alpha_i^\vee \rangle x_j, \\
\{x_i, y_j\} &= \delta_{ij} g_i, \\
\{g_i, y_j\} &= -\langle \alpha_j, \alpha_i^\vee \rangle y_j,
\end{align}

with \( \delta_{ij} \) the Kronecker delta. Sending one canonical generating set to another uniquely extends to an automorphism of \( g^c \), see [33, Thm IV.3] or [43, (II.21) & (II.22)]. Every Chevalley basis \( \{ h_1, \ldots, h_\ell, x_\alpha \mid \alpha \in \Phi \} \) contains the canonical generating set \( \{ h_i, x_i, y_i \mid i = 1, \ldots, \ell \} \) with \( x_i = x_{\alpha_i} \) and \( y_i = x_{-\alpha_i} \).

### 2.2. Real Forms.

Let \( \{ h_1, \ldots, h_\ell, x_\alpha \mid \alpha \in \Phi \} \) be a Chevalley basis of \( g^c \). It is well-known and straightforward to verify that the \( \mathbb{R} \)-span

\[ u = \text{Span}_\mathbb{R}(\{ih_1, \ldots, ih_\ell, (x_\alpha - x_{-\alpha}), (x_\alpha + x_{-\alpha}) \mid \alpha \in \Phi^+ \}) \]

is a compact form of \( g^c \), that is, a real form of \( g^c \) with negative definite Killing form; such a form is unique up to conjugacy, see [43, Cor. p. 25]. Using the decomposition \( g^c = u \oplus u \), the associated real structure (or conjugation with respect to \( u \)) is \( \tau : g^c \to g^c, x + iy \mapsto x - iy \), where \( x, y \in u \).

Let \( \theta \) be an automorphism of \( g^c \) of order 2, commuting with \( \tau \). Then \( \theta(u) \subseteq u \), and we can decompose \( u = u_+ \oplus u_- \), where \( u_\pm \) is the \( \pm 1 \)-eigenspace of the restriction of \( \theta \) to \( u \). Now \( g = \mathfrak{k} \oplus p \) with \( \mathfrak{k} = u_+ \) and \( p = u_- \) is a real Lie algebra with \( g^c = g \oplus ig \), hence a real form of \( g^c \). The associated real structure (or conjugation) is \( \sigma : g^c \to g^c, x + iy \mapsto x - iy \), where \( x, y \in g \); the maps \( \sigma, \tau \), and \( \theta \) pairwise commute and \( \tau = \theta \circ \sigma \). It is well-known that every real form of \( g^c \) can be constructed in this way, see [43]. The associated decomposition \( g = \mathfrak{k} \oplus p \) is a Cartan decomposition; the restriction of \( \theta \) to \( g \) is a Cartan involution of \( g \). Note that \( \theta \) acts on \( \mathfrak{k} \) and \( p \) by multiplication with 1 and \(-1\), respectively. We note that a real form \( g = \mathfrak{k} \oplus p \) is compact if and only if \( p = \{0\} \).

**Lemma 1.** Let \( g = \mathfrak{k} \oplus p \) be as before, with Cartan involution \( \theta \). If \( a \leq g \) is a semisimple \( \theta \)-stable subalgebra, then \( a = (a \cap \mathfrak{k}) \oplus (a \cap p) \) is a Cartan decomposition of \( a \).
PROOF. Write $\theta = \sigma \circ \tau$ where $\sigma$ is the complex conjugation associated with $g$, and $\tau$ is the compact real structure corresponding to the compact real form $u$ of $g^c$. Recall that $\xi = u_+$ and $p = u_-$ where $u_\pm$ is the $\pm 1$-eigenspace of $\tau|u$. Since $\alpha$ is stable under $\sigma$ and $\tau$, we know $\tau(\alpha) = \alpha$. Write $g^c = u \oplus u_\xi$, so that $a^c = b \oplus \xi b$ with $b = a^c \cap u$. In particular, $b$ is a real form of $a^c$ with real structure $\tau|a^c$. By a theorem of Karpelevich-Mostow (see [43, Cor. 6.1]), every Cartan involution of $b$ extends to a Cartan involution of $u$. In particular, $b$ is in fact a compact real form of $a^c$. (Since $u$ is a compact real form, the Cartan decomposition of $b$ must have a trivial 'p-part', hence $b$ is compact as well.) Clearly, $\theta|a^c$ is an automorphism of $a^c$ commuting with $\tau|a^c$, and $b = b_+ \oplus b_-$ where $b_\pm$ is the $\pm 1$-eigenspace of $\theta|b$. Now $b_+ \oplus \xi b_-$ is a real form of $a^c$ with Cartan involution $\theta|a^c$. Note that $b_+ = (a^c \cap u) \cap u_+ = (a^c \cap \xi) = a \cap \xi$ and $b_- = (a^c \cap u) \cap u_- = \tau(\xi \cap p)$, which proves the assertion. \[ \Box \]

2.3. Cartan subalgebras. Let $g$ be a real semisimple Lie algebra with adjoint group $G$. Let $g = \xi \oplus p$ be a Cartan decomposition with associated Cartan involution $\theta$. By [36, Prop. 6.59], every Cartan subalgebra of $g$ is $G$-conjugate to a $\theta$-stable Cartan subalgebra. Moreover, Kostant [38] and Sugiura [46] (using independent methods) have shown that, up to $G$-conjugacy, there are a finite number of Cartan subalgebras in $g$. We described in [14] how the methods of Sugiura yield an algorithm for constructing, up to $G$-conjugacy, all $\theta$-stable Cartan subalgebras of $g$. This algorithm has been implemented in our software package CoReLG [13] for the computer algebra system GAP [29], see [15] for more details on CoReLG.

Let $g$ be as above, with Cartan subalgebra $h$; let $\Phi$ be the root system of $g^c$ with respect to $h^c$. Define

\begin{align*}
N_{G^c}(h^c) &= \{ g \in G^c \mid g(h^c) \subseteq h^c \}, \\
Z_{G^c}(h^c) &= \{ g \in G^c \mid g(h) = h \text{ for all } h \in h^c \}.
\end{align*}

Let $W$ be the Weyl group of $\Phi$, and view $\Phi$ as subset of the dual space $(h^c)^*$. For $g \in N_{G^c}(h^c)$ and $\alpha \in \Phi$ define $\alpha^g = \alpha \circ \alpha^{-1}$; using this definition, $g(\alpha^c) = g^c(\alpha)$, in particular, $\alpha^g \in \Phi$. Hence, every $g \in N_{G^c}(h^c)$ yields a map

$$
\psi_g : \Phi \to \Phi, \quad \alpha \mapsto \alpha^g.
$$

If $g, h \in N_{G^c}(h^c)$ then $\psi_{gh}$ maps $\alpha$ to $\alpha \circ h^{-1} \circ g^{-1}$, thus $\psi_{gh} = \psi_g \circ \psi_h$. The next theorem is [47, Thm 30.6.5]; it allows us to define an action of $W$ on $h^c$.

**Theorem 2.** If $g \in N_{G^c}(h^c)$, then $\psi_g \in W$. The map $N_{G^c}(h^c) \to W, g \mapsto \psi_g$ is a surjective group homomorphism with kernel $Z_{G^c}(h^c)$. In particular, $W \cong N_{G^c}(h^c)/Z_{G^c}(h^c)$.

**Lemma 3.** If $w \in W$, then $w(h_{\alpha}) = h_{w(\alpha)}$ for all $\alpha \in \Phi$.

**Proof.** Theorem 2 shows that $w = \psi_g$ for some $g \in N_{G^c}(h^c)$, and the action of $w$ on $h^c$ is defined as $w(h) = g(h)$. If $\alpha \in \Phi$, then $g(x_{\alpha}) \in g_c(w(\alpha))$, hence $g(x_{\alpha}) = \lambda_{\alpha}(x_{w(\alpha)})$ for some $\lambda_{\alpha} \in \mathbb{C}$. It follows from (2.1) that $g(h_{\alpha}) = \lambda_{\alpha} h_{w(\alpha)}$ and $\lambda_{-\alpha} = \lambda_{\alpha}^{-1}$, hence $w(h_{\alpha}) = g(h_{\alpha}) = h_{w(\alpha)}$. \[ \Box \]

3. Computing the real Weyl group

Let $g = \xi \oplus p$ be as in the previous section, with Cartan involution $\theta = \tau \circ \sigma$ and $\theta$-stable Cartan subalgebra $h$; recall that $\tau$ is a compact real structure. Let $\Phi$ and $W$ be the root system and Weyl group associated with $h^c$; let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of simple roots and let $\{h_1, \ldots, h_\xi, x_{\alpha} \mid \alpha \in \Phi\}$ be a Chevalley basis of $q^c$. Recall the definition of $h_{\alpha} = [x_{\alpha}, x_{-\alpha}]$. We define $N_{G^c}(h)$ and $Z_{G^c}(h)$ as in the complex case, and the **real Weyl group** of $g$ relative to $h$ as

$$
W(h) = N_{G^c}(h)/Z_{G^c}(h),
$$

see [36, (7.92a)]. It follows from [36, (7.93)] that

$$
W(h) \leq W.
$$
An algorithm for finding generators of \( W(h) \), based on [3, Prop. 12.14], is implemented in the ATLAS software [1]. Here we describe a similar, but also more direct algorithm; it is based on the following theorem (see [2, Prop. 5.1] for a very similar statement).

**Theorem 4.** The real Weyl group is \( W(h) = \{ w \in W \mid \exists g \in N_{G^c}(h^c) : g \circ \theta = \theta \circ g \text{ and } w = g|_{h^c} \}. \)

**Proof.** First, we prove “\( \supseteq \)”; let \( w \in W(h) \). If \( K \) is the connected Lie subgroup of \( G \) with Lie algebra \( \mathcal{K} \), then \( W(h) = N_K(h)/Z_K(h) \) by [36, (7.92b)]. Thus, there is \( g \in N_K(h) \) whose restriction to \( h^c \) coincides with \( w \). Clearly, all elements of \( K \) commute with \( \theta \).

Second, we prove “\( \subseteq \)”; let \( w \in W \) such that \( w = g|_{h^c} \) for some \( g \in N_{G^c}(h^c) \) with \( g \circ \theta = \theta \circ g \). Consider the compact structures \( \tau \) and \( \tau' = g \circ \tau \circ g^{-1} \). The corollary to [43, Prop. 3.6] shows that \( \tau = \eta \circ \tau' \circ \eta^{-1} \) for some \( \eta \in G^c \); in particular, one can choose \( \eta = \varphi^{-1/4} \), where \( \varphi = (\tau' \circ \tau)^2 \) and \( \varphi' = \exp(t \log \varphi) \), \( t \in \mathbb{R} \), is a 1-parameter subgroup, see [43, p. 23]. Since \( \tau' \circ \tau \) commutes with \( \theta \), do \( \varphi \) and \( \eta \); the latter follows from the fact that \( \varphi \) and \( \varphi' \) have the same eigenvectors, see [43, p. 23].

If \( \alpha \in \Phi \), then \( \theta(h_\alpha) = h_{\alpha \circ \theta} \) and \( \sigma(h_\alpha) = h_{-\alpha \circ \theta} \), see [14, Lem. 6], hence \( \tau(h_\alpha) = h_{-\alpha} = -h_\alpha \). By Lemma 3 we have \( g(h_\alpha) = h_{w(\alpha)} \), implying that \( \tau' \circ \tau(h_\alpha) = h_\alpha \). Since \( \{h_1, \ldots, h_\ell\} \) with \( h_i = h_{\alpha_i} \) is a basis for \( h^c \), we get \( \tau' \circ \tau \in Z_{G^c}(h^c) \), thus \( \eta \in Z_{G^c}(h^c) \).

Now define \( \bar{g} = \eta \circ g \in G^c \), so that \( \bar{g} \) commutes with \( \theta \) and with \( \tau \); for the latter note that \( \tau = \eta \circ \tau' \circ \eta^{-1} = \eta \circ g \circ \tau \circ g^{-1} \circ \eta^{-1} = \bar{g} \circ \tau \circ \bar{g}^{-1} \). In particular, \( \bar{g} \) commutes with \( \sigma = \theta \circ \tau \), which proves \( \bar{g}(g) = g \). Thus, \( \bar{g} \in G^c(\mathbb{R}) = G \). Now \( \eta \in Z_{G^c}(h^c) \) implies that \( \bar{g} \in N_G(h) \) and that the restriction of \( \bar{g} \) to \( h^c \) coincides with the restriction \( g|_{h^c} \), hence with \( w \in W \) by the definition of \( g \). This proves \( w \in W(h) \). \( \square \)

If \( w \in W \), then \( w = \psi_g \) for some \( g \in N_{G^c}(h^c) \), see Theorem 2, and \( w \) acts on \( h^c \) as \( g \). Let

\[ W^\theta = \{ w \in W \mid \text{the action of } w \text{ on } h^c \text{ commutes with the restriction } \theta|_{h^c} \}. \]

Theorem 4 yields \( W(h) \subseteq W^\theta \). We now consider \( w \in W(h) \) and show how to construct \( g \in N_{G^c}(h^c) \) with \( w = \psi_g \). Let \( \{x_i,y_i,h_i \mid i = 1, \ldots, \ell\} \) be the canonical generating set contained in the Chevalley basis of \( g^c \). Clearly, \( \{x_{w(\alpha_i)}, x_{-w(\alpha_i)}, h_{w(\alpha_i)} \mid i = 1, \ldots, \ell\} \) is also a canonical generating set, and mapping \( (x_i,y_i,h_i) \) to \( (x_{w(\alpha_i)}, x_{-w(\alpha_i)}, h_{w(\alpha_i)}) \) for all \( i \) extends uniquely to an automorphism

\[ \eta_w : g^c \rightarrow g^c. \]

By Lemma 3, the actions of \( \eta_w \) and \( w \) on \( h^c \) coincide. Thus, \( \eta_w^{-1} \circ g \) fixes \( h^c \) pointwise; such an automorphism is inner, cf. [14, §2.3], hence \( \eta_w \) is inner.

If \( z \in Z_{G^c}(h) \), then \( z(x_\alpha) \) is a multiple of \( x_\alpha \); in particular, \( z \) is determined by nonzero parameters \( \lambda_1, \ldots, \lambda_\ell \in \mathbb{C} \) with \( z(x_i) = \lambda_i x_i \) and \( z(y_i) = \lambda_i^{-1} y_i \); conversely, for such parameters denote by

\[ \zeta_0(\lambda_1, \ldots, \lambda_\ell) \in Z_{G^c}(h) \]

the automorphism with \( z(x_i) = \lambda_i x_i, z(y_i) = \lambda_i^{-1} y_i \), and \( z(h_i) = h_i \) for all \( i \). In conclusion, we have proved the following corollary.

**Corollary.** The elements in \( N_{G^c}(h^c) \) whose restriction to \( h^c \) is \( w \in W(h) \) are exactly \( \eta_w \circ \zeta_0(\lambda_1, \ldots, \lambda_\ell) \) with nonzero \( \lambda_1, \ldots, \lambda_\ell \in \mathbb{C} \).

For each \( \alpha \in \Phi \) define scalars \( \mu_\alpha, \nu_\alpha \in \mathbb{C} \) by

\[ \theta(x_\alpha) = \mu_\alpha x_{\alpha \circ \theta} \quad \text{and} \quad \eta_w(x_\alpha) = \nu_\alpha x_{w(\alpha)}. \]

Observe that \( \mu_{-\alpha} = \mu_\alpha^{-1} \) and \( \nu_{-\alpha} = \nu_\alpha^{-1} \), and \( \theta(h_\alpha) = h_{\alpha \circ \theta} \). For nonzero \( \lambda_1, \ldots, \lambda_\ell \in \mathbb{C} \) write \( \zeta_0 = \zeta_0(\lambda_1, \ldots, \lambda_\ell) \). For \( \alpha = \sum_{i=1}^\ell a_i \alpha_i \) define \( \lambda_\alpha = \prod_{i=1}^\ell \lambda_i^{a_i} \) and \( \text{ht}(\alpha) = \sum_{i=1}^\ell a_i \), the height of \( \alpha \). An
induction on the height shows that ζ_0(x_α) = λ_α x_α. Assume w ∈ W^θ; we want to decide whether there exist nonzero λ_i such that η_w ∈ ζ_0(x) = θ ∈ η_w ∩ ζ_0. Since w and θ commute we get that η_w ∩ ζ_0(x) = θ ∈ η_w ∩ ζ_0(x_i) for all i. Secondly, η_w ∩ ζ_0(x_i) = θ ∩ η_w ∩ ζ_0(x_i) is equivalent to

(3.2) λ_{α,θ}^{-1} = ν_{α,θ}^{-1} μ_{α,θ}^{-1} η_w(α_i).

Thirdly, η_w ∩ ζ_0(y_i) = θ ∩ η_w ∩ ζ_0(y_i) is equivalent to (3.2). In conclusion, the next proposition follows.

**Proposition 6.** Let w ∈ W^θ. Then w ∈ W(h) if and only if there are nonzero λ_1, ..., λ_ℓ ∈ C satisfying (3.2) for all i.

The existence of a solution satisfying (3.2) can readily be checked using row Hermite normal forms, see [45, p. 322]. Note that the equations (3.2) are of the form λ_i = η_w ∩ ζ_0(x_i) = θ ∩ η_w ∩ ζ_0(x_i) is equivalent to (3.2). In conclusion, the next proposition follows.

**Proposition 6.** Let w ∈ W^θ. Then w ∈ W(h) if and only if there are nonzero λ_1, ..., λ_ℓ ∈ C satisfying (3.2) for all i.

Thus, to compute W(h), we conclude from Theorem 7 that it is sufficient to test whether w ∈ W(h) for w in a set of coset representatives of W_c in W_i. We remark that generators of W_c^θ are easily computed by algorithms that work for general permutation groups.

4. **Reductive subalgebras**

In this section, unless otherwise defined, g is a semisimple Lie algebra over a field of characteristic 0. Recall that g is reductive if its adjoint representation is completely reducible. This is the same as saying that g is the direct sum of its centre and its derived subalgebra, see [6, §6, no. 4, Proposition 5] or [47, Def. 20.5.1]. By the same proposition (or [47, Prop. 20.5.4]), a Lie algebra is reductive if and only if it has a finite dimensional representation with nondegenerate trace form. Following [6, §6, no. 6, Def. 5] or [47, Def.
Lemma 8. Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$.

a) The subalgebra $\mathfrak{a}$ is reductive in $\mathfrak{g}$ if and only if $\mathfrak{a}$ is reductive and $\text{ad}_\mathfrak{g}(z)$ is semisimple for all $z$ in the centre of $\mathfrak{a}$.

b) If the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{a}$ is nondegenerate and $\mathfrak{a}$ contains the semisimple and nilpotent parts of its elements, then $\mathfrak{a}$ is reductive in $\mathfrak{g}$.

c) Let $\mathfrak{a}$ be reductive in $\mathfrak{g}$. A subalgebra $\mathfrak{t} \leq \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{a}$ if and only if $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{a}$ consisting of semisimple elements of $\mathfrak{g}$.

Proof. a) This is [6, §6, no. 5, Théorème 4].

b) This is [47, Prop. 20.5.12]. In that book the ground field is assumed to be algebraically closed. However, the proof given there works over any field, replacing the reference to [47, Prop. 20.5.4 (iii)] by [6, §6, no. 4, Prop. 5d].

c) Since $\mathfrak{a}$ is reductive, $\mathfrak{a} = \mathfrak{c} \oplus \mathfrak{l}$, where $\mathfrak{l}$ is semisimple and $\mathfrak{c}$ is the centre of $\mathfrak{a}$. Since $\mathfrak{a}$ is reductive in $\mathfrak{g}$, it follows that $\text{ad}_\mathfrak{g}(z)$ is semisimple for each $z \in \mathfrak{c}$. A subspace $\mathfrak{t} \subseteq \mathfrak{a}$ is a Cartan subalgebra if and only if $\mathfrak{t} = \mathfrak{c} \oplus \mathfrak{i}$ where $\mathfrak{i}$ is a Cartan subalgebra of $\mathfrak{l}$. Furthermore, $\mathfrak{i}$ is a Cartan subalgebra of $\mathfrak{l}$ if and only if it is maximally toral; see, for example, [32, Cor. 15.3].

Lemma 9. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution $\theta$.

a) If $\mathfrak{a}$ is a $\theta$-stable subalgebra of $\mathfrak{g}$, then $\mathfrak{a}$ is reductive in $\mathfrak{g}$.

b) Let $\mathfrak{a} \leq \mathfrak{g}$ be a subalgebra; then $\mathfrak{a}$ is reductive in $\mathfrak{g}$ if and only if $\mathfrak{a}^\theta$ is reductive in $\mathfrak{g}^\theta$.

Proof. This is proved in [52, Cor. 1.1.5.4] and [6, §6, no. 10], respectively.

Remark 10. Let $\mathfrak{g}$ be a reductive Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\mathfrak{d}$ be the derived subalgebra of $\mathfrak{g}$. We note that the real Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the same as the real Weyl group of $\mathfrak{d}$ with respect to $\mathfrak{d} \cap \mathfrak{h}$. So the algorithm described in Section 3 works also in this case.

5. Toral subalgebras

In this section let $\mathfrak{g}$ be a real semisimple Lie algebra. A subalgebra $\mathfrak{a} \leq \mathfrak{g}$ is toral if it is abelian and $\text{ad}_\mathfrak{g}(x)$ is semisimple for all $x \in \mathfrak{a}$. Recall that if $\mathfrak{a}$ is reductive in $\mathfrak{g}$, then the Cartan subalgebras of $\mathfrak{a}$ are exactly the maximal toral subalgebras of $\mathfrak{a}$, see Lemma 8; in particular, every Cartan subalgebra of $\mathfrak{a}$ lies in some Cartan subalgebra of $\mathfrak{g}$. We now study toral subalgebras and their relation to Cartan decompositions.

Lemma 11. Let $\mathfrak{t} \leq \mathfrak{g}$ be a toral subalgebra and denote by $\mathfrak{t}_r, \mathfrak{t}_i \subseteq \mathfrak{t}$ the sets of elements $x \in \mathfrak{t}$ such that $\text{ad}_\mathfrak{g}(x)$ has only real and only purely imaginary eigenvalues, respectively. Let $\theta$ be a Cartan involution of $\mathfrak{g}$. Then the following hold.

a) Both $\mathfrak{t}_r$ and $\mathfrak{t}_i$ are subspaces of $\mathfrak{t}$.

b) If $\mathfrak{t}$ is $\theta$-stable, then $\mathfrak{t} = \mathfrak{t}_i \oplus \mathfrak{t}_r$ is decomposition into the 1- and $(-1)$-eigenspace of the restriction $\theta|_\mathfrak{t}$.

Proof. a) This follows from the fact that $\text{ad}_\mathfrak{g}(\mathfrak{t})$ is simultaneously diagonalisable over $\mathbb{C}$. 

b) This is [47, Prop. 20.5.12]. In that book the ground field is assumed to be algebraically closed. However, the proof given there works over any field, replacing the reference to [47, Prop. 20.5.4 (iii)] by [6, §6, no. 4, Prop. 5d].
Let $t_\pm$ be the $\pm 1$-eigenspace of $\theta$. It follows from [43, Prop. 5.1(ii)] that $t_+ \subseteq t_1$ and $t_- \subseteq t_r$. Let $x \in t_1$ and write $x = a + b$ with $a \in t_+$ and $b \in t_-; \text{ then, since } \text{ad}_g(x) \text{ has purely imaginary eigenvalues, } b \text{ has to be } 0. \text{ Thus, } t_+ = t_1, \text{ hence } t_- = t_r. \hfill \Box

If $h \leq g$ is a $\theta$-stable Cartan subalgebra, then by Lemma 11b

$$h \cap t = \{ h \in h \mid \text{ad}_g h \text{ has only purely imaginary eigenvalues}\},$$

$$h \cap p = \{ h \in h \mid \text{ad}_g h \text{ has only real eigenvalues}\}.$$

The dimension of $h \cap p$ is the noncompact dimension of $h$. Since $\text{ad}_g(h)$ and $\text{ad}_g(g(h))$ have the same eigenvalues for every $h \in h$ and $g$ in the adjoint group $G$ of $g$, it follows that the noncompact dimension is a well-defined concept also for non $\theta$-stable Cartan subalgebras: the noncompact dimension of any Cartan subalgebra $h'$ is the one of $h = g(h')$. It follows from [36, Prop. 6.61] that all Cartan subalgebras of maximal noncompact dimension are $G$-conjugate. The real rank of $g$ is the noncompact dimension of a maximally noncompact Cartan subalgebra of $g$, cf. [36, p. 424]. The next definition generalises these concepts for subalgebras reductive in $g$.

**Definition 12.** Let $a \leq g$ be reductive in $g$ and let $h \leq a$ be a Cartan subalgebra. The noncompact dimension of $h$ is $\dim h_r$, where $h_r$ is as in Lemma 11. A Cartan subalgebra $h \leq a$ is maximally noncompact if the noncompact dimension of $h$ is as large as possible. The noncompact dimension of such a Cartan subalgebra is called the real rank of $a$.

In the remainder of this section, we discuss how to describe and compute the real rank of a subalgebra which is reductive in $g$. We start with a preliminary result.

**Lemma 13.** If $x \in g$ is semisimple, then there exist unique $a, b \in g$ with

1. $x = a + b$ with both $a$ and $b$ semisimple and $[x, a] = [x, b] = [a, b] = 0$,
2. $\text{ad}_g(a)$ has purely imaginary eigenvalues only, $\text{ad}_g(b)$ has real eigenvalues only,
3. if $y \in g$ with $[x, y] = 0$, then $[a, y] = [b, y] = 0$.

The elements $x_1 = a$ and $x_r = b$ are the imaginary part and real part of $x$.

**Proof.** Let $h$ be a Cartan subalgebra of $g$ containing $x$. Let $\theta$ be a Cartan involution of $g$ stabilising $h$; this exists by [36, Prop. 6.59]. Let $g = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Now define $a = \frac{1}{2}(x + \theta(x)) \in \mathfrak{k}$ and $b = \frac{1}{2}(x - \theta(x)) \in \mathfrak{p}$, so that $x = a + b$. Since $a, b \in h$, both $a$ and $b$ are semisimple and commute with $x$; in particular, $0 = [a, x] = [a, b]$. By Lemma 11, the adjoints $\text{ad}_g(a)$ and $\text{ad}_g(b)$ only have purely imaginary and real eigenvalues, respectively. Note that if $x = a' + b'$ with the same properties, then $a' - a = b - b' \in \mathfrak{k} \cap \mathfrak{p} = \{0\}$, thus $a$ and $b$ are unique. It remains to prove (3). With respect to a Chevalley basis of $g^c$ (with respect to $h^c$), it follows that $\text{ad}_g^c(a)$ and $\text{ad}_g^c(b)$ are represented by diagonal matrices $A$ and $B$ with purely imaginary and real entries, respectively. In particular, $\text{ad}_g^c(x) = \text{ad}_g^c(a) + \text{ad}_g^c(b)$ is represented by $A + B$. This implies (3). \hfill \Box

**Lemma 14.** Let $t \leq g$ be a toral subalgebra and $\theta$ a Cartan involution of $g$. If $t$ is $\theta$-stable, then it is closed under taking real and imaginary parts. Conversely, if $t$ is closed under taking real and imaginary parts, then there is a Cartan involution $\theta'$ of $g$ stabilising $t$.

**Proof.** Suppose $t$ is $\theta$-stable, and let $x \in t$. Then $x = a + b$, where $a = \frac{1}{2}(x + \theta(x))$ and $b = \frac{1}{2}(x - \theta(x))$. The proof of Lemma 13 shows that $a, b \in t$ are the imaginary and real parts of $x$. To prove the converse, let $h$ be a Cartan subalgebra of $g$ containing $t$, and let $\theta'$ be a Cartan involution of $g$ stabilising $h$; this exists
by [36, Prop. 6.59] and [36, Cor. 6.19]. Let \( x \in t \) and let \( x = a + b \) be the decomposition into imaginary and real parts. Since \( a, b \in t \leq h \), Lemma 11 yields, \( \theta'(a) = a \) and \( \theta'(b) = -b \), so \( \theta'(x) = a - b \in t \).

The next lemma shows that Definition 12 is in line with the definition for semisimple Lie algebras.

**Lemma 15.** Let \( \mathfrak{a} \leq \mathfrak{g} \) be reductive in \( \mathfrak{g} \), and decompose \( \mathfrak{a} = \mathfrak{d} \oplus t \) where \( \mathfrak{d} = [\mathfrak{a}, \mathfrak{a}] \) and \( t \) is the centre of \( \mathfrak{a} \).

a) Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{a} \) and decompose \( \mathfrak{h} = \mathfrak{h}_d \oplus t \) with \( \mathfrak{h}_d \leq \mathfrak{d} \). Using the notation of Lemma 11, we have \( h_t = (h_d)_t \oplus t_t \).

b) A Cartan subalgebra \( \mathfrak{h} \leq \mathfrak{a} \) is maximally noncompact if and only if \( h_d \) is a maximally noncompact Cartan subalgebra of \( \mathfrak{d} \).

c) Up to conjugacy in its adjoint group, \( \mathfrak{a} \) has a unique maximally noncompact Cartan subalgebra.

**Proof.** a) Lemma 8 shows that \( \mathfrak{a} \) is reductive, so we can decompose \( \mathfrak{a} = \mathfrak{d} \oplus t \) and \( \mathfrak{h} = \mathfrak{h}_d \oplus t \). Suppose that \( t \) is not closed under taking real and imaginary parts, say \( x = x_i + x_r \in t \) with \( x_i, x_r \notin t \). Let \( t_1 \) be the subalgebra spanned by \( t, x_i \), and \( x_r \). It follows from Lemma 13 that \( t_1 \) is toral and \( [\mathfrak{d}, t_1] = 0 \). Iterating this process, we find a toral subalgebra \( t' \) containing \( t \), such that \( [\mathfrak{d}, t'] = 0 \) and \( t' \) is closed under taking real and imaginary parts. Note that \( \mathfrak{d} \cap t' = 0 \) since \( [\mathfrak{d}, t'] = 0 \) and \( \mathfrak{d} \) has trivial centre. By a theorem of Karpelevich-Mostow (see [43, Cor. 6.1]), every Cartan involution of \( \mathfrak{d} \) extends to a Cartan involution of \( \mathfrak{g} \); thus there exists a Cartan involution \( \theta \) of \( \mathfrak{g} \) which stabilises \( h_d \). Since \( \mathfrak{h} \leq \mathfrak{g} \) (hence also \( h_d \leq \mathfrak{g} \)) is toral by Lemma 8, it follows from Lemma 14 that \( h_d \) is closed under taking real and imaginary parts. Now let \( h \in \mathfrak{h}_t \) and write \( h = u + v \) with \( u \in h_d \) and \( v \in t \). Decompose \( u = u_t + u_i \) and \( v = v_t + v_i \), and note that \( u_t, u_i \in h_d \) and \( v_t, v_i \in t' \). In particular, \( h = (u_t + v_t) + (u_i + v_i) \) with \( u_t + v_t \in (h_d)_t \oplus t_t \) and \( u_i + v_i \in (h_d)_i \oplus t'_i \). But \( h \in \mathfrak{h}_t \), so \( u_i + v_i = 0 \). Since \( \mathfrak{d} \cap t' = 0 \), we get \( u_i = v_i = 0 \), hence \( h_t \subseteq (h_d)_t \oplus t_t \). The other inclusion is obvious.

b) By the proof of Part a), there is a Cartan involution \( \theta \) which stabilises \( \mathfrak{d} \) and \( h_d \). Let \( \mathfrak{d} = t' \oplus p' \) and \( \mathfrak{g} = \mathfrak{t} \oplus p \) be the corresponding Cartan decompositions; clearly, \( t' \leq t \) and \( p' \leq p \). Recall that \( \mathfrak{a} \leq \mathfrak{g} \) is reductive in \( \mathfrak{g} \), hence \( \mathfrak{h} \) is toral, hence \( h_d \) is toral. It follows from Lemma 11 that \( h_d = (h_d)_t \oplus (h_d)_i \) with \( (h_d)_t = h_d \cap t' \) and \( (h_d)_i = h_d \cap p' \). Part a) yields \( h_t = (h_d \cap p') \oplus t_t \), which shows that \( \dim h_t \) is as large as possible if and only if \( \dim (h_d \cap p') \) is as large as possible, if and only if \( h_d \leq \mathfrak{d} \) is a maximally noncompact Cartan subalgebra.

c) This follows from b) and the uniqueness of maximally noncompact Cartan subalgebras in semisimple real Lie algebras, see [36, Prop. 6.61].

6. Graded semisimple Lie algebras

Let \( \mathfrak{g} \) be a semisimple real Lie algebra. For a positive integer \( m \) let \( \mathbb{Z}_m \) be the integers modulo \( m \); in addition, define \( \mathbb{Z}_\infty = \mathbb{Z} \). A \( \mathbb{Z}_m \)-grading of \( \mathfrak{g} \) with \( m \in \mathbb{N} \cup \{ \infty \} \) is a decomposition into subspaces

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i
\]

such that \( [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \) for all \( i, j \in \mathbb{Z}_m \). This implies that \( \mathfrak{g}_0 \leq \mathfrak{g} \) is a subalgebra. As usual, write \( \mathfrak{g}_c^i = \mathfrak{g}_i \otimes \mathbb{C} \), so that \( \mathfrak{g}_c = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_c^i \). As before, denote by \( \sigma \) the conjugation of \( \mathfrak{g}_c \) with respect to \( \mathfrak{g} \).

**Lemma 16.** a) There is a Cartan involution \( \theta \) of \( \mathfrak{g} \) such that \( \theta(\mathfrak{g}_i) = \mathfrak{g}_{-i} \) for all \( i \).

b) The subalgebras \( \mathfrak{g}_0^c \) and \( \mathfrak{g}_0 \) are reductive in \( \mathfrak{g}_c \) and \( \mathfrak{g} \), respectively.

**Proof.** Part a) is [39, Thm 3.4(2)]. If \( \theta \) is as in a), then \( \theta(\mathfrak{g}_0) = \mathfrak{g}_0 \), and Lemma 9 proves that \( \mathfrak{g}_0 \) and \( \mathfrak{g}_0^c \) are reductive in \( \mathfrak{g} \) and \( \mathfrak{g}_c \), respectively. \( \square \)
Lemma 17. Let $h_0^c$ be a Cartan subalgebra of $g_0^c$. For each $i$, the weight spaces of $h_0^c$ in $g_i^c$, of nonzero weight, have dimension 1.

Proof. If $m = \infty$, then $h_0^c$ is a Cartan subalgebra of $g^c$, see for example [17, p. 370], so that weight spaces are just root spaces, and therefore they have dimension 1, see [32, Prop. 8.4]. If $m$ is a positive integer and $\omega \in \mathbb{C}$ is a primitive $m$-th root of unity, then the linear map $\varphi : g^c \to g^c$ defined by $\varphi(x) = \omega^i x$ for $x \in g_i^c$ is an automorphism of $g^c$. Clearly, the $\mathbb{Z}_m$-grading of $g^c$ coincides with the eigenspace decomposition of $\varphi$; now the assertion follows from [30, Lem X.5.4(i)]. \hfill $\square$

The following lemma is well-known, see [50, §1.4].

Lemma 18. If $g = \bigoplus_{i \in \mathbb{Z}} g_i$ is a $\mathbb{Z}$-graded real semisimple Lie algebra, then there is a unique defining element $h_0 \in g_0$ such that $g_i = \{ x \in g \mid [h_0, x] = ix \}$ for all $i$.

Here we do not try to classify the $\mathbb{Z}_m$-gradings on a real semisimple Lie algebra $g$ (for the case of $\mathbb{Z}$-gradings, see [17]); instead we describe two standard constructions of $\mathbb{Z}_m$-gradings, which yield many interesting examples. In both cases, let $g$ be the real subalgebra of $g_i^c$ generated by a canonical generating set $B = \{ h_i, x_i, y_i \mid i = 1, \ldots, \ell \}$ of $g_i^c$; then $g$ is a split real form of $g_i^c$, see [43, p. 17], with complex conjugation $B$; set defined by $\theta(h_i) = -h_i, \theta(x_i) = -y_i$, and $\theta(y_i) = -x_i$ for all $i$, see [43, Exam. 3.2]. Let $\{ h_1, \ldots, h_\ell, x_\alpha \mid \alpha \in \Phi \}$ be a Chevalley basis containing $B$, with $x_\alpha = x_i$ and $x_{-\alpha} = y_i$, for all simple roots $\alpha$. By construction, all $x_\alpha$ lie in $g$.

Example 19. We use the previous notation.

a) We construct a $\mathbb{Z}$-grading and, for this purpose, define a degree of the roots: for each $\alpha$ choose some integer $d(\alpha) \geq 0$; for a positive root $\alpha = \sum_i a_i \alpha_i$ define $d(\alpha) = \sum_i a_i d(\alpha_i)$ and $d(-\alpha) = -d(\alpha)$. Let $g_0$ be the span of $h_1, \ldots, h_\ell$ along with the $x_\alpha$ satisfying $d(\alpha) = 0$. For $i \in \mathbb{Z}$ define $g_i$ as the span of all $x_\alpha$ with $d(\alpha) = i$. Then $g = \bigoplus_{i \in \mathbb{Z}} g_i$ is a $\mathbb{Z}$-grading with $\theta(g_i) = g_{-i}$ for every $i$.

b) Now let $m \geq 1$ be an integer and $\omega \in \mathbb{C}$ a primitive $m$-th root of unity. Let $\pi$ be a permutation of $\{ 1, \ldots, \ell \}$, of order 1 or 2, such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_{\pi(i)}, \alpha_{\pi(j)} \rangle$ for $1 \leq i, j \leq \ell$, that is, $\pi$ defines a diagram automorphism and a bijection $\Phi \to \Phi$, also denoted $\pi$. We require that $m$ is even if $\pi$ has order 2. Mapping $(x_i, y_i, h_i)$ to $(x_{\pi(i)}, y_{\pi(i)}, h_{\pi(i)})$ for all $i$ defines an automorphism of $g_i^c$, also denoted $\pi$. If $k_1, \ldots, k_\ell$ are non-negative integers with $k_i = k_{\pi(i)}$ for all $i$, then $(x_i, y_i, h_i) \mapsto (\omega^{h_i}, \omega^{-k_i} y_i, h_i)$ for all $i$ defines an inner automorphism $\eta$ of $g_i^c$ which commutes with $\pi$. Suppose the $k_i$ are chosen such that $\eta$ has order $m$, thus $\varphi = \pi \circ \eta$ is an automorphism of $g_i^c$ of order $m$.

If $\alpha = \sum_j a_j \alpha_j$, then $\eta(x_\alpha) = \omega^r x_\alpha$ where $r = \sum_j a_j k_j$, and each $x_\alpha$ is an eigenvector of $\eta$. Thus, the eigenspace $V_i$ of $\eta$ with eigenvalue $\omega^i \neq 1$ is spanned by all $x_\alpha$ with $\eta(x_\alpha) = \omega^i x_\alpha$; the 1-eigenspace $V_0$ of $\eta$ is spanned by $h_1, \ldots, h_\ell$ and $x_\alpha$ with $\eta(x_\alpha) = x_\alpha$. This and the definition of $\theta$ imply that $\theta(V_i) = V_{-i}$ for all $i$. Since $x_\alpha \in g$ for all roots $\alpha$, we also have that $\sigma(V_i) = V_i$ for all $i$.

Let $g_i^c = g_+^c \oplus g_-^c$ be the $\pm 1$-eigenspace decomposition of $\pi$. By construction, $\pi$ and $\theta$ commute, hence $\theta$ fixes $g_+^c$ and $g_-^c$. Also, $\pi(x_\alpha) = x_{\pi(\alpha)}$ for every $\alpha \in \Phi$, and it is easy to see that there exist bases of $g_+^c$ and $g_-^c$ which are fixed by $\sigma$, hence $\sigma(g_+^c) = g_+^c$ and $\sigma(g_-^c) = g_-^c$.

Let $g_i^c$ be the eigenspace of $\varphi$ with eigenvalue $\omega^i$ for $i = 0, 1, \ldots, m - 1$. Since $\pi$ and $\eta$ commute, $\eta(g_i^c) = g_{\pi(i)}^c$ and $\pi(V_i) = V_{\pi(i)}$ for all $i$, hence $V_i = (V_i \cap g_+^c) \oplus (V_i \cap g_-^c)$. Clearly, $V_i \cap g_+^c \leq g_+^c$ and $V_i \cap g_-^c \leq g_-^c$. Note that in the latter case $\omega^{m/2} = -1$. Thus, either $\pi = 1$ and $g_i^c = V_i$, or $\pi$ has order 2, $m$ is even, and

\[ g_i^c = (g_+^c \cap V_i) \oplus (g_-^c \cap V_{m/2+i}). \]
Since \( \theta(V_i) = V_{-i} \) for all \( i \), and \( g^c \) and \( g^c_i \) are fixed by \( \theta \), in both cases we have \( \theta(g^c_i) = g^c_{-i} \) and \( \sigma(g^c_i) = g^c_i \) for all \( i \). In particular, if \( g_i = g^c_i \cap g \), then \( g = \bigoplus_{i \in \mathbb{Z}_m} g_i \), and \( \theta(g_i) = g_{-i} \) for all \( i \).

6.1. A representation associated with the grading. Let \( g = \bigoplus_{i \in \mathbb{Z}_m} g_i \) be a real semisimple Lie algebra with complexification \( g^c = \bigoplus_{i \in \mathbb{Z}_m} g^c_i \). It is customary to associate a representation of an algebraic group to the grading of \( g^c \), see [49, 50]. Namely, as in Section 1.2, let \( G^c \) be the adjoint group of \( g^c \), and define \( G^c_0 \) as the connected algebraic subgroup of \( G^c \) with Lie algebra \( \text{ad}g^c(g^c_0) \). This group acts on \( g^c_i \), yielding a representation

\[ \rho^c: G^c_0 \to \text{GL}(g^c_i). \]

We note that the differential of this representation is \( d\rho^c: g^c_0 \to g^c_i \), given by \( d\rho^c(x)(y) = [x, y] \). In the literature, \( G^c_0 \) is called a "\( \theta \)-group" and \( \rho^c \) a "\( \theta \)-representation", but we avoid using this terminology here as we already use "\( \theta \)" to denote a Cartan involution.

Again, as in Section 1.2, let \( G \) be the adjoint group of \( g \). Note that \( G^c_0 \) and \( \rho^c \) are defined over \( \mathbb{R} \), so if we define \( G_0 \) to be the Lie subgroup of \( G \) with Lie algebra \( \text{ad}g(g_0) \), then \( G_0 = G^c_0(\mathbb{R}) = \{ g \in G^c \mid g(g) = g \} \). Furthermore, we can restrict \( \rho^c \) to \( G_0 \) to obtain a representation \( \rho: G_0 \to \text{GL}(g_1) \).

7. Listing semisimple regular \( \mathbb{Z} \)-graded subalgebras

Let \( g = \bigoplus_{i \in \mathbb{Z}_m} g_i \) be a real semisimple Lie algebra with complexification \( g^c = \bigoplus_{i \in \mathbb{Z}_m} g^c_i \). By Lemma 16, throughout this section, we suppose that the Cartan involution \( \theta \) of \( g \) satisfies

\[ \theta(g_i) = g_{-i} \quad \text{for all } i. \]

A \( \mathbb{Z} \)-graded subalgebra \( s^c \) of \( g^c \) is a subalgebra with \( \mathbb{Z} \)-grading \( s^c = \bigoplus_{k \in \mathbb{Z}} s^c_k \) such that \( s^c_k \subset g^c \mod m \) for all \( k \); here \( k \mod m = k \) if \( m = \infty \). It is a regular subalgebra of \( g^c \) if it is normalised by some Cartan subalgebra \( h^c_0 \) of \( g^c_0 \). If we want to specify the particular Cartan subalgebra, then we say that \( s^c \) is \( h^c_0 \)-regular. We define the same concepts for subalgebras of \( g \). The following is an easy observation.

Lemma 20. Let \( s^c \leq g^c \) be an \( h^c_0 \)-regular subalgebra, where \( h^c_0 \leq g^c_0 \) is a Cartan subalgebra. If \( h^c_0 \) contains the (unique) defining element \( h \) of \( s^c \), then \( [h^c_0, s^c] \subseteq s^c \) for all \( i \).

Proof. If \( k \in h^c_0 \), then \( [k, s^c] \subseteq s^c \) since \( s^c \) is \( h^c_0 \)-regular. Thus, if \( x \in s^c \), then \( [h, [k, x]] = -[k, [x, h]] - [x, [k, h]] = [k, [h, x]] = i[k, x] \), and \( [k, x] \in s^c \) follows. Hence, \( [h^c_0, s^c] \subseteq s^c \) for all \( i \). \( \square \)

In this section, we describe an algorithm for the following task: Given a semisimple \( \mathbb{Z} \)-graded regular subalgebra \( a^c \) of \( g^c \), list, up to \( G_0 \)-conjugacy, all semisimple \( \mathbb{Z} \)-graded regular subalgebras \( s \) of \( g \) such that \( s^c \) is \( G^c_0 \)-conjugate to \( a^c \). First we introduce some notation. Let \( h^c_0 \) be a Cartan subalgebra of \( g_0 \), so \( h^c_0 \) is a Cartan subalgebra of \( g^c_0 \). For \( \lambda \in (h^c_0)^* \) and \( k \in \mathbb{Z}_m \) define

\[ g^c_{k, \lambda} = \{ x \in g^c \mid \forall h \in h^c_0: [h, x] = \lambda(h)x \}. \]

By Lemma 17, if \( \lambda \neq 0 \), then \( g^c_{k, \lambda} \) is \( 0 \) or \( g^c_{k, \lambda} \) has dimension 1. Let

\[ P(g^c) = \{ (k, \lambda) \mid k \in \mathbb{Z}_m, \lambda \in (h^c_0)^*, \lambda \neq 0, g^c_{k, \lambda} \neq 0 \}. \]

Let \( s \) be an \( h^c_0 \)-regular \( \mathbb{Z} \)-graded semisimple subalgebra of \( g \), and set \( s^c_{k, \lambda} = g^c_{k, \lambda} \cap s^c \) for all \( k \) and \( \lambda \). Thus, either \( s^c_{k, \lambda} = 0 \) or \( s^c_{k, \lambda} \neq 0 \) and \( s^c_{k, \lambda} \) is a weight space of \( s^c \) of weight \( (k, \lambda) \). Since \( s^c \) is \( h^c_0 \)-regular, it is the sum of its weight spaces. Let

\[ P(s^c) \subseteq P(g^c) \]
be the set of all weights \((k, \lambda)\) of \(s^c\) with \(\lambda \neq 0\). Since \(s^c_{k,\lambda} = s^c_{k+l,\lambda+\mu}\), weights are added componentwise: \((k, \lambda) + (l, \mu) = (k + l, \lambda + \mu)\). If \(\kappa\) is the Killing form of \(s^c\), then \(\kappa(s^c_{k,\lambda}, s^c_{k,\lambda}) = 0\) unless \(l = -k, \mu = -\lambda\). As \(\kappa\) is nondegenerate, we have
\[
(k, \lambda) \in P(s^c) \iff (-k, -\lambda) \in P(s^c).
\]
Let \(W_0^c = N_{G_0}(h_0^c)/Z_{G_0}(h_0^c)\) be the Weyl group of \(g_0^c\) relative to \(h_0^c\). The group \(W_0^c\) acts on \(P(g^c)\) as follows: if \(w \in W_0^c\) with \(g \in N_{G_0}(h_0^c)\) projecting to \(w\), then
\[
w \cdot (k, \lambda) = (k, \lambda^w);
\]
recall that \(\lambda^w = \lambda \circ g^{-1}\). Note that \(W_0^c\) is the Weyl group of the root system of \(g_0^c\) relative to \(h_0^c\), see Theorem 2; hence, by Lemma 3, we know how \(W_0^c\) acts on \((h_0)^*\) without computing a \(g \in G_0^c\) for a given \(w \in W_0^c\). Similarly, \(W_0(h_0) = N_{G_0}(h_0)/Z_{G_0}(h_0)\) is the real Weyl group relative to \(h_0\); note that \(W_0(h_0) \leq W_0^c\), see Section 3.

**Proposition 21.** Let \(h_0\) be a \(\theta\)-stable Cartan subalgebra of \(g_0\), and let \(s\) be an \(h_0\)-regular \(\mathbb{Z}\)-graded semisimple subalgebra of \(g\). Then the following hold.

a) The algebra \(s\) is \(\theta\)-stable.

b) The normaliser \(n_{g_0}(s) = \{x \in g_0 \mid [x, s] \subseteq s\}\) is reductive in \(g\).

**Proof.** a) Recall that \(g = \mathfrak{k} \oplus \mathfrak{p}\) and \(h_0 = (h_0 \cap \mathfrak{k}) \oplus (h_0 \cap \mathfrak{p})\), and note that \(\sigma\) leaves \(s\) and \(s^c\) invariant. Let \((k, \lambda)\) be a weight of \(s^c\) with \(\lambda \neq 0\), thus \(s^c_{k,\lambda} = g_{k,\lambda}\), and define the linear map \(\mu : h_0^c \rightarrow \mathbb{C}\) by
\[
\mu(h) = \lambda(h) \ 	ext{if} \ h \in h_0^c \cap \mathfrak{k}^c,
\]
\[
-\lambda(h) \ 	ext{if} \ h \in h_0^c \cap \mathfrak{p}^c.
\]
Note that \(\sigma\) maps \(g_{k,\lambda}^c\) to itself whereas \(\theta(g_{k,\lambda}^c) = g_{-k,\lambda}^c\). Using this and Lemma 11, we see that \(\sigma(s_{k,\lambda}^c) = g_{k,\mu}^c\) and \(\theta(s_{k,\lambda}^c) = g_{-k,\mu}^c\). Since \(\sigma(s^c) = s^c\), we conclude that \((k, -\mu) \in P(s^c)\) and \(g_{k,\mu}^c = s_{k,-\mu}^c\). From what is said above, \((-k, \mu) \in P(s^c)\), hence \(\theta(s_{k,\lambda}^c) = s_{k,-\mu}^c\). Therefore, \(s^c\), and hence \(s\), is \(\theta\)-stable.

b) Since \(s\) is \(\theta\)-stable, the same holds for \(n_{g_0}(s)\), and the assertion follows from Lemma 9; recall that \(\theta(g) = g^{-1}\) for all \(i\), hence \(g_0\) is \(\theta\)-stable by assumption. \(\square\)

**Definition 22.** Let \(h_0 \leq g_0\) be a Cartan subalgebra. An \(h_0\)-regular subalgebra \(s \leq g\) is **strongly \(h_0\)-regular** if \(h_0\) is maximally noncompact in \(n_{g_0}(s)\).

Proposition 21 shows that \(g_0\) and \(n_{g_0}(s)\) both are reductive in \(g\). Thus, by Lemma 15, there is, up to conjugacy, a unique maximally noncompact Cartan subalgebra of \(n_{g_0}(s)\).

We end this section with another useful result on \(\mathbb{Z}\)-graded semisimple subalgebras.

**Lemma 23.** If \(s \leq g\) is a \(\mathbb{Z}\)-graded semisimple subalgebra, then \(\delta_{g_0}(s)\) is reductive in \(g\).

**Proof.** We first show that \(C = \delta_g(s)\) is reductive in \(g\). Let \(\kappa\) be the Killing form of \(g\); we consider \(g\) as an \(s\)-module. By Weyl’s Theorem, see [27, Thm 4.4.6], the submodule \(C\) has a complement, say \(g = C \oplus U\). Similarly, \([s, U]\) is a submodule of \(U\) and there is a submodule \(V \subseteq U\cap C = \{0\}\) proves \([s, U] = U\), thus every \(v \in U\) has the form \(v = [u, v']\) with \(v' \in U\) and \(u \in s\). If \(y \in C\), then \(\kappa(y, v) = \kappa([y, u], v') = \kappa([y, u], v') = 0\), thus \(\kappa|_{C \times U} = 0\). Since \(\kappa\) is nondegenerate, this implies that the restriction of \(\kappa\) to \(C\) must be nondegenerate. Let \(g \in C\) with Jordan decomposition \(g = s + n\) in \(g\). Since \(g\) centralises \(s\), so do \(s\) and \(n\), see [27, Prop. A.2.6], thus \(s, n \in C\). It follows from Lemma 8 that \(C\) is reductive in \(g\).
The next step is to show that \( C_0 = \mathfrak{g}_{00}(s) \) is reductive in \( \mathfrak{g} \). For \( i \geq 0 \) define \( C_i = C \cap \mathfrak{g}_i \), and \( D = \bigoplus_{i \in \mathbb{Z}_m} C_i \); clearly, \( D \leq C \). Write \( c \in C \) as \( c = \bigoplus_{i \in \mathbb{Z}_m} c_i \) with \( c_i \in \mathfrak{g}_i \). Since \( c \in C \), we have \( 0 = [c, s] = \bigoplus_{i \in \mathbb{Z}_m} [c_i, s] \) for all \( s \in \mathfrak{g}_j \), and \( [c_i, s] \in \mathfrak{g}_{i+j} \mod m \) implies that \( [c_i, s] = 0 \) for all \( i \). Since this holds for all \( s \in \mathfrak{g}_j \), it follows that \( c_i \in C_i \). We get \( C = D \), and \( C \) is a reductive subalgebra of \( \mathfrak{g} \) with the inherited \( \mathbb{Z}_m \)-grading. As noted above, the restriction of \( \kappa \) to \( C \) is nondegenerate. If \( a, b \in C_0 \) and \( b \in C_i \), then \( \text{ad}_a(a) \circ \text{ad}_a(b) \) maps \( \mathfrak{g}_k \) into \( \mathfrak{g}_{k+i} \), thus \( \kappa(a, b) = 0 \); this implies that the restriction of \( \kappa \) to \( C_0 \) is nondegenerate. Let \( \varphi \) be the automorphism of \( \mathfrak{g}^\circ \) associated with the grading; in particular, note that \( \mathfrak{g}_{00} = \{ x \in \mathfrak{g} \mid \varphi(x) = x \} \). If \( g \in C_0 \) has Jordan decomposition \( g = n + s \) in \( \mathfrak{g} \), then \( n + s = g = \varphi(g) = \varphi(n) + \varphi(s) \), and the uniqueness of Jordan decomposition proves \( n = \varphi(n) \) and \( s = \varphi(s) \), hence \( n, s \in \mathfrak{g}_0 \). As above, \( n, s, C \in C_0 \), and Lemma 8 proves the assertion.

7.1. Computing strongly regular \( \mathbb{Z} \)-graded subalgebras. We are ready to state some algorithms based on the previous results. Throughout this section, we continue with the assumptions that \( \theta \) is a Cartan involution with \( \theta(\mathfrak{g}_i) = -\mathfrak{g}_{-i} \) for all \( i \); let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the corresponding Cartan decomposition, and let \( \mathfrak{h}_0 \) be a \( \theta \)-stable Cartan subalgebra of \( \mathfrak{g}_0 \).

The next proposition yields an algorithm to decide whether two \( \mathbb{Z} \)-graded semisimple strongly \( \mathfrak{h}_0 \)-regular subalgebras are \( G_0 \)-conjugate.

**Proposition 24.** Let \( \mathfrak{h}_0 \) be a \( \theta \)-stable Cartan subalgebra of \( \mathfrak{g}_0 \) with real Weyl group \( W_0(\mathfrak{h}_0) \). Let \( s \) and \( s' \) be \( \mathbb{Z} \)-graded semisimple strongly \( \mathfrak{h}_0 \)-regular subalgebras of \( \mathfrak{g} \). Then \( s \) and \( s' \) are \( G_0 \)-conjugate if and only if \( P(s^c) \) and \( P((s')^c) \) are in the same \( W_0(\mathfrak{h}_0) \)-orbit.

**Proof.** Suppose \( a(s) = s' \) for some \( a \in G_0 \). First, we show that \( g(s) = s' \) for some \( g \in N_{G_0}(\mathfrak{h}_0) \). To prove this claim, note that \( \mathfrak{h}_0 \) and \( a(\mathfrak{h}_0) \) both are maximally noncompact Cartan subalgebras of \( \mathfrak{g}_0(s') \). Thus, \( ba(\mathfrak{h}_0) = \mathfrak{h}_0 \) for some \( b \in N_{G_0}(s') \), and \( g = ba \in N_{G_0}(\mathfrak{h}_0) \) satisfies \( g(s) = s' \). Recall that \( W_0(\mathfrak{h}_0) = N_{G_0}(\mathfrak{h}_0)/Z_{G_0}(\mathfrak{h}_0) \), and let \( w = gZ_{G_0}(\mathfrak{h}_0) \). If \( (k, \lambda) \in P(s^c) \), then \( g(\mathfrak{g}_{k, \lambda}^c) = \mathfrak{g}_{k, \lambda}^c \), which shows that \( w \cdot (k, \lambda) = (k, \lambda^\theta) \in P((s')^c) \); in particular, \( P((s')^c) = w \cdot P(s^c) \), and \( P(s^c) \) and \( P((s')^c) \) are conjugate under \( W_0(\mathfrak{h}_0) \).

Conversely, let \( w \cdot P(s^c) = P((s')^c) \) for some \( w \in W_0(\mathfrak{h}_0) \); write \( w = gZ_{G_0}(\mathfrak{h}_0) \) with \( g \in N_{G_0}(\mathfrak{h}_0) \). Now \( w \cdot (k, \lambda) = (k, \lambda^\theta) \) and \( g(\mathfrak{g}_{k, \lambda}^c) = (s')^c_{k, \lambda^\theta} \) for every weight \( (k, \lambda) \), which proves that \( g(s) = s' \).

The following proposition yields an algorithm to decide whether an \( \mathfrak{h}_0 \)-regular semisimple \( \mathbb{Z} \)-graded subalgebra is strongly \( \mathfrak{h}_0 \)-regular.

**Proposition 25.** Let \( s \) be an \( \mathfrak{h}_0 \)-regular semisimple \( \mathbb{Z} \)-graded subalgebra of \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k} \). Then \( s \) is strongly \( \mathfrak{h}_0 \)-regular if and only if \( \mathfrak{g}_{00}(\mathfrak{h}_0 \cap \mathfrak{p}) \cap \mathfrak{g}_{00}(s) \cap \mathfrak{p} = \mathfrak{h}_0 \cap \mathfrak{p} \).

**Proof.** Write \( n_0 = \mathfrak{g}_{00}(s) \). It follows from Proposition 21 that \( n_0 \) is \( \theta \)-stable and reductive in \( \mathfrak{g} \). Thus, \( n_0 = b \oplus c \), where \( b \) is the derived subalgebra and \( c \) is the centre of \( n_0 \); both \( b \) and \( c \) are \( \theta \)-stable, hence we can decompose

\[
n_0 = (b \cap \mathfrak{t}) \oplus (b \cap \mathfrak{p}) \oplus (c \cap \mathfrak{t}) \oplus (c \cap \mathfrak{p})
\]

Every Cartan subalgebra of \( n_0 \) is the direct sum of \( c \) and a Cartan subalgebra of \( b \); a maximally noncompact Cartan subalgebra of \( n_0 \) is the direct sum of \( c \) and a maximally noncompact Cartan subalgebra of \( b \). Since \( \mathfrak{h}_0 \) is \( \theta \)-stable, we can write

\[
\mathfrak{h}_0 = (\mathfrak{h}_0 \cap b \cap \mathfrak{t}) \oplus (\mathfrak{h}_0 \cap b \cap \mathfrak{p}) \oplus (c \cap \mathfrak{t}) \oplus (c \cap \mathfrak{p})
\]

note that \( \mathfrak{h}_0 = b \cap \mathfrak{h}_0 \) is a Cartan subalgebra of \( \mathfrak{b} \). Since \( \mathfrak{b} = (b \cap \mathfrak{t}) \oplus (b \cap \mathfrak{p}) \) is a Cartan decomposition of \( \mathfrak{b} \). Clearly, \( s \) is strongly \( \mathfrak{h}_0 \)-regular if and only if \( \mathfrak{h}_0 \) is a maximally noncompact Cartan subalgebra of \( \mathfrak{b} \). By [36, Prop. 6.47] and the remarks in [36, p. 386], the
Proposition 26. Let \( h_0 \cap p \) be a maximal abelian subspace of \( b \cap p \). This is the same as saying that the centraliser of \( h_0 \cap p \) in \( b \cap p \) is equal to \( h_0 \cap p \), which is equivalent to \( z_{h_0}(v) \cap p = v \) with \( v = h_0 \cap p \). \( \square \)

To state the main algorithm of this section, we need one more piece of notation. As before, let \( h_0 \) be a \( \theta \)-stable Cartan subalgebra of \( g_0 \), and \( \theta(g_i) = g_i \) for all \( i \). Let \( W_0^\prime \) be the Weyl group of \( g_0 \) relative to \( h_0 \). Let \( s^c \) be an \( h_0 \)-regular \( Z \)-graded semisimple subalgebra of \( g^c \). For \( w \in W_0 \) denote by

\[
w \cdot s^c
\]

the \( h_0 \)-regular semisimple \( Z \)-graded subalgebra of \( g^c \) whose weight system is \( w(P(s^c)) \); then \( s^c \) and \( w \cdot s^c \) are called \( W_0 \)-equivalent. (Note that if \( w \) is induced by \( g \in G^c \), then \( w \cdot s^c = g(s^c) \).) A semisimple \( Z \)-graded subalgebra \( s^c \) is \( \sigma \)-stable, if each \( s_i^c \) is \( \sigma \)-stable.

Algorithm 1 below constructs a list \( L \) of strongly \( h_0 \)-regular semisimple \( Z \)-graded subalgebras of \( g \) such that each \( \tilde{s} \in L \) is \( G_0^c \)-conjugate to \( s^c \), and every strongly \( h_0 \)-regular semisimple \( Z \)-graded subalgebra of \( g \) whose complexification is \( G_0^c \)-conjugate to \( s^c \) is \( G_0 \)-conjugate to a unique element in \( L \).

**Algorithm 1: StronglyRegularSubalgebras(\( g, h_0, s^c \))**

```plaintext
/* s^c is a h_0^c-regular semisimple Z-graded subalgebra of g^c, where h_0 is a \( \theta \)-stable Cartan subalgebra of g_0 and \( \theta \) is a Cartan involution with \( \theta(g_i) = g_i \), for all \( i \). Return a list \( L \) of strongly h_0-regular semisimple Z-graded subalgebras of g with
   (1) the complexification of each \( \tilde{s} \in L \) is G_0^c-conjugate to s^c,
   (2) every strongly h_0-regular semisimple Z-graded subalgebra of g whose complexification is G_0^c-conjugate to s^c is G_0-conjugate to a unique \( \tilde{s} \in L \).
*/
begin
  compute the root system \( \Phi_0 \) of \( g_0^c \) with respect to \( h_0^c \), and generators of its Weyl group \( W_0 \);
  compute the real Weyl group \( W_0(h_0) \leq W_0 \) (Section 3);
  compute a set \( w_1, \ldots, w_s \) of representatives of the right cosets of \( W_0(h_0) \) in \( W_0 \);
  set \( L = \emptyset \) and let \( \sigma \) be the real structure defined by \( g \);
  for \( 1 \leq i \leq s \) do
    set \( \tilde{s}^c = w_i \cdot s^c \);
    if \( \tilde{s}^c \) is \( \sigma \)-stable then
      set \( \tilde{s} = \{ x \in \tilde{s}^c \mid \sigma(x) = x \} \);
      if \( \tilde{s} \) is strongly \( h_0 \)-regular (Proposition 25) then add \( \tilde{s} \) to \( L \);
    end
  end
  remove \( G_0 \)-conjugate copies in \( L \) (Proposition 24);
return \( L \);
end
```

**Proposition 26.** Algorithm 1 is correct.

**Proof.** Denote by \( L \) the output of the algorithm; clearly, \( L \) contains no \( G_0 \)-conjugate subalgebras.

Let \( s' \leq g \) be a strongly \( h_0 \)-regular semisimple \( Z \)-graded subalgebra such that \( (s')^c \) is \( G_0^c \)-conjugate to \( s^c \); we have to show that \( s' \) is \( G_0 \)-conjugate to an element of \( L \). By [50, Prop. 4(2)], there exists \( w' \in N_{G_0^c}(h_0^c) \) with \( w'(s') = (s')^c \), thus its projection \( w \in W_0^c \) satisfies \( w \cdot s^c = (s')^c \); write \( w = u w_j \) for some \( u \in W_0(h_0) \) and \( w_j \) as in Line 4 of the algorithm. In particular, \( w_j \cdot s^c \) is \( W_0 \)-equivalent to \( (s')^c \). If \( i = j \) in the iteration in Line 6, then \( \tilde{s}^c = w_j \cdot s^c \) is constructed. Since \( (s')^c \) is \( \sigma \)-stable and \( (s')^c \) is \( W_0(h_0) \)-equivalent to \( \tilde{s}^c \), it follows that the latter is \( \sigma \)-stable as well; the real subalgebra \( \tilde{s} \) is constructed in
Let \( g \) and \( \theta \) be as before, in particular, \( \theta(g_1) = g_{i-1} \) for all \( i \). Let \( s^c \) be a semisimple \( \mathbb{Z} \)-graded subalgebra of \( g^c \), and let \( h_0^i, \ldots, h_0^t \), up to \( G_0 \)-conjugacy, be the \( \theta \)-stable Cartan subalgebras of \( g_0 \). Up to \( G_0 \)-conjugacy, the regular semisimple \( \mathbb{Z} \)-graded subalgebras of \( g \) that are \( G_0^\circ \)-conjugate to \( s^c \) are \( L = L_1 \cup \ldots \cup L_t \), where each \( L_i \) is the output of Algorithm 1 with input \( (g, h^i_0, s^c) \).

**Proposition 27.**

We remark that the \( \theta \)-stable Cartan subalgebras of \([g_0, g_0]\) (hence also those of \( g_0 \)) can, up to conjugacy, be constructed using our algorithms in [14, §4.3].

### 8. Regular subalgebras of real simple Lie algebras

Let \( g \) be a real simple Lie algebra, with trivial grading. Let \( h \) be a Cartan subalgebra of \( g \). Let \( s^c \leq g^c \) be an \( h^c \)-regular semisimple subalgebra with trivial \( \mathbb{Z} \)-grading, that is, \( s^c_0 = s^c \). Algorithm 1 with input \( h \) and \( s^c \) returns a list of strongly \( h \)-regular semisimple subalgebras of \( g \) such that each strongly \( h \)-regular semisimple subalgebra of \( g \) is \( G \)-conjugate to exactly one element of the list.

Dynkin [20, §5] has given an algorithm for listing the \( h^c \)-regular semisimple subalgebras of \( g^c \), up to \( G^c \)-conjugacy. As mentioned above, we can compute a list of \( \theta \)-stable Cartan subalgebras \( h_1, \ldots, h_5 \) of \( g \) such that each Cartan subalgebra of \( g \) is \( G \)-conjugate to exactly one of them. We perform Algorithm 1 for each \( h_i \) and each \( h_i^c \)-regular semisimple subalgebra \( s^c \) of \( g^c \). Taking the union of all outputs we get a list of all regular semisimple subalgebras of \( g \), up to \( G \)-conjugacy; for this, note that every regular semisimple subalgebra \( s \) of \( g \) is strongly \( h_i \)-regular for a unique \( i \).

As an example, in Table I we display the outcome of our computations for the real form \( g \) of type \( E_6 \) of the simple complex Lie algebra \( g^c \) of type \( E_6 \). Recall that \( g = \mathfrak{k} \oplus \mathfrak{p} \) where \( \mathfrak{k} \) is simple of type \( C_4 \) and \( \mathfrak{p} \) has dimension 42. Up to conjugacy, \( g \) has five Cartan subalgebras \( h_1, \ldots, h_5 \) where \( h_i \) has noncompact dimension 7. Let \( \Phi_i \) denote the root system of \( g^c \) with respect to \( h_i^c \). The first set of rows in Table I, labelled 'real rts', 'im. rts', and 'cpt. im. rts', gives the subsystems of real roots, of imaginary roots, of compact imaginary roots of \( \Phi_i \), respectively (see Section 3). The second set of rows gives the cardinality of the real Weyl group \( W(h_i) \) and the index \([W^c : W(h_i)]\); subsequently, the runtimes (in seconds) are given for computing \( W(h_i) \) and for constructing all strongly \( h_i \)-regular subalgebras of \( g \) up to conjugacy; all runtimes have been obtained on a 3.16GHz machine. The rows below list these regular subalgebras. Up to \( G^c \)-conjugacy there are 19 regular subalgebras in \( g^c \); we have assigned a number to each of them (without intending any sort of order). Since \( h_1 \) is the split Cartan subalgebra of \( g \), the column of \( h_1 \) has in each row exactly one real subalgebra, which is the split form of the corresponding complex regular subalgebra. The columns of \( h_i \) have, on row \( j \), the strongly \( h_i \)-regular subalgebras which are \( G_0^\circ \) equivalent to the \( j \)-th complex regular subalgebra. On many occasions there is simply nothing, meaning that there are no such subalgebras, and in other places there are more than one – for those we have just added a row to the table without repeating the label of the complex regular subalgebra.
Concerning the runtimes, we see that the time needed to compute \( W(\mathfrak{h}_i) \) has no impact on the total time. Also, it is seen that the runtimes increase sharply if the index \( [W^c : W(\mathfrak{h})] \) increases: this is to be expected as the main iteration in Algorithm 1 runs over a set of coset representatives.

**Table I.** Regular subalgebras of the real form of type EI of \( E_6 \)

<table>
<thead>
<tr>
<th>CSA</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th>( h_4 )</th>
<th>( h_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>real rts</td>
<td>( E_6 )</td>
<td>( A_5 )</td>
<td>( A_3 )</td>
<td>( A_1 )</td>
<td>0</td>
</tr>
<tr>
<td>im. rts</td>
<td>0</td>
<td>( A_1 )</td>
<td>2( A_1 )</td>
<td>3( A_1 )</td>
<td>( D_4 )</td>
</tr>
<tr>
<td>cpt. im. rts</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 4A_1 )</td>
</tr>
<tr>
<td>(</td>
<td>W(\mathfrak{h})</td>
<td>)</td>
<td>51840</td>
<td>1440</td>
<td>192</td>
</tr>
<tr>
<td>( [W^c : W(\mathfrak{h})] )</td>
<td>1</td>
<td>36</td>
<td>270</td>
<td>540</td>
<td>135</td>
</tr>
<tr>
<td>time ( W(\mathfrak{h}) )</td>
<td>0.03</td>
<td>0.17</td>
<td>0.72</td>
<td>0.99</td>
<td>1.61</td>
</tr>
<tr>
<td>total time</td>
<td>28</td>
<td>459</td>
<td>3191</td>
<td>10949</td>
<td>3864</td>
</tr>
</tbody>
</table>

**Regular subalgebras of \( E_6 \)**

1. \( sl_2(\mathbb{R}) \)
2. \( 2sl_2(\mathbb{R}) \)
3. \( sl_2(\mathbb{R}) \oplus sl_3(\mathbb{R}) \)
4. \( sl_2(\mathbb{R}) \)
5. \( so(5, 5) \)
6. \( sl_2(\mathbb{R}) \oplus sl_5(\mathbb{R}) \)
7. \( 2sl_2(\mathbb{R}) \oplus sl_3(\mathbb{R}) \)
8. \( sl_2(\mathbb{R}) \oplus 2sl_3(\mathbb{R}) \)
9. \( sl_2(\mathbb{R}) \oplus sl_4(\mathbb{R}) \)
10. \( 3sl_2(\mathbb{R}) \)
11. \( sl_3(\mathbb{R}) \)
12. \( sl_4(\mathbb{R}) \)
13. \( sl_5(\mathbb{R}) \)
14. \( 2sl_3(\mathbb{R}) \)
15. \( so(4, 4) \)
16. \( sl_2(\mathbb{R}) \oplus sl_6(\mathbb{R}) \)
17. \( 2sl_3(\mathbb{R}) \oplus sl_4(\mathbb{R}) \)
18. \( 4sl_2(\mathbb{R}) \)
19. \( 3sl_3(\mathbb{R}) \)

**9. Carrier algebras**

We continue with our assumption that \( \theta \) is a Cartan involution of \( \mathfrak{g} \) with \( \theta(\mathfrak{g}_i) = \mathfrak{g}_{-i} \) for all \( i \).

**Definition 28.** A semisimple \( \mathbb{Z} \)-graded subalgebra \( \mathfrak{s}^c \) of \( \mathfrak{g}^c \) is a **carrier algebra** in \( \mathfrak{g}^c \) if

1. \( \mathfrak{s}^c \) is regular, that is, normalised by some Cartan subalgebra of \( \mathfrak{g}_0 \),
(2) \( s^e \) is complete, that is, not a proper subalgebra of a reductive \( \mathbb{Z} \)-graded regular subalgebra of \( g^c \) of the same rank as \( s^e \).

(3) \( s^e \) is locally flat, that is, \( \dim s_0^e = \dim s_i^e \).

A \( \mathbb{Z} \)-graded subalgebra \( s \) of \( g \) is defined to be a carrier algebra in \( g \) if \( s^e \) is a carrier algebra in \( g^c \). A carrier algebra \( s \) of \( g \) is principal if \( s_0 \) is a torus, that is, if it is a Cartan subalgebra of \( s \).

Results of Vinberg [50] relate carrier algebras to nilpotent elements. To describe this, we need more notation and preliminary results. Let \( e \in g_1 \) be nilpotent (and nonzero), that is, \( \text{ad}_g(e) \) is nilpotent. A variant of the Jacobson-Morozov Theorem, see [39, Thm 2.1], states that there are \( h \in g_0 \) and \( f \in g_{-1} \) such that \( [h, e] = 2e, [h, f] = -2f \), and \( [e, f] = h \). Such a triple \( (h, e, f) \) is a homogeneous \( sl_2 \)-triple, and \( h \) is a characteristic of \( e \). The next lemma follows from [39, Thm 2.1].

**Lemma 29.** Two nonzero nilpotent \( e, e' \in g_1 \) lying in homogeneous \( sl_2 \)-triples \( (h, e, f) \) and \( (h', e', f') \), respectively, are \( G_0 \)-conjugate if and only if the triples are \( G_0 \)-conjugate.

**Lemma 30.** Let \( (h, e, f) \) be an homogeneous \( sl_2 \)-triple in \( g \); let \( \alpha \leq g \) be the subalgebra generated by \( \{h, e, f\} \), and let \( v \) be the subspace spanned by \( e \). If \( h_z \) is a Cartan subalgebra of \( 3g_0(\alpha) \), then \( \text{Span}_R(h) \oplus h_z \) is toral, and a maximal toral subalgebra of \( n_{g_0}(v) \).

**Proof.** By Lemma 8 and the definition of toral, it is obvious that \( t = \text{Span}_R(h) \oplus h_z \) is a toral subalgebra; it remains to show that it is maximal. First, note that \( n_{g_0}(v) = \text{Span}_R(h) \oplus c \), where \( c = 3g_0(\alpha) \): if \( u \in n_{g_0}(v) \), then \([u, e] = ce \) for some \( c \in R \), hence \( u = c'h + u - c'h \) with \( c' = c/2 \) and \( h - c'h \in c \). If \( u \in c \), then \([h, u], e = [h, [u, e]] + [u, [e, h]] \), hence \( [h, u] \in c \), and \( c \) has a basis of eigenvectors of \( h \):

Since \( [e, c] = 0 \), it follows from \( sl_2 \)-theory that the eigenvalues of \( h \) on \( C \) are non-negative integers. Write \( c_i \) for the eigenspace with eigenvalue \( i \). If \( u \in c_0 \), then \([f, u], e = [u, h] = 0 \), hence \([f, u] \in c_0 \); this proves that \( c_0 \) is an \( \alpha \)-module. By \( sl_2 \)-theory, \( c_0 \) is a direct sum of 1-dimensional \( \alpha \)-modules, whence \( c_0 = 3g_0(\alpha) \).

Suppose, for a contradiction, that \( t' \neq t \) is a toral subalgebra of \( n_{g_0}(v) \) containing \( t \). Let \( t \in t' \setminus t \). We may suppose that \( t \in c \), hence \( t = \sum t_i \) with \( t_i \in c_i \). Since \( t' \) is abelian, \([h, t_i] = 0 \), which implies that \( t_i = 0 \) for all \( i > 0 \). Thus, \( t = t_0 \in c_0 = 3g_0(\alpha) \). This, the subalgebra of \( 3g_0(\alpha) \) generated by \( h_z \) and \( t \) is toral. Since \( h_z \) is a Cartan subalgebra of \( 3g_0(\alpha) \), we must have \( t \in h_z \), a contradiction to \( t \notin t' \).

We now describe a construction of carrier algebras which in the complex case is due to Vinberg [50].

**Definition 31.** Let \( e \in g_1 \) be nilpotent with homogeneous \( sl_2 \)-triple \( (h, e, f) \) in \( g \). Let \( a \) be the subalgebra of \( g \) generated by \( \{h, e, f\} \). Let \( h_z \) be a maximally noncompact Cartan subalgebra of \( 3g_0(\alpha) \), see Lemma 15. By Lemma 30, the subalgebra \( t = \text{Span}_R(h) \oplus h_z \) is maximal toral in \( n_{g_0}(\text{Span}_R(e)) \); define \( \lambda : t \to R \) by \([t, e] = \lambda(t)e \), and note that \( \lambda(h) = 2 \) and \( \lambda(t) = 0 \) for \( t \in h_z \). For \( k \in \mathbb{Z} \) define

\[
(9.1) \quad g_k(t, \lambda) = \{ x \in g_k \mid [t, x] = k\lambda(t)x \text{ for all } t \in t \} \quad \text{and} \quad g(t, \lambda) = \bigoplus_{k \in \mathbb{Z}} g_k(t, \lambda);
\]

here \( g_k = g_k \mod m \) if \( m < \infty \). Let \( c(e, h, h_z) \) be the derived subalgebra of \( g(t, \lambda) \).

The same construction can be performed in \( g^c \) and, by results of Vinberg [50], the resulting algebra \( c^c(e, h, h_z) \) is a carrier algebra in \( g^c \). Since \( c^c(e, h, h_z) = c(e, h, h_z) \otimes_R \mathbb{C} \), we have the following proposition; in particular, \( g(t, \lambda) \) is reductive in \( g \) and \( c(e, h, h_z) \) is semisimple.

**Proposition 32.** Using the notation of Definition 31, the algebra \( c(e, h, h_z) \) is a carrier algebra of \( g \).

We call \( c = c(e, h, h_z) \) a carrier algebra of \( e \). It depends on \( e \), on the choice of characteristic \( h \), and on the choice of a maximally noncompact Cartan subalgebra \( h_z \). However, it does not depend on the choice
of the third element of the \( s_2 \)-triple, as that one is uniquely determined by \( h \) and \( e \), see [39, Thm 2.1]. By [50, Thm 2], the element \( e \) is in general position in \( c_1 \), that is, \( [c_0, e] = c_1 \).

**Remark 33.** Let \( c \) be a carrier algebra with defining element \( h_0 \in c_0 \), see Lemma 18. Let \( e' \in c_1 \) be in general position. Since \( c \) is locally-flat and \( [c_0, e'] = c_1 \), it follows that \( \tilde{z}_{00}(e') = 0 \), hence \( n_{00}(e') = S_{\tilde{z}_{00}}(h_0) \). Thus, there is a unique \( h = 2h_0 \in c_0 \) with \([h, e'] = 2e'\). Moreover, by [39, Thm 2.1], there is a unique homogeneous \( s_2 \)-triple \((h, e', f')\) in \( c \). This proves that every nilpotent element in general position lies in a unique homogeneous \( s_2 \)-triple in \( c \) with characteristic \( h = 2h_0 \). Thus, if \((h', e', f')\) is a homogeneous \( s_2 \)-triple in \( c \) with \( e' \) in general position, then \( h'/2 \) is the defining element of \( c \).

**Proposition 34.** Let \( e, e' \in g_1 \) be nilpotent with carrier algebras \( c(e, h, h_2) \) and \( c(e', h', h'_2) \). If \( e \) and \( e' \) are \( G_0 \)-conjugate, then \( c(e, h, h_2) \) and \( c(e', h', h'_2) \) are \( G_0 \)-conjugate.

**Proof.** Let \( g \in G_0 \) with \( g(c) = e' \). By Lemma 29, we can assume that \( g(e) = e' \) and \( g(h) = h' \). Now \( g(c(e, h, h_2)) = c(g(e), g(h), g(h_2)) = c(e', h', g(h_2)) \), and we have to show that \( c(e', h', g(h_2)) \) and \( c(e', h', h'_2) \) are \( G_0 \)-conjugate. Let \( a' \) be the subalgebra of \( g \) generated by \([h', e', f']\). By definition, \( g(h_2) \) and \( h'_2 \) are maximally noncompact Cartan subalgebras of \( \tilde{z}_{00}(a') \); by Lemma 15, they are conjugate under \( Z_{G_0}(a') \). Since the elements of the latter group leave \( e' \) and \( h' \) invariant, the assertion follows.

By Proposition 34, we have a well-defined map \( \psi \) from the set of \( G_0 \)-orbits of nilpotent elements in \( g_1 \) to the set of \( G_0 \)-conjugacy classes of carrier algebras in \( g \), mapping the \( G_0 \)-orbit of a nilpotent \( e \) to the \( G_0 \)-conjugacy class of the carrier algebra \( c(e, h, h_2) \). Similarly, we have a map \( \psi^c \) from the set of nilpotent \( G_0^c \)-orbits in \( g_1^c \) to the set of \( G_0^c \)-conjugacy classes of carrier algebras in \( g_0^c \).

In the complex case, \( \psi^c \) is a bijection, and the inverse of \( \psi^c \) is obtained by mapping a carrier algebra \( c^e \) to an element \( e \in c_1^c \) in general position: by [50, Thm 4], there exist a characteristic \( h \) and torus \( h_2 \) such that \( c^e = c^e(e, h, h_2) \). In the real case, \( \psi \) is not injective: If \( c \leq g \) is a carrier algebra and \( e, e' \in c_1 \) are in general position, then it is not necessarily true that \( e \) and \( e' \) are \( G_0 \)-conjugate. Moreover, it is not necessarily true that \( c \) is a carrier of \( e \) or \( e' \). In general, the map \( \psi \) is not even surjective: it is possible that for a given carrier algebra \( c \leq g \) there is no \( e \in c_1 \) in general position such that \( e = c(e, h, h_2) \) is a carrier algebra of \( e \).

Algorithms are known to list the \( G_0 \)-conjugacy classes of carrier algebras in \( g_0^c \), see [28, 40, 50]. In combination with Proposition 27, this gives an immediate algorithm to list the \( G_0 \)-conjugacy classes of carrier algebras in \( g \). We note that all carrier algebras constructed by this procedure are strongly \( h_0 \)-regular for some \( \theta \)-stable Cartan subalgebra \( h_0 \leq g_0 \), so they are \( \theta \)-stable by Proposition 21. We also mention that, using this procedure, the defining element of each such carrier algebra \( c \) lies in \( h_0 \), so that \([h_0, c_1] \leq c_i \) for all \( i \) by Lemma 20.

10. Nilpotent Orbits

We continue with the previous notation and suppose \( g \) is a \( \mathbb{Z}_m \)-graded semisimple Lie algebra with Cartan involution \( \theta \) reversing the grading. As shown in the previous section, we have an algorithm to compute a list \( L \) of all carrier algebras in \( g \), up to \( G_0 \)-conjugacy. Using this algorithm, each such carrier algebra \( c \) is strongly \( h_0 \)-regular for a (known) \( \theta \)-stable Cartan subalgebra \( h_0 \leq g_0 \) with \([h_0, c_1] \leq c_i \) for all \( i \). Since the map \( \psi \) of Section 9 is not necessarily injective or surjective, this does not immediately yield the nilpotent \( G_0 \)-orbits in \( g_1 \). In this section, we discuss what needs to be done to obtain the nilpotent orbits from the list \( L \) of carrier algebras.

A fundamental problem that we encounter here is to decide, for a given carrier algebra \( c \) and \( e \in c_1 \) in general position, whether \( c \) is a carrier algebra of \( e \), that is, whether \( c = c(e, h, h_2) \) for some \( h_2 \); note that \( h/2 \) must be the unique defining element of \( c \), see Remark 33. For that we use the next result.
Proposition 35. Let \( c \) be a carrier algebra in \( g \) with defining element \( h/2 \). Let \( e \in c_1 \) be in general position, lying in the homogeneous \( \mathfrak{sl}_2 \)-triple \((h, e, f)\). Let \( a \) be the subalgebra spanned by \( \{h, e, f\} \).

a) If \( \tilde{h} \leq \tilde{3}_g(\epsilon) \) is a Cartan subalgebra of \( \tilde{3}_g(\epsilon) \), then \( \tilde{h} \) is a Cartan subalgebra of \( \tilde{3}_g(\epsilon) \).

b) The subalgebra \( c \) is a carrier algebra of \( e \) if and only if the real ranks of \( \tilde{3}_g(\epsilon) \) and \( \tilde{3}_g(\epsilon) \) coincide.

c) Write \( \tilde{3}_g(\epsilon) = \mathfrak{d} \oplus \mathfrak{t} \), where \( \mathfrak{d} \) is the derived subalgebra and \( \mathfrak{t} \) is the centre. Let \( \tilde{h} \) be a maximally noncompact Cartan subalgebra of \( \tilde{3}_g(\epsilon) \). Then \( c \) is a carrier algebra of \( e \) if and only if \( \tilde{h} \cap \mathfrak{d} \) is a maximally noncompact Cartan subalgebra of \( \mathfrak{d} \).

d) Let \( e' \in c_1 \) be in general position. If \( e \) and \( e' \) are \( G_0 \)-conjugate, then \( c \) is a carrier algebra of \( e \) if and only if it is a carrier algebra of \( e' \).

PROOF. By Remark 33, the triple \((h, e, f)\) exists and is uniquely defined by \( e \). Recall that each of \( \tilde{3}_g(\epsilon) \) and \( \tilde{3}_g(\epsilon) \) is reductive in \( g \), see Lemma 23. We make use of Lemma 8c) and Lemma 15; the main ideas in the proof of a) are borrowed from [50].

a) Set \( \mathfrak{b} = \text{Span}_{\mathbb{R}}(h) \oplus \tilde{h} \), and define the linear map \( \varphi : \mathfrak{b} \to \mathbb{R} \) by \( \varphi(h) = 1 \) and \( \varphi(u) = 0 \) for \( u \in \tilde{h} \).

b) Set \( \varphi \) be defined as in (9.1). Now [50, Prop. 2] shows that \( \varphi \mathfrak{b} = \mathfrak{s} \oplus \tilde{h} \), where \( \mathfrak{s} \) is a complete regular semisimple \( \mathbb{Z} \)-graded subalgebra, and \( \tilde{h} \) is the centre. (In [50], the complex case is discussed, but the same result follows for real algebras.) Moreover, \( \mathfrak{s} \) contains \( c \) and has the same rank as \( c \). As \( c \) is a carrier algebra (and therefore complete), we get \( \varphi \mathfrak{b} = \tilde{h} \). Note that \( \tilde{3}_g(\epsilon) \leq \tilde{3}_g(\epsilon) \), hence \( \mathfrak{h} \) is contained in a Cartan subalgebra \( \tilde{h}_z \) of \( \tilde{3}_g(\epsilon) \). It follows easily from the definition of \( \varphi \mathfrak{b} = \varphi \mathfrak{h} \cap \mathfrak{g}_0 = \mathfrak{c}_0 \oplus \tilde{h} \).

Note that \( \tilde{h}_z \) centralises \( h \) and \( \tilde{h} \), hence \( \tilde{h}_z \leq \tilde{3}_g(\epsilon) \). If the first inclusion would be proper, then there exists a nonzero \( v \in \tilde{h}_z \cap \tilde{h} \) with \( v \in \mathfrak{c}_0 \); now \( v \in \tilde{3}_g(\epsilon) \cap \mathfrak{c}_0 = \{0\} \) yields a contradiction. (Note that \( \tilde{3}_g(\epsilon) = \{0\} \) since \( c \) is locally-flat and \( e \) is in general position.) This proves that \( \tilde{h}_z = \tilde{h} \).

c) Suppose that \( \tilde{3}_g(\epsilon) \) is a carrier algebra of \( e \) and \( \tilde{h} \) is a maximally noncompact Cartan subalgebra of \( \tilde{3}_g(\epsilon) \). We use the same ideas as in a) to prove that \( \tilde{3}_g(\epsilon) \) is a carrier algebra of \( e \).

d) Suppose \( e' = g(e) \) with \( g \in G_0 \), and note that \( e' \) lies in a unique \( \mathfrak{sl}_2 \)-triple \((h, e', f')\) in \( c \), see Remark 33. It follows from Lemma 29 that \((h, e, f)\) and \((h, e', f')\) are \( Z_{G_0}(h) \)-conjugate, thus we can assume that \( g(h) = h \) and \( g(e) = e' \). Let \( a \) and \( a' \) be the algebras spanned by \( \{h, e, f\} \) and \( \{h, e', f'\} \), respectively; note that \( a' = g(a) \), hence \( \tilde{3}_g(\epsilon) \) and \( \tilde{3}_g(\a') \) have the same real rank. Now b) proves the assertion. \( \square \)
Remark 36. Using our algorithms in [14] for computing a Cartan decomposition and a maximally non-compact Cartan subalgebra of a real semisimple Lie algebra, Proposition 35c) gives an immediate algorithm to decide whether \( c \) is a carrier of a given \( e \in c_1 \).

In summary, an approach for computing (representatives of) the nilpotent \( G_0 \)-orbits in \( g_1 \) is as follows. Recall the definition of the representation \( \rho : G_0 \to GL(g_1) \), see Section 6.1.

(a) Use the methods of Section 9 to compute the list \( L \) of carrier algebras in \( g_\), up to \( G_0 \)-conjugacy.
(b) For each carrier algebra \( \epsilon \in L \) with defining element \( h/2 \) do the following:
   (b1) Use linear algebra methods to determine the set \( G \) of elements \( e \in c_1 \) in general position.
   (b2) Use elements of \( \rho(G_0) \) to find a small finite set \( G' \subseteq G \), such that every element of \( G \) is \( G_0 \)-conjugate to at least one element of \( G' \). The objective is to make \( G' \) as small as possible; ideally, we want to reduce \( G \) up to \( G_0 \)-conjugacy.
   (b3) Let \( C \) be set of those \( e \in G' \) such that \( e \) is a carrier algebra of \( e \), see Remark 36.
   (b4) Use \( G_0 \)-invariants to show that the elements of \( C \) are not \( G_0 \)-conjugate.
(c) The union of all sets \( C \) that have been obtained is a complete and irredundant set of \( G_0 \)-orbit representatives of nilpotent elements in \( g_1 \), see Proposition 34.

This is not an algorithm, but more a structured programme of work: it is not immediately clear how to perform steps (b1), (b2), and (b4). We use the previous notation and comment on each step below.

10.1. The set of elements in general position. Let \( \epsilon \) be a carrier algebra, and let \( \{ x_1, \ldots, x_s \} \) and \( \{ y_1, \ldots, y_s \} \) be bases of \( c_0 \) and \( c_1 \) respectively; recall that \( \text{dim} \ c_0 = \text{dim} \ c_1 \) since \( \epsilon \) is locally-flat. Write \( e = c_1y_1 + \ldots + c_sy_s \in c_1 \) and recall that \( e \) is in general position if and only if \( [c_0, e] = c_1 \), that is, if and only if \( \text{ad}_g(e) \) induces a bijection \( c_0 \to c_1 \). For each \( y_i \), let \( M_i \) be the matrix describing \( \text{ad}_g(y_i) : c_0 \to c_1 \), \( x \mapsto [y_i, x] \). Then \( \text{ad}_g(e)|_{c_0} \) is represented by the matrix \( M(c_1, \ldots, c_s) = c_1M_1 + \ldots + c_sM_s \), which is a bijection if and only if \( f(c_1, \ldots, c_s) = \det(M(c_1, \ldots, c_s)) \) is non-zero. It follows that the set \( G \) of elements in \( c_1 \) in general position can be described as
\[
G = \{ c_1y_1 + \ldots + c_sy_s \mid f(c_1, \ldots, c_s) \neq 0 \}.
\]
By itself, the condition \( f(c_1, \ldots, c_s) \neq 0 \) is not very revealing. However, on many occasions we can factorise \( f \), and the factors tend to give useful information.

10.2. Finding elements of \( \rho(G_0) \). Let \( G \) be as in the previous section; we want to reduce \( G \) to a small finite subset \( G' \) such that each \( e \in G \) is \( G_0 \)-conjugate to at least one element in \( G' \). The problem is that, in general, we do not know \( \rho(G_0) \). Here we discuss how to construct elements of \( \rho(G_0) \) which often can be used to bring elements in \( G \) into the form \( c_1y_1 + \ldots + c_sy_s \) where some of the coefficients \( c_i \) are 0, while others are \( \pm 1 \).

10.2.1. Nilpotent elements. If \( x \) is a nilpotent element of \( c_0 \), then \( \exp(\text{ad}_g(x)) \) lies in \( G_0 \). We know from [36, Thm 0.23] that \( \rho(\exp(\text{ad}_g(x))) = \exp(\text{dp}(x)) \), and \( \text{dp}(x) \) is the restriction of \( \text{ad}_g(x) \) to \( g_1 \). Thus, \( \rho(\exp(\text{ad}_g(x))) \) stabilises \( c_1 \), and its restriction to \( c_1 \) is \( \exp(y) \), where \( y \) is the restriction of \( \text{ad}_g(x) \) to \( c_1 \). Typically, we can use such elements to show that some coefficients of \( e \in G \) may be assumed to be 0.

10.2.2. Semisimple elements. By construction, \( \epsilon \) is \( h_0 \)-regular, where \( h_0 \) is a \( \theta \)-stable Cartan subalgebra of \( g_0 \). Note that \( c_1' \) is spanned by \( h_0^\theta \)-weight vectors. So \( c_1' \) is normalised by \( h_0^\theta \), and therefore \( c_1 \) is normalised by \( h_0 \). Let \( H_0 \) be the connected Lie subgroup of \( G_0 \) with Lie algebra \( h_0 \); we describe how to 'parametrise' \( H_0 \) and find elements in \( \rho(H_0) \). Recall that \( h_0 \) is a toral subalgebra of \( g_1 \); we start with a general construction for such subalgebras.
Let $t^c$ be a toral subalgebra of $g^c$, which we assume to be algebraic, that is, there is an algebraic subgroup $T^c$ of $G^c$ whose Lie algebra is $\text{ad}_{g^c}(t^c) \cong t^c$; let $n = \dim g^c$. First, suppose that $\text{ad}_{g^c}(t^c)$ consists of diagonal matrices. Let $\Lambda \subseteq \mathbb{Z}^n$ be the lattice defined by

$$\Lambda = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid e_1 \alpha_1 + \ldots + e_n \alpha_n = 0 \text{ for all } \text{diag}(\alpha_1, \ldots, \alpha_n) \in \text{ad}_{g^c}(t^c)\};$$

here $\text{diag}(\alpha_1, \ldots, \alpha_n)$ denotes the diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. If we define

$$T^c = \{\text{diag}(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid \alpha_1^{e_1} \cdots \alpha_n^{e_n} = 1 \text{ for all } e = (e_1, \ldots, e_n) \in \Lambda\},$$

then $T^c$ is a connected algebraic subgroup of $\text{GL}(n, \mathbb{C})$ with Lie algebra $\text{ad}_{g^c}(t^c)$, see [5, §II.7.3]; the set-up considered in [5] is slightly different, but the proof is the same; alternatively, see [8, §13, Prop. II.3]. Let $E = \{e^1, \ldots, e^s\}$ with $e^k = (e_1^k, \ldots, e_n^k)$ be a $\mathbb{Z}$-basis of $\Lambda$, and define $L \subseteq \mathbb{Z}^n$ as the lattice consisting of all $(d_1, \ldots, d_n)$ with $d_1 e_1^k + \ldots + d_n e_n^k = 0$ for all $1 \leq k \leq r$. Let $\{d^1, \ldots, d^s\}$ be a basis of $L$ and define

$$\eta: \mathbb{C}^* \to T^c, \ t \mapsto \text{diag}(t^{d_1}, \ldots, t^{d_n}).$$

Now the map $\eta: (\mathbb{C}^*)^s \to T^c, (t_1, \ldots, t_s) \mapsto \eta(t_1) \cdots \eta(t_s)$, is an isomorphism of algebraic groups. On the other hand, if $t^c$ is diagonalisable, but not diagonal, then there is $A \in \text{GL}(n, \mathbb{C})$ such that $t^c = A \text{ad}_{g^c}(t^c)A^{-1}$ consists of diagonal matrices. The construction above, applied to $t^c$, yields an isomorphism $\tilde{\eta}: (\mathbb{C}^*)^s \to \tilde{T}^c$ and we define $\eta: (\mathbb{C}^*)^s \to T^c, a \mapsto A^{-1}\tilde{\eta}(a)A$. Thus, in both cases, we obtain an isomorphism

$$\eta: (\mathbb{C}^*)^s \to T^c.$$

We apply this procedure to $t^c = h_0^c$ and $T^c = H_0^c \subset G_0^c$. Then $\eta((\mathbb{C}^*)^s) \cap \text{GL}(n, \mathbb{R}) \subset G_0$, and restricting elements of this set to $g_{1}$ yields elements of $\rho(G_0)$, normalising $c_1$.

It turns out that it is a good idea to apply this construction to the toral subalgebras $h_0^c \cap t^c$ and $h_0^c \cap p^c$ separately. For the latter algebra, the diagonalising matrix $A$ is defined over $\mathbb{R}$, hence we can simply restrict $\eta$ to $(\mathbb{R}^*)^s$ to get a large set of semisimple elements of $\rho(G_0)$. For $h_0^c \cap t^c$ the diagonalising matrix $A$ is not defined over $\mathbb{R}$, and it may not be straightforward to find the subset of $(\mathbb{C}^*)^s$ which is mapped by $\eta$ into $G_0$. However, typically, the matrix $\eta_j(t)$ has a block diagonal form with $2 \times 2$-blocks of the form $B(t^k)$ for some $k \in \mathbb{Z}$, where

$$B(t) = \begin{pmatrix} \frac{1}{2}(t+t^{-1}) & \frac{1}{2}(t-t^{-1}) \\ -\frac{1}{2}(t-t^{-1}) & \frac{1}{2}(t+t^{-1}) \end{pmatrix},$$

see the comment below. Note that $B(s)B(t) = B(st)$, and $B(t)$ has coefficients in $\mathbb{R}$ if and only if $t = x+iy$ with $x^2 + y^2 = 1$; in which case

$$B(t) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is the matrix describing rotation about the origin of $\mathbb{R}^2$ about the angle $\arccos(x)$. If $k \neq 0$, then $t = x+iy$ runs through the circle defined by $x^2 + y^2 = 1$ and only if $t^k$ does so. The nonzero orbits of the group consisting of all $B(t)$, with $t = x+iy$ and $x^2 + y^2 = 1$, acting on $\mathbb{R}^2$, have representatives $(\alpha, 0), \alpha \geq 0$.

In all examples we considered, the matrices of $\eta_j(t)$ had this block diagonal form; this is no coincidence, in view of the normal form theorem for orthogonal real matrices [34, Thm VI.9], and the fact that the elements of the subgroup of $G$ corresponding to $\mathfrak{f}$ are orthogonal transformations, cf. [43, Prop. 5.1(i)].

**10.3. $G_0$-invariants.** We describe some methods for establishing that two nilpotent $e_1, e_2 \in g_1$ are not $G_0$-conjugate. A very powerful invariant is the carrier algebra: if the carrier algebras of $e_1$ and $e_2$ are not isomorphic, then $e_1$ and $e_2$ cannot be $G_0$-conjugate. However, as we construct nilpotent elements in $g_1$ by first listing the $G_0$-conjugacy classes of carrier algebras, we may assume that both $e_1$ and $e_2$ have the same carrier algebra $c$. In particular, viewed as elements of $g_1^c$, they are $G_0^c$-conjugate.
(i) If \( e_1, e_2 \) are \( G_0 \)-conjugate, then their centralisers \( z_g(e_i) \) are as well. Thus, if these centralisers contain different \( G \)-classes of nilpotent elements, then the \( e_i \) cannot be \( G_0 \)-conjugate. This can be checked using the Kostant-Sekiguchi correspondence, see [9, §9.5], and [16] for a computational approach. Similarly, we can study the centralisers \( z_{g_0}(e_i) \).

(ii) Let \( (h_i, e_i, f_i) \) be a homogeneous \( sl_2 \)-triple, spanning the subalgebra \( a_i \); by Remark 33, we have \( h_1 = h_2 = h \), where \( h/2 \) is the unique defining element of the carrier algebra of \( e_1 \) and \( e_2 \). Recall that \( e_1 \) and \( e_2 \) are \( G_0 \)-conjugate if and only if these triples are; thus we can consider the centralisers \( z_g(a_1) \) and \( z_g(a_2) \). We can check whether they are isomorphic over \( \mathbb{R} \), or whether they contain different \( G \)-classes of nilpotent elements. If any of these tests fails, then \( e_1 \) and \( e_2 \) cannot be \( G_0 \)-conjugate. Similarly, we can study the centralisers \( z_{g_0}(a_i) \).

(iii) Using the notation of (ii), we also investigate centralisers of the semisimple elements \( e_i - f_i \) or \( e_i + f_i \).

11. Examples of nilpotent orbit computations

We discuss in detail three interesting examples, motivated by the literature; the gradings we use are constructed as in Example 19. In particular, this construction already gives as a Cartan involution which reverses the grading, which is supposed in most of our main algorithms.

Example 37. We consider a \( \mathbb{Z} \)-graded simple Lie algebra of type \( E_8 \), as in Example 19a). The degrees of the simple roots are given by the diagram

![Diagram](image)

where the simple root corresponding to the black node has degree 1, and all others have degree 0. This example has also been considered by Djoković [18], and \( G_0 \cong GL(8, \mathbb{R}) \) and \( g_1 \cong \mathbb{R}^8 \wedge \mathbb{R}^8 \wedge \mathbb{R}^8 \) as \( G_0 \)-module. Djoković classified the nilpotent \( G_0 \)-orbits in \( g_1 \) using an approach based on Galois cohomology. Concerning the results obtained with our methods, we remark the following:

- There are 53 \( G_0 \)-conjugacy classes of real carrier algebras (whereas there are 22 \( G_0^c \)-conjugacy classes of carrier algebras in \( g^p \)); our program needed 2567 seconds to list them.
- Although there are real carrier algebras which contain representatives of more than one nilpotent \( G_0 \)-orbit, in all cases all (but at most one) are ruled out by the condition of Proposition 35c). So each real carrier algebra corresponds to at most one nilpotent orbit. In particular, in this example, there is no need to use the criteria outlined in Section 10.3.
- Exactly 34 carrier algebras correspond to exactly one nilpotent orbit, the others correspond to no nilpotent orbit.
- We find that each complex orbit splits in exactly the same number of real nilpotent orbits as in [18]; so our classification and the one of Djoković are equivalent. Moreover, we checked the representatives given by Djoković, and they all turned out to lie in the correct nilpotent orbit.
- Djoković also computed the isomorphism types of the centralisers \( z_{g_0}(a) \), where \( a \) is the subalgebra spanned by a homogeneous \( sl_2 \)-triple. Here, on some occasions, we get different results. For example, the complex orbit labelled “VI” in [18] has only one real representative, and Djoković claims that \( z_{g_0}(a) \) is isomorphic to \( 3s_2(\mathbb{R}) \); we find that \( z_{g_0}(a) \) is isomorphic to \( sp_0(\mathbb{R}) \oplus t_2 \), where \( t_2 \) is a 2-dimensional toral subalgebra lying inside \( p \).

Example 38. We consider a \( \mathbb{Z}_2 \)-graded simple Lie algebra of type \( G_2 \), as in Example 19b). The complex simple Lie algebra of type \( G_2 \) has only one (conjugacy class of) involution, which we use to construct the grading; this grading is also studied in the physics literature, see [35]. Here \( g_0 \) has four (classes of) Cartan
subalgebras, but strongly regular carrier algebras only exist with respect to the split Cartan subalgebra. There are five of these carrier algebras, like in the complex case. Three of them yield one nilpotent orbit, and two correspond to two nilpotent orbits. This leads to an equivalent classification as derived in [35].

We now study the carrier algebra c that is isomorphic to g (as Lie algebra) in a more detail: it has dimension 14, and dim c₀ = dim c₁ = 4. Let \{y₁, \ldots, y₄\} be a fixed basis of c₁, as computed by our programs. Using the approach of Section 10.1, an element \(\sum_i c_i y_i \in c_1\) is in general position if and only if

\[
(11.1) \quad c₁^2 c₂^2 - 6c₁c₂c₃c₄ + 4c₁a₃ + 4c₂^3c₄^2 - 3c₂^2c₃^2 = 0.
\]

Unfortunately, this polynomial is irreducible over \(\mathbb{Q}\). The semisimple part of c₀ is isomorphic to sl₂(\(\mathbb{R}\)), so it has two nilpotent basis elements, denoted by \(u\) and \(v\). The exponentials of their adjoints, restricted to c₁, are

\[
\exp(s \text{ad}_g(u)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 \\ s^3 & -2s^2 & 1 & 0 \\ -s^4 & 3s^3 & -3s & 1 \end{pmatrix} \quad \text{and} \quad \exp(t \text{ad}_g(v)) = \begin{pmatrix} 1 & -3t & 3t^2 & -t^3 \\ 0 & 1 & -2t & t^3 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

where \(s, t \in \mathbb{R}\). Let \(e = \sum_i c_i y_i\) be an element in general position; we now use Section 10.2.1 and act with \(\exp(s \text{ad}_g(u))\) and \(\exp(t \text{ad}_g(v))\). First, after acting with an \(\exp(s \text{ad}(u))\) (if required), we can assume \(c₄ \neq 0\). Note that \(\exp(t \text{ad}_g(v)) e = \sum_i c_i^* y_i\), where \(c₁ = c₁ - 3tc₂ + 3t^2c₃ - t^3c₄\) is a polynomial in \(t\) of degree 3; this polynomial has a real zero \(t₀\), and, by acting with \(\exp(t₀ \text{ad}_g(v))\), we may assume that \(c₁ = 0\). Now it follows from (11.1) that \(c₂ \neq 0\). Write \(\exp(s \text{ad}_g(u)) e = \sum_i c_i^* y_i\), so that \(c₁ = 0, c₂ = c₂\) and \(c₃ = -2se₂ + c₃\). We can choose \(s\) so that \(c₃ = 0\); in conclusion, we may assume that \(c₁ = c₃ = 0, c₂ \neq 0 \neq c₄\) by (11.1).

Using Section 10.2.2, the group corresponding to the split Cartan subalgebra of \(g₀\) acts on \(c₁\) as

\[
\text{diag}(a₁^{3}a₂^{8}, a₁^{2}a₂^{5}, a₁a₂^{2}, a₂^{-1}).
\]

where each \(a_i \in \mathbb{R}^*\). Acting with this group on \(e = c₂y₂ + c₄y₄\), we can multiply \(c₄\) by an arbitrary \(b \in \mathbb{R}^*\), and \(c₂\) by an \(a \in \mathbb{R}^*\), provided that \(ab\) is a square in \(\mathbb{R}^*\). We get two possible nilpotent orbits, represented by \(e₁ = y₂ + y₄\) and \(e₂ = -y₂ + y₄\) respectively. For \(i = 1, 2\) let \(a_i\) denote the subalgebra spanned by the homogeneous \(sl₂\)-triple \((h, e_i, f_i)\). It turns out that \(₃g₀(a₁) = 0\) and \(₃g₀(c) = 0\), so the condition of Proposition 35c) is trivially satisfied. Now consider \(u_i = e_i - f_i\); its centraliser in \(g₀\) is 1-dimensional and spanned by a semisimple element \(t_i\). The minimal polynomials of \(ad_{g₀}(t₁)\) and \(ad_{g₀}(t₂)\) are \((X - 6)(X - 2)X(X + 2)(X + 6)\) and \(X(X^2 + 4)(X^2 + 36)\), respectively. So \(t₁\) is not \(G₀\)-conjugate to \(λt₂\) for all \(λ \in \mathbb{R}\). Thus, the centralisers of the \(u_i\) are not \(G₀\)-conjugate, and hence neither are \(e₁\) and \(e₂\).

**Example 39.** We consider a Z-graded simple Lie algebra of type \(E₈\), as in Example 19a). The degrees of the simple roots are given by the diagram

```
\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {1};
  \node (B) at (1,0) {2};
  \node (C) at (2,0) {3};
  \node (D) at (3,0) {4};
  \node (E) at (4,0) {5};
  \node (F) at (5,0) {6};
  \node (G) at (6,0) {7};
  \node (H) at (7,0) {8};

  \draw (A) -- (B);
  \draw (B) -- (C);
  \draw (C) -- (D);
  \draw (D) -- (E);
  \draw (E) -- (F);
  \draw (F) -- (G);
  \draw (G) -- (H);
\end{tikzpicture}
\end{center}
```

where the simple root corresponding to the black node has degree 1, and all others have degree 0. Over the complex numbers this grading has been considered in [26], where it is shown that there are nine nonzero orbits, and for each orbit a representative is given. Here \(G₀^₆ \cong \text{Spin}_{14}(\mathbb{C}) \times \mathbb{C}^*\) and \(g₀^₆\) is (as \(G₀\)-module) the 64-dimensional spinor representation of \(\text{Spin}_{14}(\mathbb{C})\). (The orbits of \(\text{Spin}_{14}(\mathbb{C})\) on this space have also been classified by Popov in [44]; but without making use of graded Lie algebras.) We have computed the orbits of \(G₀\) acting on \(g₁\), and we remark the following:

- It took 4409 seconds to complete the classification of the carrier algebras.
- There are 10 Cartan subalgebras in \(g₀\) (up to \(G₀\)-conjugacy).
- There are 26 carrier algebras (up to \(G₀\)-conjugacy), and all of them are principal.
- Each carrier algebra corresponds to at most one orbit; this yields 15 orbits in total.
In Table II we summarise the results of our computations. In this table, the first column lists the label of the corresponding complex orbit; we use the same numbering as in [26]. In particular, the rows of the table correspond to the complex orbits. The second column contains the dimension of the orbit. The third column lists a representative of the orbit. These representatives are given as spinors; for the notation we refer to [26, §2.0]. If a complex orbit splits into more than one real orbit, then the third and fourth column have extra lines in the corresponding row, with the corresponding information for each real orbit. The fourth column contains the isomorphism type of the centraliser in \( g_0 \) of a homogeneous \( \mathfrak{s}\mathfrak{e}_2 \)-triple containing the given representative. We use the following notation: \( t_{r,s} \) denotes a \( \theta \)-stable toral subalgebra such that \( \dim(t \cap t_{r,s}) = r \) and \( \dim(p \cap t_{r,s}) = s \) where \( g = t \oplus p \) as usual; the Lie algebras \( G_2, G_2(\mathbb{C}), \) and \( G_2^{\text{cpt}} \) are the split real form of the Lie algebra if type \( G_2 \), the complex Lie algebra of type \( G_2 \) seen as real, and the compact real form of \( G_2 \), respectively.

**Table II.** Real orbits of \( \text{Spin}_{14}(\mathbb{R}) \times \mathbb{R}^* \)

<table>
<thead>
<tr>
<th>orbit</th>
<th>dim</th>
<th>representative</th>
<th>( \mathfrak{g}_0(a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>22</td>
<td>( f_1f_5f_6f_7 )</td>
<td>( \mathfrak{sl}<em>7(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>II</td>
<td>35</td>
<td>( f_1f_5f_6f_7 - f_2f_4f_5f_6 )</td>
<td>( \mathfrak{sl}_5(\mathbb{R}) \oplus \mathfrak{so}<em>7(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>III</td>
<td>43</td>
<td>( f_1f_2 + f_1f_6f_7 + f_2f_3f_5f_6 )</td>
<td>( \mathfrak{sp}<em>6(\mathbb{R}) \oplus t</em>{0,2} )</td>
</tr>
<tr>
<td>IV</td>
<td>44</td>
<td>( f_1f_2f_6f_5 + f_2f_3f_5f_7 - f_1f_2f_5f_7 - f_1f_4f_5f_6 )</td>
<td>( \mathfrak{sl}<em>6(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>V</td>
<td>50</td>
<td>( f_1f_2 + f_1f_4f_6f_7 + f_1f_5f_6f_7 + f_2f_3f_4f_5 )</td>
<td>( \mathfrak{sl}<em>4(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>VI</td>
<td>54</td>
<td>( f_1f_5f_6f_7 - f_2f_4f_5f_6 )</td>
<td>( 2\mathfrak{sl}<em>3(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>VII</td>
<td>59</td>
<td>( f_1f_2 + f_1f_4f_6f_7 + f_2f_3f_4f_5 - f_3f_4f_5f_6 )</td>
<td>( \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}<em>5(\mathbb{R}) \oplus t</em>{0,1} )</td>
</tr>
<tr>
<td>VIII</td>
<td>63</td>
<td>( f_1f_2 + f_1f_4f_6f_7 + f_1f_5f_6f_7 + f_2f_3f_4f_5 - f_3f_4f_5f_6 )</td>
<td>( \text{su}(2) \oplus \mathfrak{so}(1,4) \oplus t_{0,1} )</td>
</tr>
<tr>
<td>IX</td>
<td>64</td>
<td>( f_1f_4 + f_1f_5f_6f_7 - f_2f_3f_4f_5f_6 )</td>
<td>( G_2 \oplus t_{0,1} )</td>
</tr>
</tbody>
</table>

**References**


The GAP Group. GAP – groups, algorithms, and programming. v.4.7.2. Available at gap-system.org.


**SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, VIC 3800, AUSTRALIA**

*E-mail address*: heiko.dietrich@monash.edu

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TRENTO, POVO (TRENTO), ITALY**

*E-mail address*: faccin@science.unitn.it

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TRENTO, POVO (TRENTO), ITALY**

*E-mail address*: degraaf@science.unitn.it