Radical subgroups of the finite exceptional groups of Lie type E_6

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ABSTRACT. We consider the finite exceptional groups of Lie type $E_6^{+1}(q) = E_6(q)$ and $E_6^{-1}(q) = {}^2E_6(q)$, both the universal versions. We classify, up to conjugacy, the maximal p-local subgroups and radical p-subgroups of $G = E_6^{\varepsilon}(q)$ for $p \geq 5$ with $p \nmid q$ and $q \equiv \varepsilon \bmod p$, and for $p = 3$ with $3 \nmid q$ and $q \equiv -\varepsilon \bmod 3$. As an application, the essential p-rank of the Frobenius category $\mathcal{F}_D(G)$ is determined, where D is a Sylow p-subgroup of G. Moreover, if $p = 3$, then we show that there is a subgroup $H = F_4(q)$ of G containing D such that $\mathcal{F}_D(G) = \mathcal{F}_D(H)$, that is, H controls 3-fusion in G.

1. Introduction

A nontrivial subgroup R of a finite group G is (p-)radical if $R = O_p(N_G(R))$, the largest normal p-subgroup of $N_G(R)$. Radical subgroups play an important role in modular representation theory: for example, a defect group of a block is radical, the subgroup R of a weight (R, φ) is radical, and the first nontrivial subgroup in any radical chain is radical. Radical subgroups are also heavily used in the studies of the Dade Conjecture and the Alperin Weight Conjecture (see [26]). If the radical subgroups of G are known, then the essential rank of the Frobenius category $\mathcal{F}_D(G)$, $D \leq G$ a Sylow subgroup, can be readily determined, cf. [1,2]. Another application of radical subgroups (mentioned in [22]) is the study of so-called p-local geometries.

The classification of radical subgroups therefore is an interesting (and open) research problem; classifications are known for the the symmetric, classical, sporadic, and the exceptional groups ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, ${}^2B_2(q)$, see [5] and the references given there. This paper is the fourth one of a current research project which considers the remaining exceptional groups of Lie type, especially when p is small and a *bad* prime. In the first paper [4], the maximal 2-local subgroups of $F_4(q)$ and $E_6^{\pm 1}(q)$ are classified for q odd; here we write $E_6^{+1}(q) = E_6(q)$ and $E_6^{-1}(q) = {}^2E_6(q)$ for the universal versions. The radical 3-subgroups of $F_4(q)$ are classified in [3, 5]. The aim of this paper is to classify the radical p-subgroups of $E_6^{\varepsilon}(q)$. Recall that these groups are known by the Borel-Tits Theorem [19, Corollary 3.1.5] if p is the defining characteristic, thus we assume $p \nmid q$. First, we consider $p = 3$ with $3 \mid q + \varepsilon$; in this case $E_6^{\varepsilon}(q)$ is the simple adjoint version. Second, we consider $E_6^{\varepsilon}(q)$ and $p \geq 5$ with $p \mid q - \varepsilon$. The cases $3 \mid q - \varepsilon$ and $p \mid q + \varepsilon$ with $p \geq 5$ are not discussed in this paper.

A subgroup $M \leq G$ is *p-local* if it is the normaliser of a nontrivial p-subgroup of G. It is *maximal p-local* if M is maximal with respect to inclusion among all p-local subgroups of G. It is *local maximal* if it is p-local for some prime p and maximal among all subgroups of G. If $R \leq G$ is p-radical, then $N_G(R)$ is p-local and $N_G(R) \leq N_G(C)$ for every characteristic subgroup $C \leq R$. In particular, $N_G(R)$ is contained in some maximal p-local $M \leq G$, so that $N_G(R) = N_M(R)$ and R is p-radical in M. Hence, every radical p-subgroup of G is radical in some maximal p-local subgroup of G.

A strategy for classifying radical p-subgroups is as follows: Firstly, classify all maximal p-local $M \leq G$ up to conjugacy. Secondly, classify the radical p-subgroups R of each such M. Thirdly, decide which of these R

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are p-radical in G, and fuse these in G. We used this approach in [3, 5] to classify the radical subgroups of $F_4(q)$. We remark that Cohen et al. [10] classified local maximal subgroups of exceptional groups of Lie type. However, not every maximal p-local subgroup is local maximal, and the details obtained in our classification of maximal p-local subgroups of $F_4(q)$ were necessary for our classification of the p-radical subgroups.

The approach we use in the paper is different to our recent work [3, 5]. If $p \geq 5$, then p is *good* for $E_6^\varepsilon(q)$ (cf. [9, p. 28]), and we classify the radical p-subgroups directly; the main tool in our proof is the work of Fleischmann & Janiszczak [14] on semisimple conjugacy classes of E_6. If $p = 3$, then $G = E_6^\varepsilon(q)$ with $3 \mid q + \varepsilon$ has a subgroup $H = F_4(q)$ containing a Sylow 3-subgroup of G. Applying our results of [3, 5] to H, we first classify the elementary abelian 3-subgroups of G and determine their local structure; it is then straightforward to determine the maximal 3-local subgroups of G. The classification of the radical 3-subgroups of G also reduces to our results in [3, 5].

1.1. Main results. The main result of this paper is the following classification theorem; it is proved for $p \geq 5$ in Section 3 and for $p = 3$ in Sections 4 and 5. Our results confirm the constructions we did for $q \in \{2, 4\}$ using the computer algebra system Magma [7]. Throughout, $G = E_6^\varepsilon(q)$ is of universal type.

Theorem A. *Let $G = E_6^\varepsilon(q)$ and either $p = 3$ with $3 \mid q + \varepsilon$, or $p \geq 5$ with $p \mid q - \varepsilon$. Up to conjugacy, the maximal p-local subgroups of G and the radical p-subgroups of G are as given in Theorems 3.6, 4.15, and 5.7, and Corollary 3.4.*

Let $G = E_6^\varepsilon(q)$ and write $q = r^f$ for a prime r. Let $\mathcal{F} = \mathcal{F}_D(G)$ be the Frobenius category, where $D \leq G$ is a nontrivial Sylow p-subgroup of G. As an application of Theorem A, we determine the essential rank of $\mathcal{F}$, which coincides with the number of conjugacy classes of p-essential subgroups of G; more details and definitions are given in Section 6. If $p = r$, then the essential rank of $\mathcal{F}$ is 6 by [2, Theorem 1.4]. If $p \neq r$ and $p \geq 7$ (or $p = 5$ and $5 \nmid q - \varepsilon$), then D is abelian, $N_G(D)$ controls p-fusion in G by a theorem of Burnside, and the essential rank is 0. Having classified the radical 3- and 5-subgroups of G, it is an easy exercise to determine the essential rank of $\mathcal{F}$ for $p \in \{3, 5\}$. The following is proved in Section 6.

Theorem B. *Let $G = E_6^\varepsilon(q)$ and either $p = 3$ with $3 \mid q + \varepsilon$, or $p = 5$ with $5 \mid q - \varepsilon$. Let $D \leq G$ be a Sylow p-subgroup. If $p = 5$, then the essential rank of $\mathcal{F}_D(G)$ is 2; if $p = 3$, then its essential rank is 2 or 3, depending on whether $9 \nmid q^2 - 1$ or $9 \mid q^2 - 1$.*

It is well-known that $G = E_6^\varepsilon(q)$ with $3 \mid q+\varepsilon$ has a subgroup $H = F_4(q)$ which contains a Sylow 3-subgroup of G, cf. Section 4.2. Our results (together with a result of Benson et al. [6]) imply that H controls 3-fusion in G; we prove the following in Corollary 4.14.

Theorem C. *Let $G = E_6^\varepsilon(q)$ with $3 \mid q+\varepsilon$. There is a subgroup $H = F_4(q)$ of G containing a Sylow 3-subgroup D of G. If $1 \neq A \leq D$ and $A^g \leq D$ with $g \in G$, then $g = ch$ for some $c \in C_G(A)$ and $h \in H$ with $A^g = A^h$; also, $\mathcal{F}_D(G) = \mathcal{F}_D(H)$, and H controls 3-fusion in G.*

In the notation of Theorem C, let $\mathcal{R}_3(H)/H$ be a set of H-conjugacy class representatives of 3-radical subgroups of H. Our results compared with [3, 5] show that one can choose $\mathcal{R}_3(H)/H = \mathcal{R}_3(G)/G$ if and only if $q + \varepsilon$ is not a 3-power – even though H controls 3-fusion in G for all q.

2. Notation and preliminary results

Our notation for simple groups, group extensions, and conjugacy classes are as in [3,5], thus taken from [11,19]. If not indicated by brackets, then we read group extensions $A.B.C$ from the left, that is, $A.B.C = (A.B).C$. If n, m are positive integers, then n^m denotes the direct product of m copies of cyclic groups of order n. This notation is ambiguous if n is written as a power itself, for example, $n = 3^a$ for some indeterminate a. There are only a few cases in this paper where this occurs, but the meaning should always follow from the context.

Let pX be the conjugacy class of an element of order p in a finite group G. An elementary abelian p-subgroup $E \leq G$ of order p^n is called *pX-pure* if all its nontrivial elements lie in pX; in this case we say that E

has *type* pX^n and write $E = pX^n$. Analogously, we write $E = pX_{m_1}Y_{m_2}$ if E has exactly m_1 and m_2 nontrivial elements lying in pX and pY, respectively, and $m_1 + m_2 = p^n - 1$; similarly, $E = pX_{m_1}Y_{m_2}Z_{m_3}$ is defined. Note that if $E = pX$, then E is uniquely defined up to conjugacy; we pose this condition for the other types as well, that is, if we write $E = pX_{m_1}Y_{m_2}Z_{m_3}$ (with possibly $m_2 = 0$ or $m_3 = 0$), then E is uniquely defined up to conjugacy in G. If for some type $pX_{m_1}Y_{m_2}Z_{m_3}$ there are non-conjugate groups, then we label these groups and write $E = (pX_{m_1}Y_{m_2}Z_{m_3})_i$, $i = 1, 2, \ldots$.

For an integer $n \neq 0$ denote by n_p the largest p-power dividing n; for $\eta \in \{\pm 1\}$ we define $n_\eta = \gcd(n, q-\eta)$, where the prime power q is always the size of the underlying field of $G = E_6^\varepsilon(q)$.

If $K, H \leq G$, then we write $K \leq_G H$ whenever there exists $x \in G$ with $K^x \leq H$. Analogously, $H =_G K$ and $y \in_G H$ with $y \in G$ are defined. Recall the notation $\mathrm{SL}_n^\varepsilon(q)$ and $\mathrm{GL}_n^\varepsilon(q)$: if $\varepsilon = 1$, then these are the special linear and general linear group of degree n over the field $\mathrm{GF}(q)$; if $\varepsilon = -1$, then these are the corresponding special unitary and unitary group, respectively, defined over $\mathrm{GF}(q^2)$. Recall that $\mathrm{SL}_2(q) = \mathrm{SL}_2^{-1}(q) = \mathrm{Sp}_2(q)$.

We denote by $\mathcal{R}_p(G)$ the set of all p-radical subgroups of G and write

$$\mathrm{Out}_G(H) = N_G(H)/HC_G(H).$$

If $H_1, H_2 \leq G$ and $Z \leq Z(H_1) \cap Z(H_2)$, then $H_1 \circ_Z H_2$ is the central product of H_1 and H_2 over Z; we also write $H_1 \circ H_2 = H_1 \circ_{Z(H_1)\cap Z(H_2)} H_2$. If $E \leq G$ is a p-subgroup and $N_G(E)$ is maximal p-local, then

$$N_G(E) = N_G(\Omega_1(Z(O_p(N_G(E))))),$$

where $Z(G)$ is the centre of G and $\Omega_i(G) = \langle g \in G \mid |g| = p^i \rangle$; note that $\Omega_1(Z(O_p(N_G(E))))$ is an elementary abelian p-group. This shows that every maximal p-local M has the form $M = N_G(E)$ with E in

$$\mathcal{ER}_p(G) = \{E \leq G \mid 1 \neq E = \Omega_1(Z(O_p(N_G(E))))\}.$$

The following two lemmas are [21, Satz III.13.10] and [28, Lemma 4.6].

Lemma 2.1. *Let G be a p-group, $p \geq 3$. If every abelian characteristic subgroup is cyclic, but not G, then $G = E \circ_Z Z(G)$ where E is an extra-special p-group of exponent p and $Z = \Omega_1(Z(G)) = Z(E)$.*

Lemma 2.2. *If $R \leq G$ is p-radical, then $O_p(G) \leq R$.*

The next lemma gives a criterion on when a group $N_G(E)$ with $E \in \mathcal{ER}_p(G)$ is maximal p-local. Note that if $E, F \in \mathcal{ER}_p(G)$ with $N_G(E)$ and $N_G(F)$ maximal p-local, then $N_G(E) =_G N_G(F)$ if and only if $E =_G F$.

Lemma 2.3. *Let $E \in \mathcal{ER}_p(G)$ and $R = O_p(N_G(E))$. Then $N_G(E)$ is maximal p-local if and only if $N_G(E) = N_G(Y)$ for every nontrivial elementary abelian p-subgroup Y of $\Omega_1(R)$ which is normal in $N_G(E)$; in particular, if R is abelian, then $Y \leq E$.*

Proof. If $N_G(E)$ is maximal p-local, then clearly $N_G(E) = N_G(Y)$ for all Y as in the lemma. Now consider the converse and suppose $N_G(E) \leq N$ for some p-local $N \leq G$. We can write $N = N_G(Y)$ for some nontrivial elementary abelian p-subgroup $Y \leq N$. Recall that $E = \Omega_1(Z(R))$, hence $N_G(R) = N_{N_G(E)}(R) = N_G(E)$ and R is radical in $N_G(E)$. In particular, $R = O_p(N_G(R)) = O_p(N_N(R))$, and R is radical in N; now Lemma 2.2 yields $Y \leq O_p(N) \leq R$, hence $Y \leq \Omega_1(R)$; clearly, Y is normal in $N_G(R)$. By assumption, $N_G(E) = N_G(Y)$, hence $N_G(E)$ is maximal p-local. Note that if R is abelian, then $Y \leq E$. □

The next lemma is [25, Theorem 25.14].

Lemma 2.4. *Let $G = E_6^\varepsilon(q)$. If $p \nmid q$ is a prime with $p \geq 7$, or $p = 5$ and $5 \nmid q - \varepsilon$, then every p-Sylow subgroup of G is abelian.*

Proof. A direct approach is to decompose $|G| = q^{36}(q^2-1)(q^5-\varepsilon)(q^6-1)(q^8-1)(q^9-\varepsilon)(q^{12}-1)$, see [19, Table 2.2], into factors which pairwise have a greatest common divisor less than p, and then make a case distinction on what factor is divisible by p. In every case one can then find a maximal torus of G containing an (abelian) Sylow p-subgroup of G, see [9, 13]. □

2.1. Maximal tori in classical groups. We recall the structure of maximal tori in some classical groups. First, let $G = \mathrm{GL}_d^\varepsilon(q)$ with $\varepsilon \in \{\pm 1\}$. The conjugacy classes of maximal tori of G are parametrised by partitions of d: the partition $(b_1, \ldots, b_m)$ corresponds to $T = (q^{b_1} - \varepsilon^{b_1}) \times \ldots \times (q^{b_m} - \varepsilon^{b_m})$. Over the algebraic closure, each direct factor $q^b - \varepsilon^b$ corresponds to a subgroup generated by a diagonal matrix with diagonal entries $z, z^{\varepsilon q}, z^{\varepsilon^2 q^2}, \ldots, z^{\varepsilon^{b-1} q^{b-1}}$ and all other diagonal entries equal to 1, where z has order $q^b - \varepsilon^b$. Moreover, $G' = \mathrm{SL}_d^\varepsilon(q)$ and $T \cap G'$ is a maximal torus of G'. Now let $G = \mathrm{Sp}_d(q)$ with $d = 2n$. The conjugacy classes of maximal tori of G are parametrised by signed partitions of n: the signed partition $(b_1^-, \ldots, b_i^-, c_1^+, \ldots, c_j^+)$ of n corresponds to $T = (q^{b_1^-} + 1) \times \ldots \times (q^{b_i^-} + 1) \times (q^{c_1^+} - 1) \times \ldots \times (q^{c_j^+} - 1)$. Over the algebraic closure, each $q^b \pm 1$ of T corresponds to a subgroup generated by a diagonal matrix with $2b$ nontrivial diagonal entries. The maximal tori of $G = \mathrm{SO}_d(q)$ with $d = 2n+1$ are as for $\mathrm{Sp}_{2n}(q)$. The maximal tori of $\mathrm{SO}_{2n}^-(q)$ and $\mathrm{SO}_{2n}^+(q)$ are as for $\mathrm{Sp}_{2n}(q)$, with the condition that i is odd and even, respectively. The commutator subgroup of $\mathrm{SO}_{2n}^\varepsilon(q)$ is $\Omega_{2n}^\varepsilon(q)$, and $\mathrm{PSpin}_{2n}^\varepsilon(q) = \mathrm{P}\Omega_{2n}^\varepsilon(q)$; the maximal tori of $\mathrm{Spin}_{2n}^-(q)$ are those of $\mathrm{SO}_{2n}^-(q)$.

3. Radical 5-subgroups

In this section, let $G = E_6^\varepsilon(q)$ with $\varepsilon \in \{\pm 1\}$ such that $5 \mid q - \varepsilon$; recall that G has an abelian Sylow 5-subgroup if $5 \nmid q - \varepsilon$. In a preliminary section, the subgroups of order 5 are determined. Subsequently, the abelian radical 5-subgroups of G are classified, and then the non-abelian ones. Also the maximal 5-local subgroups of G are determined. The classification of the radical p-subgroups with $p \geq 7$ and $p \mid q - \varepsilon$ follows in Corollary 3.4.

3.1. Weyl group action and subgroups of order 5. Let $T = (q-\varepsilon)^6$ be a maximal torus of G, and $E = \Omega_1(O_5(T)) = 5^6$. Let $W = N_G(T)/T$ be the corresponding Weyl group. Every maximal torus of G isomorphic to $(q-\varepsilon)^6$ is conjugate to T, see [13]. We have $C_G(E) = C_G(T) = T$, cf. Lemma 3.2, which implies $N_G(E) \leq N_G(T)$, thus $N_G(E) = N_G(T)$ and $\mathrm{Out}_G(E) = \mathrm{Out}_G(T) = W \leq \mathrm{Aut}(E) = \mathrm{GL}_6(5)$. Recall that

$$W = W(E_6) \cong \mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{PSp}_4(3).2.$$

We denote by S_k the symmetric group of degree k.

Proposition 3.1. *Let $H = \mathrm{GL}_6(5)$ with underlying vector space $V = 5^6$, and consider $W = N_G(T)/T \leq H$.*

a) *The group H has two conjugacy classes of subgroups isomorphic to W, with representatives W_1 and W_2; both, W_1 and W_2, have the same action on the subspaces of V.*

b) *There are nine W-orbits of lines in V, with representatives $L_{\mathbf{A}}, L_{\mathbf{B}}, \cdots, L_{\mathbf{I}}$; the centraliser $C_W(L_X)$ and normaliser $N_W(L_X)$ for each line L_X are given in Table I.*

c) *Identifying E with V, there are exactly nine G-conjugacy classes of subgroups of order 5 in E, denoted also by L_X; each subgroup of G of order 5 is conjugate to some L_X.*

d) *The centraliser $C_G(L_X)$ for each X is given in Table I.*

Proof. a) By [27, Theorem 4.1(v)], up to conjugacy, $\mathrm{GL}_6(\mathbb{Z})$ has a unique subgroup $W \times \langle -1_6 \rangle$, where $1_6 \in H$ is the identity. Thus, H contains a subgroup $N = W \times \langle -1_6 \rangle$, and so $[N, N] = \mathrm{PSp}_4(3) \leq H$. By [17, p. 62], the group $\mathrm{PSp}_4(3)$ has a unique irreducible 5-Brauer character of degree 6, which implies that H contains a unique conjugacy class of subgroups isomorphic to $\mathrm{PSp}_4(3)$. A simple computation shows that $N_H(\mathrm{PSp}_4(3)) = W \times Z(H) = W \times 4$. Since $x^2 \in \mathrm{PSp}_4(3)$ for any $x \in W$, it follows that H contains exactly two conjugacy classes of subgroups isomorphic to W, with representatives W_1 and W_2. In particular, the actions of both W_i on the subspaces of V are the same. Alternatively, part a) can be checked directly by computer.

b) This follows from a direct computation in $\mathrm{GL}_6(5)$, for example, using Magma [7].

c) By b), there are exactly nine W-orbits of order 5 subgroups in E, also denoted by L_X. Since $C_G(T) = T$ and T is abelian, $T \leq C_G(L_X)$, $C_{C_G(L_X)}(T) = T$, hence $C_W(L_X) \cong N_{C_G(L_X)}(T)/T$ for each X. It follows from [14, pp. 94 & 98] (with $J = A_1 + A_5$) that G contains a subgroup $K = (K_1 \circ_{2_\varepsilon} K_2).2_\varepsilon$ where $K_1 = \mathrm{SL}_2(q)$ and $K_2 = \mathrm{SL}_6^\varepsilon(q)$; in fact, if q is odd, then $K = C_G(i)$ for some involution $i \in G$ by [19, Table 4.5.1]. The group K contains a Sylow 5-subgroup S of G; recall that $|S| = 5^{6a+1}$ where 5^a is the largest 5-power dividing $q - \varepsilon$. If $z \in G$ has order 5, then we may suppose $z \in S$ and hence $z = z_1 : z_2$ with each $z_i \in K_i$. In particular,

$C_{K_1}(z_1) \in \{K_1, (q-\varepsilon)\}$, and we can suppose $T \le C_G(z)$. By the uniqueness of maximal tori, $z \in_G T$ for every $z \in G$ of order 5; thus $\langle z \rangle =_G L_X$ for some X.

d) We continue with the notation of c) and determine $C_G(L_{\mathbf{B}})$. We can suppose each $T_1 = T \cap K_1 = q - \varepsilon$ and $T_2 = T \cap K_2 = (q-\varepsilon)^5$ consists of diagonal matrices of K_1 and K_2, respectively. Let $\lambda \in \mathrm{GF}(q^2)^\times$ be a non-square if q is odd, and $\lambda = 1$ otherwise. Then we can define $K = \langle K_1 \circ_{2_\varepsilon} K_2, x\rangle$, where $x = x_1{:}x_2$, and x_1 and x_2 act as $\mathrm{diag}(1,\lambda) \in \mathrm{GL}_2(q)$ and $\mathrm{diag}(1,1,1,1,1,\lambda) \in G_2$ on K_1 and K_2, respectively. Thus, $C_K(T_1 \circ_{2_\varepsilon} T_2) = \langle T_1 \circ_{2_\varepsilon} T_2, x\rangle$ has order $(q-\varepsilon)^6$, and $T \le C_K(T_1 \circ_{2_\varepsilon} T_2)$ implies that

$$T = \langle T_1 \circ_{2_\varepsilon} T_2, x\rangle.$$

If $y \in G$ is a generator of $\Omega_1(O_5(T_1))$, then $C_K(y) = (q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_6^\varepsilon(q).2_\varepsilon)$ and $|N_{C_K(y)}(T)/T| = |\mathrm{S}_6| = 720$. Since $N_{C_K(y)}(T)/T \le N_{C_G(y)}(T)/T \cong C_W(y)$, the orders $|C_W(L_X)|$ given in b) imply that $y \in_G L_{\mathbf{B}}$ and, moreover, $N_{C_K(y)}(T) = N_{C_G(y)}(T)$. The centralisers of semisimple elements in G are discussed in [12, 14], and it follows from $C_K(y) \le C_G(y)$ that the only possibility for $C_G(y)$ is $C_G(y) = C_K(y)$; we comment in more detail on [14] below. Thus,

$$C = C_G(L_{\mathbf{B}}) = (q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_6^\varepsilon(q).2_\varepsilon).$$

Note that $p = 5$ is good for $G = E_6^\varepsilon(q)$, and every $s \in G$ of order 5 is of so-called parabolic type; in addition, we have $5 \nmid |Z(G)|$, see [19, Table 2.2]. Now [19, Theorem 4.2.2] shows that $C = C_G(s) = L.U$ where $L = L_1 \times \cdots \times L_s$ is a direct product of groups of Lie type in characteristic r, and U is an abelian r'-group inducing inner-diagonal automorphisms on each L_i. Moreover, the Dynkin diagram Δ_L of L is a proper subdiagram of E_6, that is, it lies in

$$(3.1) \qquad \{D_5, A_5, A_4+A_1, A_3+A_1, 2A_2+A_1, 2A_2, A_2+2A_1, A_4, A_2+A_1, D_4, 3A_1, A_3, 2A_1, A_2, A_1, \emptyset\}.$$

Let Π be a basis of simple roots of the root system of G, and denote by δ_0 the highest (long) root. The centraliser $C_G(s)$ of a semisimple $s \in G$ is determined in [12], see also [14]: in the notation of [14, p. 94], each $C_G(s)$ is parametrised by a pair $(J, [w])$ where J is a proper subset of $\Pi_0 = \Pi \cup \{-\delta_0\}$ and $[w]$ runs over the conjugacy class representatives of the normaliser $N_W(J)$. Now suppose s has order 5. As shown in the proof of part c), we can assume that $T \le C_G(s) = C_G(L_X)$ for some X. In the notation of [14], we have $|C_G(L_X)| = |Z_J^{nF}||S_J^{nF}|$, where S_J^{nF} is the semisimple component of $C_G(L_X)$, and $Z_J^{nF} = Z(C_G(L_X)) \le T$, hence $Z_J^{nF} = (q-\varepsilon)^t$ for some $t \ge 1$. (Here F is a Frobenius automorphism and n lies in the simply connected algebraic group corresponding to G, mapping on w.) The orders $|Z_J^{nF}|$ and $|S_J^{nF}|$ are determined in [14], and the important observation is that only for $n = 1$ (that is, $w = 1$) the order of $C_G(L_X)$ is divisible by $(q-\varepsilon)^6$. Since $T \le C_G(s)$, this implies that the centralisers $C_G(s)$ with s of order 5 are parametrised by pairs $(J, [1])$. The comment above also shows that we have in fact $J \subseteq \Pi$, and J is uniquely determined by $C_W(L_X) = N_{C_G(L_X)}(T)/T$.

Recall that $C = C_G(s) = L.U$, and write this as $C = Z(C) \circ_{Z(C)\cap Z(L)} (L.Q) = H_1 \circ_D (H_2.Q)$, where $H_1 = Z(C)$, $H_2 = L$, $D = H_1 \cap Z(H_2)$, and Q induces outer-diagonal automorphisms on H_2. As mentioned above, we also have $|C_G(L_X)| = |Z_J^F||S_J^F|$ with $Z_J^F = (q-\varepsilon)^t \le T$ for some $t \ge 1$; in particular, $H_1 = Z(C_G(L_X)) = Z_J^F$, $\Delta_L = J$, and $|C_W(L_X)| = |N_{C_G(L_X)}(T)/T|$. Thus $H_2 = L$ is uniquely determined by J, and Q is a subgroup of the outer-diagonal automorphism group of H_2 of order $|D|$. Running over all possibilities for J, cf. (3.1), this gives the structure of $C_G(L_X)$ for all X; we give two examples:

If $J = A_4 + A_1$, then $(J, [1])$ gives $H_2 = L = \mathrm{SL}_2(q) \times \mathrm{SL}_5^\varepsilon(q)$, thus $H_1 = (q-\varepsilon)$, $D = 5 \times 2_\varepsilon$, and $Q = 5 \times 2_\varepsilon$; this yields the centraliser $C_G(s) = (q-\varepsilon) \circ_{5\times 2_\varepsilon} (\mathrm{SL}_2(q) \times \mathrm{SL}_5^\varepsilon(q)).(5 \times 2_\varepsilon)$.

If $J = 2A_2$, then $(J,[1])$ gives $H_2 = \mathrm{SL}_3^\varepsilon(q) \times \mathrm{SL}_3^\varepsilon(q)$. Now [14, p. 100] with $[w] = [1]$ shows that $|H_1| = |Z_J^F| = (q-\varepsilon)^2$, and we define $C = (q-\varepsilon)^2 \circ_{3_\varepsilon} (\mathrm{SL}_3^\varepsilon(q) \times \mathrm{SL}_3^\varepsilon(q)).3_\varepsilon$. Note that $N_C(T)/T \cong \mathrm{S}_3 \times \mathrm{S}_3$, but there is no X with $|C_W(L_X)| = |N_C(T)/T|$. Thus C is not a centraliser of an order 5 element. □

3.2. Abelian radical 5-subgroups. Recall that $5 \mid q - \varepsilon$ and $T = (q-\varepsilon)^6$ is a maximal torus of $G = E_6^\varepsilon(q)$ with Weyl group $W = N_G(T)/T$. Let 5^a be the largest 5-power dividing $q - \varepsilon$, and define

$$\mathcal{AR}_5(G) = \{R \mid R \le G \text{ is abelian and 5-radical in } G\}.$$

L_X	$C_W(L_X)$	$N_W(L_X)$	$C_G(L_X)$
$L_{\mathbf{A}}$	$2^4.\mathrm{S}_5$	$C_W(L_{\mathbf{A}})$	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{Spin}^\varepsilon_{10}(q).2_\varepsilon)$
$L_{\mathbf{B}}$	S_6	$C_W(L_{\mathbf{B}}).2$	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{SL}^\varepsilon_6(q).2_\varepsilon)$
$L_{\mathbf{C}}$	$2\times\mathrm{S}_5$	$C_W(L_{\mathbf{C}})$	$(q-\varepsilon)\circ_{5\times 2_\varepsilon}((\mathrm{SL}_2(q).2_\varepsilon)\times(\mathrm{SL}^\varepsilon_5(q).5))$
$L_{\mathbf{D}}$	$2\times\mathrm{S}_3\times\mathrm{S}_3$	$C_W(L_{\mathbf{D}}).2$	$(q-\varepsilon)\circ_{6_\varepsilon}((\mathrm{SL}_2(q).2_\varepsilon)\times((\mathrm{SL}^\varepsilon_3(q))^2.3_\varepsilon))$
$L_{\mathbf{E}}$	$2^{1+4}_+.\mathrm{S}_3$	$C_W(L_{\mathbf{E}}).2$	$(q-\varepsilon)^2\circ_{(2_\varepsilon)^2}(\mathrm{Spin}^\varepsilon_8(q).(2_\varepsilon)^2)$
$L_{\mathbf{F}}$	S_5	$C_W(L_{\mathbf{F}})$	$(q-\varepsilon)^2\circ_5(\mathrm{SL}^\varepsilon_5(q).5)$
$L_{\mathbf{G}}$	$2\times\mathrm{S}_4$	$C_W(L_{\mathbf{G}})$	$(q-\varepsilon)^2\circ_{(2_\varepsilon)^2}((\mathrm{SL}_2(q)\times\mathrm{SL}^\varepsilon_4(q)).(2_\varepsilon)^2)$
$L_{\mathbf{H}}$	$2^2\times\mathrm{S}_3$	$C_W(L_{\mathbf{H}}).2$	$(q-\varepsilon)^2\circ_{(2_\varepsilon)^2}((\mathrm{SL}_2(q)^2\times\mathrm{SL}^\varepsilon_3(q)).(2_\varepsilon)^2)$
$L_{\mathbf{I}}$	S_4	$C_W(L_{\mathbf{I}}).4$	$(q-\varepsilon)^3\circ_{4_\varepsilon}(\mathrm{SL}^\varepsilon_4(q).4_\varepsilon)$

TABLE I. Subgroups of order 5 of $G=E^\varepsilon_6(q)$, where $5\mid q-\varepsilon$ and $n_\eta=\gcd(n,q-\eta)$

Lemma 3.2. *If E is an elementary abelian 5-subgroup of G, then $E\le_G T$ and $C_G(E)=H_1\circ(H_2.Q)$, where $H_1=(q-\varepsilon)^t=Z(C_G(E))$ for some $t\le 6$, $H_2=L_1\times\cdots\times L_s$ with each L_i a group of Lie type (universal version) in characteristic r, the Dynkin diagram Δ_{H_2} lies in (3.1), and Q acts as a group of outer-diagonal automorphisms on each L_i. We may suppose $H_1\le T$, hence, if $Y=O_5(C_G(E))$ and $X=\Omega_1(Y)\le\Omega_1(O_5(T))$, then $C_G(Y)=C_G(X)=C_G(E)$,*

$$N_G(Y)=N_G(X)=C_G(Y).(N_W(X)/C_W(X)),\tag{3.2}$$

and $O_5(N_G(Y))=O_5(C_G(Y))=Y$. In particular, Y is an abelian radical subgroup of G.

PROOF. Let $\overline{G}$ be a simply-connected algebraic group with Frobenius automorphism F such that $G=\overline{G}^F$. It follows from [8, Proposition 2.1(ii)] that $C^\circ_{\overline{G}}(E)$ is a Levi subgroup of $\overline{G}$, cf. [19, Definition 2.6.6]; in the notation of [14, p. 94], we have $C^\circ_{\overline{G}}(E)=G_J$ for some $J\subseteq\Pi$ as in (3.1). By [8, Proposition 2.1(iii)],

$$C_G(E)=(C_{\overline{G}}(E))^F=(C^\circ_{\overline{G}}(E))^F=G^F_J;$$

in the notation of [14, p. 94], we have $G^F_J=Z^F_J S^F_J$ where $S_J=[G_J,G_J]$ is the semisimple part of G_J, and Z_J is the 1-component of the centre $Z(G_J)$; the orders $|G^F_J|=|Z^F_J||S^F_J|$ are given in [14], and we have $|Z^F_J|=(q-\varepsilon)^t$ for some t. Since $\overline{G}$ is universal, so is S_J, see [19, Theorem 1.13.3(c)]; moreover, $p=5$ is a good prime for G and each Lie component of S^F_J. Thus, by [19, Theorem 2.6.5(f)], we have $C_G(E)=TS^F_J=H_1\circ(H_2.Q)$ where $H_1=Z^F_J=(q-\varepsilon)^t\le_G T$ and $H_2=S^F_J=L_1\times\ldots\times L_s$ are as in the lemma, and $Q=H_1\cap Z(H_2)$ acts as outer-diagonal automorphisms. Note that $E\le Z(C_G(E))=H_1$, which proves that $E\le_G T$. In the following, we assume that $E\le H_1\le T\le C_G(E)$. Let

$$Y=O_5(C_G(E))=O_5(H_1)\quad\text{and}\quad X=\Omega_1(Y),$$

so that $E\le X\le Y$, hence $C_G(Y)\le C_G(X)\le C_G(E)$. Now $Y\le Z(C_G(E))$ yields $C_G(Y)=C_G(X)=C_G(E)$. Every $x\in N_G(X)$ normalises $Z(C_G(X))=H_1$. Therefore T^x is a maximal torus of G centralising E, and $T^x\le C_G(E)$. If $T_2=T\cap H_2$ and $D=H_1\cap T_2$, then $\tilde{T}=H_1\circ_D T_2\le T$ with $C_G(\tilde{T})=T$. Both T_2 and T_2^x are maximal tori of H_2, and, by uniqueness, $T_2^{xy}=T_2$ for some $y\in H_2\le C_G(X)$. Also y normalises H_1, thus $xy\in N_G(H_1\circ_D T_2)$ and $xy\in N_G(C_G(H_1\circ_D T_2))=N_G(C_G(\tilde{T}))=N_G(T)$. In particular, $xy\in N_{N_G(T)}(X)$ and $N_G(X)=C_G(X)N_{N_G(T)}(X)$. Since $N_{N_G(T)}(X)\cap C_G(X)=N_{C_G(X)}(T)=C_{N_G(T)}(X)$, it follows that

$$N_G(X)/C_G(X)=(N_{N_G(T)}(X)/T)/(C_{N_G(T)}(X)/T)=N_W(X)/C_W(X)$$

and therefore $N_G(X)=C_G(X).(N_W(X)/C_W(X))$. Recall that $N_G(Y)\le N_G(X)$ by construction. Now $C_G(X)=C_G(Y)=C_G(E)$ shows $Y=O_5(Z(C_G(E)))=O_5(Z(C_G(X)))$, which implies that $N_G(Y)=N_G(X)$. It remains to show that Y is 5-radical in G.

If $t=6$, then $C_G(E)=H_1=T$; recall that $N_G(T)=N_G(O_5(T))=N_G(\Omega_1(O_5(T)))$, see Section 3.1, and so $O_5(N_G(Y))=O_5(T)=Y$. Similarly, if $t=5$, then $H_1=(q-\varepsilon)^5$ and $H_2=\mathrm{SL}_2(q)$, and therefore $C_G(X)=C_G(E)=H_1\circ_{2_\varepsilon}(H_2.2_\varepsilon)$. A direct computation as in the proof of Proposition 3.1 shows that $O_5(N_W(X))=1$ for every $X=5^5$, and so $O_5(N_G(Y))=O_5(C_G(X))=O_5(H_1)=Y$.

Now let $t \leq 4$. Note that W has a Sylow 5-subgroup P of order 5; if we assume that $T \leq C_G(L_{\mathbf{B}})$, then $N_{C_G(L_{\mathbf{B}})}(T)/T = \mathrm{S}_6$ contains a Sylow 5-subgroup of W, which shows that P acts on $T = (q-\varepsilon)^6$ as a 5-cycle permuting the direct factors. Recall that $X \leq Y \leq H_1 \leq T$, thus, if $t \leq 4$, then $5 \nmid |N_W(X)|$ and $O_5(N_G(Y)) = O_5(C_G(X)) = Y$ follows from $N_G(Y) = N_G(X) = C_G(X).(N_W(X)/C_W(X))$. □

Let $W \leq \mathrm{GL}_6(5)$ be the Weyl group of $E_6^\varepsilon(q)$, and write $W = \langle s_1, \cdots, s_6\rangle$ where each s_j is a reflection. A subgroup $U \leq W$ is a *parabolic subgroup* of W if $U = \langle s_{i_1}, \cdots, s_{i_m}\rangle$ for some $s_{i_j} \in \{s_1, \cdots, s_6\}$. If Δ is a subdiagram of the Dynkin diagram of G with respect to which W is defined (for example, if Δ is as in (3.1)), then $W(\Delta)$ is the parabolic subgroup generated by the simple roots occurring in Δ. In the following proposition, let $\mathcal{PS}(W)$ be the set of all parabolic subgroups of W.

Up to conjugacy, every elementary abelian 5-subgroup $E \leq E_6^\varepsilon(q)$ lies in the maximal torus T; define

$$W_E = N_{C_G(E)}(T)/T \quad \text{and} \quad V_E = N_W(E)/C_W(E).$$

Proposition 3.3. *Let $G = E_6^\varepsilon(q)$, $T = (q-\varepsilon)^6 \leq G$, and $W = N_G(T)/T$.*

a) *Up to conjugacy, the abelian radical 5-subgroups of G and their centralisers are R_i and $C_G(R_i)$, $i \in \{1, \ldots, 18\}$, as in Table II. If $i \leq 16$, then $N_G(R_i) = N_G(\Omega_1(R_i))$ and $C_G(R_i) = C_G(\Omega_1(R_i))$.*

b) *If $E \leq G$ is an elementary abelian 5-group, then, up to conjugacy, $C_G(E) = C_G(R_i)$ for some $i \leq 16$; moreover, $E \leq \Omega_1(R_i)$, and $C_G(R_i) \leq N_G(E) \leq N_G(R_i)$. Thus, if $M \leq G$ is maximal 5-local, then $M =_G N_G(R_i)$ for some $i \leq 16$.*

c) *Let $R \in \mathcal{AR}_5^*(G) = \{R \in \mathcal{AR}_5(G) \mid R =_G R_i \text{ for } i \leq 16\}$ and suppose $T \leq C_G(R)$. Then $\Omega_1(R) \in \mathcal{ER}_5(G)$ and $W_{\Omega_1(R)} \in \mathcal{PS}(W)$ is a parabolic subgroup of W. The map $R \mapsto W_{\Omega_1(R)}$ induces a bijection between the G-conjugacy classes of groups in $\mathcal{AR}_5^*(G)$ and the W-conjugacy classes of groups in $\mathcal{PS}(W)$.*

d) *If $i \leq 16$, then $N_G(R_i) = C_G(R_i).V_{\Omega_1(R_i)}$ is given in Table II.*

e) *Up to conjugacy, the maximal 5-local subgroups of G are the groups in Table II displayed in boldface.*

Proof. a) Let $R \in \mathcal{R}_5(G)$ be abelian, and let $E = \Omega_1(R)$. Using the notation of Lemma 3.2, we can write $C_G(E) = H_1 \circ (H_2.Q)$ with $H_1 = (q-\varepsilon)^t = Z(C_G(E))$. Observe that $N_G(R)$ normalises $C_G(E)$, thus it also normalises $Y = O_5(H_1) = O_5(Z(C_G(E)))$ and the 5-subgroup $YR \leq G$. This shows that $N_G(R)$ normalises $N_{YR}(R)$, hence $R \leq N_{YR}(R) \leq O_5(N_G(R)) = R$. If we would have $YR > R$, then $N_{YR}(R) > R$ by p-group theory; therefore we must have $YR = R$, and $Y \leq R$. But $\Omega_1(R) = E \leq Y \leq R$, so $\Omega_1(R) = E = \Omega_1(Y)$ and $|E| = 5^t$. Since $R \leq C_G(E)$ and $Y \leq R$, it follows that $R = Y \circ_A \hat{R}$ for some $\hat{R} \leq H_2.Q$ and $A = H_1 \cap Z(H_2)$.

First, suppose $A = 1$, that is, $R = Y \times \hat{R}$ and

$$E = \Omega_1(R) = \Omega_1(Y) \times \Omega_1(\hat{R}) = E \times \Omega_1(\hat{R}),$$

which implies that $\Omega_1(\hat{R}) = 1$, hence $\hat{R} = 1$. In the notation of Lemma 3.2, we have $R = Y = O_5(C_G(E))$ and $E = X = \Omega_1(Y)$, in particular $R = O_5(N_G(R)) = O_5(C_G(E)) = O_5(Z(C_G(E)))$. This yields $C_G(E) \leq C_G(R)$, thus $C_G(R) = C_G(E) = H_1 \circ (H_2.Q)$; the Dynkin diagram Δ_{H_2} corresponding to H_2 is given by (3.1). Up to conjugacy, the corresponding groups $C_G(E) = G^F_{\Delta_{H_2}} = H_1 \circ (H_2.Q)$ are given in [14], which yields $R_1, \ldots, R_{16}$; the corresponding centralisers $C_G(R)$ in Table II are obtained as in the proof of Proposition 3.1. In the notation of Lemma 3.2, we have $R = Y$, thus $C_G(R) = C_G(Y) = C_G(X) = C_G(E)$, and the normalisers $N_G(R) = N_G(Y) = N_G(X) = N_G(\Omega_1(Y)) = N_G(E)$ are given by (3.2), see part e) of this proposition for more details. In particular, the groups $R_1, \ldots R_{16}$ exist and are indeed radical in G.

Now suppose $R = Y \circ_A \hat{R}$ with $A \neq 1$, hence $H_1 \cap Z(H_2)$ is not a $5'$-group. Note that $\Omega_1(\hat{R}) \leq A$ since $\Omega_1(R) = \Omega_1(Y)$, and $t \leq 4$ by the proof of Lemma 3.2. The Dynkin diagram associated with H_2 lies in (3.1), and it follows that the only possibility for Δ_{H_2} with $Z(H_2)$ of order divisible by 5 is $\Delta_{H_2} \in \{A_1 + A_4, A_4\}$. Thus, we can assume that $A = 5$. Since $\Omega_1(R) = \Omega_1(Y)$, we must have $\Omega_1(\hat{R}) = 5 = A$, hence $\hat{R}$ is cyclic. We may also suppose that $\hat{R} \neq 5$ since otherwise $\hat{R} \leq Y$ and $R = Y$, which has been considered above.

If $\Delta_{H_2} = A_1 + A_4$, then $C_G(E) = (q-\varepsilon) \circ_{10_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}_5^\varepsilon(q).5))$, which implies $\hat{R} \leq \mathrm{SL}_5^\varepsilon(q).5$, $Y = O_5((q-\varepsilon)) = 5^a$, $E = \Omega_1(R) = 5$, and R is cyclic. If $a > 1$ and $|\hat{R}| \leq 5^a$, then $R = Y \circ_5 \hat{R}$ has exponent 5^a, and therefore cannot be cyclic since $|R| = 5^{2a-1} > 5^a$; this implies that $|\hat{R}| = 5^{a+1}$. If $a = 1$, then $\hat{R} > 5^a$ since we assume that $\hat{R} \neq 5$. Thus, we have proved that $R = \hat{R} = 5^{a+1} = R_{17}$ in this case. Note that $H_1 \circ_5 (\mathrm{SL}_5^\varepsilon(q).5) = \mathrm{GL}_5^\varepsilon(q)$ with $C_{\mathrm{GL}_5^\varepsilon(q)}(\hat{R}) = q^5 - \varepsilon$ and $N_{\mathrm{GL}_5^\varepsilon(q)}(\hat{R}) = (q^5 - \varepsilon).5$. Since $\Omega_1(R) = E = 5$, we have $C_G(R) = C_{C_G(E)}(R)$, thus $C_G(R) = (q^5 - \varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$. Similarly, $N_G(R) = N_{N_G(E)}(R)$. In the notation of Lemma 3.2, we have $E = \Omega_1(R) = \Omega_1(Y) = X$, hence $N_G(E) = C_G(E).(N_W(E)/C_W(E))$. Since $C_G(E) =_G C_G(L_{\mathbf{C}})$, it follows from Table I that $N_G(R) = N_{C_G(E)}(R)$, which implies that $N_G(R) = C_G(R).5$.

If $\Delta_{H_2} = A_4$, then $C_G(E) = (q - \varepsilon)^2 \circ_5 (\mathrm{SL}_5^\varepsilon(q).5)$, thus $\hat{R} \leq \mathrm{SL}_5^\varepsilon(q).5$, $Y = (5^a)^2$, and $E = \Omega_1(R) = \Omega_1(Y) = 5^2$. If $5 < |\hat{R}| \leq 5^a$, then R has exponent 5^a and so $R = Y = (5^a)^2$ by the fundamental theorem of finite abelian groups, a contradiction to $|Y \circ_5 \hat{R}| > 5^{2a}$, which proves that $\hat{R} = 5^{a+1}$. As before, note that $C_G(E) = (q-\varepsilon)^2 \circ_5 (\mathrm{SL}_5^\varepsilon(q).5) = (q-\varepsilon) \times \mathrm{GL}_5^\varepsilon(q)$, thus $R = 5^a \times 5^{a+1} = R_{18}$ and $C_G(R) = C_{C_G(E)}(R) = (q-\varepsilon) \times (q^5 - \varepsilon)$. We can assume that $E = \Omega_1(R_5)$, hence $N_G(R) \leq N_G(E) = N_G(R_5)$ and $N_G(R) = N_{N_G(R_5)}(R)$; this proves $N_G(R) = C_G(R).10_\varepsilon$. The local structure proves that $R_1, \ldots, R_{18}$ are pairwise not conjugate in G

b) By the proof of Lemma 3.2, we can assume that $C_G(E) = C_G(R_i)$ for some $i \leq 16$. The proof also shows that $R_i = O_5(C_G(E))$, thus R_i is normalised by $N_G(E)$, hence $C_G(R_i) \leq N_G(E) \leq N_G(R_i)$. Clearly, $E \leq \Omega_1(O_5(C_G(E))) = \Omega_1(R_i)$. Thus, if M is maximal 5-local, then $M = N_G(\Omega_1(Z(O_5(M)))) =_G N_G(R_i)$ for some $i \leq 16$.

c) Let $R \in \mathcal{AR}_5^*(G)$ and write $E = \Omega_1(R)$. We have $R =_G R_i$ for some $i \leq 16$, and part a) shows that $N_G(R) = N_G(E)$, hence $E \in \mathcal{ER}_5(G)$. Write $C_G(E) = H_1 \circ (H_2.Q)$ as in Lemma 3.2, and suppose that $H_1 \leq T \leq C_G(E)$. Now clearly $W_E = N_{C_G(E)}(T)/T = W(\Delta_{H_2}) \in \mathcal{PS}(W)$. If $R =_G S$, then $\Omega_1(R)^g = \Omega_1(S)$ for some $g \in G$. We can suppose T centralises R and S, hence T^g is a maximal torus of $C_G(S)$ and $T^g = T^h$ for some $h \in C_G(S)$. Now $gh^{-1} \in N_G(T)$, and $C_G(R)^{gh^{-1}} = C_G(S)$ and $W_{\Omega_1(R)}^{gh^{-1}T} = W_{\Omega_1(S)}$ with $gh^{-1}T \in W$. Since $R \leq S$ if and only if $\Omega_1(R) \leq \Omega_1(S)$ if and only if $C_G(\Omega_1(S)) \leq C_G(\Omega_1(R))$, it follows that $R \leq S$ if and only if $W_{\Omega_1(S)} \leq W_{\Omega_1(R)}$. A simple computation, for example, by computer, shows that, up to conjugacy, W has 16 parabolic subgroups. Since we have already found 16 non-isomorphic parabolic subgroups $W_{\Omega_1(R)}$, the map $R \mapsto W_{\Omega_1(R)}$ defines a bijection between the G-classes in $\mathcal{AR}_5^*(G)$ and the W-classes in $\mathcal{PS}(W)$.

d) If $i \in \{17, 18\}$, then $N_G(R_i)$ is given in the proof of a), so let $R = R_i$ and $E = \Omega_1(R)$ with $i \leq 16$ in the following. Part a) and (3.2) show that $C_G(R) = C_G(E)$ and $N_G(R) = N_G(E) = C_G(R).(N_W(E)/C_W(E)) = C_G(R).V_E$, where $V_E = N_W(E)/C_W(E) = N_G(E)/C_G(E)$, see the proof of Lemma 3.2. By c), $W_E = N_{C_G(E)}(T)/T$ is a parabolic subgroup of $W \leq \mathrm{GL}_6(5)$, acting on $V = 5^6$. The proof of c) also shows that W has 16 conjugacy classes of parabolic subgroups, and, using a computer, we obtain explicit representatives of these subgroup classes. If $H \leq W$ is such a parabolic subgroup, then its structure readily determines a unique $i \leq 16$ such that $H =_W W_{\Omega_1(R_i)}$ and $E_i =_G C_V(H) = \{v \in V \mid vH = v\} \leq V$. Thus, up to conjugacy, we can construct $E_1, \ldots, E_{16} \leq V$, which allows us to compute $V_E = N_W(E)/C_W(E)$, and hence $N_G(R) = C_G(R).V_E$. The results of these computations are given in Table II.

e) If $R_i \in \mathcal{AR}_5^*(G)$, then we can suppose $T \leq C_G(R_i)$, and we define $E_i = \Omega_1(R_i)$, $W_i = N_{C_G(E_i)}(T)/T$, and $V_i = N_W(E_i)/C_W(E_i)$. If $M \leq G$ is maximal 5-local, then we can assume that $M = N_G(R_i)$ for some $i \leq 16$, see part b), hence it remains to determine the maximal 5-local subgroups among $N_G(R_1), \ldots, N_G(R_{16})$. Let $i \leq 16$ in the following, hence $N_G(R_i) = N_G(E_i)$ by part a), and $E_i \in \mathcal{ER}_5(G)$ by part c). Note that $N_G(R_i) = (H_1 \circ H_2.Q).V_i$ with $H_1 = (q - \varepsilon)^t$, $R_i = O_5(H_1)$, $E_i = 5^t$, and $V_i \leq \mathrm{Aut}(E_i) = \mathrm{GL}_t(5)$, see Table II and Lemma 3.2. By Lemma 2.3, the group $N_G(R_i) = N_G(E_i)$ is maximal 5-local if there is no nontrivial $E < E_i$ fixed by V_i. As in part d), we can explicitly compute V_i, and it follows that V_i acts irreducibly on E_i if and only if $i \in \{1, 2, 3, 4, 6, 9, 13, 15, 16\}$; thus, for these i, the group $N_G(R_i)$ is maximal 5-local. Also, we find that V_i fixes a proper 1-dimensional subspace of E_i if and only if $i \in \{5, 7, 8, 10, 11, 12, 14\}$; thus, for these i, we obtain $N_G(R_i) <_G N_G(R_j)$ for some $j \leq 4$, and $N_G(R_i)$ is not maximal 5-local. □

3.3. Radical p-subgroups with $p \geq 7$. The previous results allow us to classify the radical p-subgroups of $G = E_6^\eta(q)$ for $p \geq 7$ and $p \mid q - \eta$; every Sylow p-subgroup of G is abelian, see Lemma 2.4. Denote by b the largest p-power dividing $q - \eta$.

	R	$C_G(R)$	W_E	$N_G(R) = C_G(R).V_E$
R_1	5^a	$(q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{Spin}^\varepsilon_{10}(q).2_\varepsilon)$	$2^4.\mathrm{S}_5$	$\boldsymbol{C_G(R)}$
R_2	5^a	$(q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}^\varepsilon_6(q).2_\varepsilon)$	S_6	$\boldsymbol{C_G(R).2}$
R_3	5^a	$(q-\varepsilon) \circ_{10_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}^\varepsilon_5(q).5))$	$2 \times \mathrm{S}_5$	$\boldsymbol{C_G(R)}$
R_4	5^a	$(q-\varepsilon) \circ_{6_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times ((\mathrm{SL}^\varepsilon_3(q))^2.3_\varepsilon))$	$2 \times (\mathrm{S}_3)^2$	$\boldsymbol{C_G(R).2}$
R_5	$(5^a)^2$	$(q-\varepsilon)^2 \circ_5 (\mathrm{SL}^\varepsilon_5(q).5)$	S_5	$C_G(R).2$
R_6	$(5^a)^2$	$(q-\varepsilon)^2 \circ_{(2_\varepsilon)^2} (\mathrm{Spin}^+_8(q).(2_\varepsilon)^2)$	$2^{1+4}_+.\mathrm{S}_3$	$\boldsymbol{C_G(R).\mathrm{S}_3}$
R_7	$(5^a)^2$	$(q-\varepsilon)^2 \circ_{4_\varepsilon \times 2_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}^\varepsilon_4(q).4_\varepsilon))$	$2 \times \mathrm{S}_4$	$C_G(R).2$
R_8	$(5^a)^2$	$(q-\varepsilon)^2 \circ_{3_\varepsilon \times (2_\varepsilon)^2} ((\mathrm{SL}_2(q).2_\varepsilon)^2 \times (\mathrm{SL}^\varepsilon_3(q).3_\varepsilon))$	$2^2 \times \mathrm{S}_3$	$C_G(R).2$
R_9	$(5^a)^2$	$(q-\varepsilon)^2 \circ_{(3_\varepsilon)^2} ((\mathrm{SL}^\varepsilon_3(q).3_\varepsilon)^2)$	$(\mathrm{S}_3)^2$	$\boldsymbol{C_G(R).D_{12}}$
R_{10}	$(5^a)^3$	$(q-\varepsilon)^3 \circ_{6_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}^\varepsilon_3(q).3_\varepsilon))$	$2 \times \mathrm{S}_3$	$C_G(R).\mathrm{S}_3$
R_{11}	$(5^a)^3$	$(q-\varepsilon)^3 \circ_{4_\varepsilon} (\mathrm{SL}^\varepsilon_4(q).4_\varepsilon)$	S_4	$C_G(R).D_8$
R_{12}	$(5^a)^3$	$(q-\varepsilon)^3 \circ_{(2_\varepsilon)^3} ((\mathrm{SL}_2(q).2_\varepsilon)^3)$	2^3	$C_G(R).D_{12}$
R_{13}	$(5^a)^4$	$(q-\varepsilon)^4 \circ_{3_\varepsilon} (\mathrm{SL}^\varepsilon_3(q).3_\varepsilon)$	S_3	$\boldsymbol{C_G(R).3^2.D_8}$
R_{14}	$(5^a)^4$	$(q-\varepsilon)^4 \circ_{(2_\varepsilon)^2} ((\mathrm{SL}_2(q).2_\varepsilon)^2)$	2^2	$C_G(R).2^3.\mathrm{S}_3$
R_{15}	$(5^a)^5$	$(q-\varepsilon)^5 \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$	2	$\boldsymbol{C_G(R).\mathrm{S}_6}$
R_{16}	$(5^a)^6$	$(q-\varepsilon)^6$	1	$\boldsymbol{T.W}$
R_{17}	5^{a+1}	$(q^5-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$		$C_G(R).5$
R_{18}	$5^a \times 5^{a+1}$	$(q-\varepsilon) \times (q^5-\varepsilon)$		$C_G(R).10_\varepsilon$.

TABLE II. Abelian radical 5-subgroups of $G = E^\varepsilon_6(q)$ with $5 \mid q-\varepsilon$ and $n_\eta = \gcd(n, q-\eta)$

Corollary 3.4. *Let $p \geq 7$ be a prime and $G = E^\eta_6(q)$ the universal version with $7 \mid q-\eta$. Up to conjugacy, the radical p-subgroups of G are $R_1, \ldots, R_{16}$ with local structure given in Table II, with $(5, \varepsilon, a)$ replaced by (p, η, b). Up to conjugacy, the maximal p-local subgroups of G are the groups $N_G(R_i)$ in Table II displayed in boldface.*

PROOF. First, Lemma 3.2 also holds when 5 is replaced by p, with some modifications in the last paragraph of its proof; note that $p \nmid |W(E_6)|$. Second, if $R \leq G$ is p-radical, then R is abelian by Lemma 2.4. Now the proof of Proposition 3.3a) carries over and shows that $R \cong R_i$ for some $i \in \{1, \ldots, 16\}$. (The proof also shows that R_{17} and R_{18} do not exist since H_2 with Δ_{H_2} in (3.1) always satisfies $p \nmid |Z(H_2)|$.) It remains to show that each R_i exists. For this purpose, consider $K_i = C_G(R_i)$, which exists by [14], and write $K_i = H_1 \circ (H_2.Q)$ such that $H_1 = Z(K_i) = (q-\eta)^t$, $E_i = \Omega_1(O_p(Z(K_i))) = p^t$, and Δ_{H_2} as in (3.1). Clearly, E_i is elementary abelian and $K_i \leq C_G(E_i)$. Lemma 3.2 shows that, $C_G(E_i) = A_1 \circ_S (A_2.S)$ with $A_1 = Z(C_G(E_i)) = (q-\eta)^s$, $E_i = \Omega_1(O_p(A_1)) = p^s$, and Δ_{A_2} as in (3.1). Up to conjugacy, the groups $C_G(E_i) = G^F_{\Delta_{A_2}} = A_1 \circ_S (A_2.S)$ are given in [14], and it follows that $K_i = C_G(E_i)$. This proves that $O_p(C_G(E_i)) = R_i$ exists and $E_i = \Omega_1(R_i)$; Lemma 3.2 now shows that $O_p(N_G(R_i)) = O_p(C_G(R_i)) = O_p(C_G(E_i)) = R_i$, so R_i is p-radical. In particular, the proof of Proposition 3.3a) also shows that $N_G(R_i) = N_G(O_p(H_1))$. Since $C_G(O_p(H_1)) = H_1 \circ (H_2.Q)$ and $H_1 = Z(C_G(O_p(H_1)))$, it follows that $N_G(O_p(H_1)) \leq N_G(H_1)$, hence $N_G(R_i) = N_G(H_1)$, which is independent of p since $H_1 = (q-\eta)^t$. Therefore, the local structure of R_i is as for $p = 5$ as given in Table II. If $M \leq G$ is maximal p-local, then, as in the proof of Proposition 3.3e), we can assume that $M = N_G(R_i)$ with $i \leq 16$. In particular, $N_G(R_i)$ is maximal p-local if and only if $N_G(R_i) \not<_G N_G(R_j)$ for $i \neq j$. As shown above, $N_G(R_i)$ is independent of p, and the assertion follows from our results on $p = 5$. □

3.4. Non-abelian radical 5-subgroups. We continue our classification; let $R_1, \ldots, R_{18}$ be the groups in Table II. The next preliminary lemma shows that $N_G(R_i) = C_G(R_i).V_i = (H_1.V_i) \circ (H_2.Q)$ for $i \in \{5, 15\}$; this is required for the subsequent main theorem.

Lemma 3.5. *We have $N_G(R_5) = ((q-\varepsilon)^2.2) \circ_5 (\mathrm{SL}^\varepsilon_5(q).5)$ and $N_G(R_{15}) = ((q-\varepsilon)^5.\mathrm{S}_6) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$.*

PROOF. Proposition 3.3 proves $N_G(R_3) = N_G(\Omega_1(R_3)) = Q_1 \circ_{10_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}^\varepsilon_5(q).5))$ where $Q_1 = (q-\varepsilon)$ and $R_3 = O_5(Q_1)$. Let $Q_2 = (q-\varepsilon) \leq \mathrm{SL}_2(q).2_\varepsilon$ be a maximal torus such that $\hat{R} = O_5(Q_1 \times Q_2)$ satisfies

$C_G(\hat{R}) = C_{C_G(R_3)}(\hat{R}) = (q-\varepsilon)^2 \circ_5 (\mathrm{SL}_5^\varepsilon(q).5) \cong C_G(R_5)$ and $N_{N_G(R_3)}(\hat{R}) = ((q-\varepsilon)^2.2) \circ_5 (\mathrm{SL}_5^\varepsilon(q).5)$. An argument as in the proof of Lemma 3.2 proves that $\hat{R} =_G R_5$, namely, [8, Proposition 2.1] shows that $C_G(\hat{R})$ is a Levi subgroup of G, and the list of groups in [14, p. 101] implies that we can assume $C_G(\hat{R}) = C_G(R_5)$. But $\hat{R} \leq O_5(C_G(R_5)) = R_5$, hence $\hat{R} = R_5$. By Table II, we know $N_G(R_5) = C_G(R_5).2$, thus $N_G(R_5) =_G N_G(\hat{R}) = N_{N_G(R_3)}(\hat{R})$.

Let $K \leq G$ be as in the proof of Proposition 3.1c), that is, $K = (K_1 \circ_{2_\varepsilon} K_2).2_\varepsilon \leq G$ with $K_1 = \mathrm{SL}_2(q)$ and $K_2 = \mathrm{SL}_6^\varepsilon(q)$. Let $Q_1 = (q-\varepsilon)^5 \leq K_2$ be a maximal torus so that $N_K(Q_1) = (K_1 \circ_{2_\varepsilon} Q_1.\mathrm{S}_6).2_\varepsilon$ and $C_K(O_5(Q_1)) = (\mathrm{SL}_2(q).2_\varepsilon) \circ_{2_\varepsilon} Q_1 \leq C_G(O_5(Q_1))$. Since $Q_1 \leq Z(C_G(O_5(Q_1)))$, it follows from [14] that $C_K(O_5(Q_1)) = C_G(O_5(Q_1)) \cong C_G(R_{15})$. As before, we deduce that $C_G(R_{15}) =_G C_G(O_5(Q_1))$, hence $R_{15} =_G O_5(Q_1)$. Now Table II implies that $N_G(R_{15}) = (((q-\varepsilon)^5.\mathrm{S}_6) \circ_{2_\varepsilon} \mathrm{SL}_2(q)).2_\varepsilon$. □

Theorem 3.6. *Let $G = E_6^\varepsilon(q)$ with $5 \mid q-\varepsilon$. Up to conjugacy, the abelian and non-abelian radical 5-subgroups are $R_1, \ldots, R_{22}$ as given in Tables II and Table III, respectively. Up to conjugacy, the maximal 5-local subgroups of G are the groups in Table II displayed in boldface.*

Proof. By Proposition 3.3, we may suppose that R is non-abelian. Let $E = \Omega_1(Z(R))$, so that $R \leq C_G(E)$. In particular, $C_G(E)$ has a non-abelian Sylow 5-subgroup and Lemma 3.2 implies $C_G(E) \in_G \{C_1, C_2, C_3, C_5\}$, where $C_i = C_G(R_i)$ with R_i as in Table II. Recall that 5^a is the largest 5-power dividing $q-\varepsilon$.

(1) Suppose $C_G(E) \in_G \{C_1, C_2\}$, so that $C_G(E) = H_1 \circ_{2_\varepsilon} (H_2.Q)$, where $H_1 = (q-\varepsilon)$, $Q = 2_\varepsilon$, and $H_2 = \mathrm{Spin}_{10}^\varepsilon(q)$ or $H_2 = \mathrm{SL}_6^\varepsilon(q)$. Thus $R = T_1 \times T_2$ with $T_1 \leq H_1$ and $T_2 \leq H_2$. Since R is non-abelian, it follows that T_2 is non-abelian, in particular, $\Omega_1(Z(T_2)) \neq 1$. This yields $E = \Omega_1(Z(R)) = \Omega_1(T_1) \times \Omega_1(Z(T_2))$, a contradiction to $E \leq H_1$. Thus we must have $C_G(E) \in_G \{C_3, C_5\}$, according as $Z(R)$ is cyclic or non-cyclic.

(2) Suppose R contains a non-cyclic abelian characteristic subgroup A; choose A such that $F = \Omega_1(AZ(R))$ has maximal possible order. Note that F is non-cyclic and $N_G(R) \leq N_G(F)$, thus R is radical in $N_G(F)$ and $R \geq O_5(N_G(F))$ by Lemma 2.2. In particular, the Sylow 5-subgroup of $N_G(F)$ is non-abelian. Since F is non-cyclic, $C_G(F) = C_i$ and $N_G(R) \leq N_G(F) \leq N_G(R_i)$ for some $i \in \{5, 15, 16\}$, see Proposition 3.3b).

If $C_G(F) = C_{16}$, then $C_G(F) = T$ and $O_5(C_G(F)) = (5^a)^6 \leq R$ as R is a radical subgroup of $N_G(F)$. But R is non-abelian, so $R = R_{19} = 5^a \times 5^a \wr 5 = S$, a Sylow 5-subgroup of G. Note that $Z(S) = (5^a)^2$, and $C_G(Z(S)) \leq C_G(\Omega_1(Z(S)))$ contains a Sylow 5-subgroup of G. By Proposition 3.3b), we can assume that $C_G(S) \leq C_G(\Omega_1(Z(S))) = C_G(R_5) = C_5$ with $R_5 \leq S$ and $\Omega_1(Z(S)) = \Omega_1(R_5)$. Thus, $C_G(S) = C_{C_5}(S) = (q-\varepsilon)^2$, and, similarly, $N_G(S) = N_{N_G(R_5)}(S)$. Lemma 3.5 shows that $N_G(R_5) = ((q-\varepsilon)^2.2) \circ_5 (\mathrm{SL}_5^\varepsilon(q).5)$, hence $\mathrm{Out}_G(S) = 2 \times \mathrm{Out}_{\mathrm{SL}_5^\varepsilon(q)}(\hat{S}) = 2 \times 4$ where $\hat{S} = (5^a)^4 \rtimes 5 \leq S$ is a Sylow 5-subgroup of $\mathrm{SL}_5^\varepsilon(q)$.

If $C_G(F) = C_{15}$, then $N_G(F) \leq N_G(R_{15})$, and R is radical in $N_G(R_{15}) = C_G(R_{15}).\mathrm{S}_6$; recall that $N_G(R_{15}) = ((q-\varepsilon)^5.\mathrm{S}_6) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$ by Lemma 3.5. Since $O_5(N_G(F)) \leq R$, we have $(5^a)^5 = O_5(C_G(F)) \leq R$. Hence $R = T_1 \times T_2$ with $T_1 \leq (q-\varepsilon)^5.\mathrm{S}_6$ and $T_2 \leq \mathrm{SL}_2(q)$. By Lemma 2.2, we have $O_5(N_G(R_{15})) \leq R$; since R is non-abelian, this implies that $T_1 = 5^a \wr 5$ and $T_2 = 1$ or $T_2 = 5^a$. In the latter case, $R = R_{19}$ (Sylow 5-subgroup), so let us assume that $R = T_1 = 5^a \wr 5 \leq (q-\varepsilon)^5.\mathrm{S}_6$. This yields $R = R_{20}$, and $C_G(R) \leq C_G(F) = (q-\varepsilon)^5 \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$ implies $C_G(R) = (q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$, where $(q-\varepsilon)$ is a diagonally embedded in $(q-\varepsilon)^5$. Also, we have $N_G(R) = N_{N_G(R_{15})}(R) = (C_G(R)R).4$.

If $C_G(F) = C_5 = (q-\varepsilon)^2 \circ_5 (\mathrm{SL}_5^\varepsilon(q).5)$, then $N_G(F) \leq N_G(R_5) = C_G(F).2$, thus $R = (5^a)^2 \circ_5 \hat{R}$ for some radical subgroup $\hat{R} \leq \mathrm{SL}_5^\varepsilon(q).5$; recall that R is radical in $N_G(F)$. In particular, $Z(R) = (5^a)^2 \circ_5 Z(\hat{R})$, and $\hat{R}$ is non-abelian. If $\hat{A}$ is an abelian characteristic subgroup of $\hat{R}$, then $Z(R)\hat{A}$ is an abelian characteristic subgroup R. By Proposition 3.3b), we have $F \leq (q-\varepsilon)^2$; since A is chosen such that $F = \Omega_1(Z(R)A)$ has maximal order, it follows that all the abelian characteristic subgroups of $\hat{R}$ are cyclic. Now Lemma 2.1 shows that $\hat{R} = 5_+^{1+2s} \circ_5 5^\alpha$ with $Z(\hat{R}) = 5^\alpha$ for some $\alpha \geq 1$ and $s \geq 1$. Note that $(q-\varepsilon) \circ_5 (\mathrm{SL}_5^\varepsilon(q).5) = \mathrm{GL}_5^\varepsilon(q)$ for some $(q-\varepsilon) \leq Z(C_5) = (q-\varepsilon)^2$, so that $(5^a) \circ_5 \hat{R}$ is a radical 5-subgroup of $\mathrm{GL}_5^\varepsilon(q)$. These radical subgroups are classified in [1, Theorem 2.3], and it follows that $\alpha = s = 1$, hence $\hat{R} = 5_+^{1+2}$ and $R = R_{21} = (5^a)^2 \circ_5 5_+^{1+2}$. Up to conjugacy, $\mathrm{GL}_5^\varepsilon(q)$ contains a unique radical subgroup $O_5(Z(\mathrm{GL}_5^\varepsilon(q)))\hat{R} = 5^a \circ_5 \hat{R}$, and we have $\mathrm{Out}_{\mathrm{GL}_5^\varepsilon(q)}(\hat{R}) = \mathrm{Sp}_2(5) = \mathrm{SL}_2(5)$, cf. [1, p. 152], thus $\mathrm{Out}_G(R) = \mathrm{SL}_2(5)$.

(3) Suppose all abelian characteristic subgroups of R are cyclic, thus $R = 5^\alpha \circ_5 5_+^{1+2s}$ for some $\alpha, s \geq 1$ by Lemma 2.1. Moreover, $E = \Omega_1(Z(R)) = 5$, and (1) shows that $C_G(E) = C_3 = (q-\varepsilon) \circ_{10_\varepsilon} ((\mathrm{SL}_2(q).2_\varepsilon) \times (\mathrm{SL}_5^\varepsilon(q).5)) = N_G(E)$. Note that R is radical in $C_G(E)$, hence $5^a = O_5(C_G(E)) \leq Z(R)$, and $R = 5^a \circ_5 (T_2 \times T_3)$, where $T_2 \in \mathcal{R}_5(\mathrm{SL}_2(q).2_\varepsilon)$ and $T_3 \in \mathcal{R}_5(\mathrm{SL}_5^\varepsilon(q).5)$. Since $Z(R)$ is cyclic and R is non-abelian, we must have $T_2 = 1$ and $T_3 \neq 1$ non-abelian. Note that all abelian characteristic subgroup of T_3 are cyclic by assumption, thus $T_3 = 5^\beta \circ_5 5_+^{1+2t}$ with $\beta, t \geq 1$. It follows as in (2) that $\beta = t = 1$, thus $T_3 = 5_+^{1+2}$, and $R = R_{22} = 5^a \circ_5 5_+^{1+2}$ is unique up to conjugacy. The local structure of R_{22} is determined as in the last paragraph of (2). □

	R	$C_G(R)$	$N_G(R)$
R_{19}	$5^a \times 5^a \wr 5$	$(q-\varepsilon)^2$	$(C_G(R)R).(4 \times 2)$
R_{20}	$5^a \wr 5$	$(q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$	$(C_G(R)R).4$
R_{21}	$(5^a)^2 \circ_5 5_+^{1+2}$	$(q-\varepsilon)^2$	$(C_G(R)R).\mathrm{SL}_2(5)$
R_{22}	$5^a \circ_5 5_+^{1+2}$	$(q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$	$(C_G(R)R).\mathrm{SL}_2(5)$

Table III. Non-abelian radical 5-subgroups of $G = E_6^\varepsilon(q)$ with $5 \mid q-\varepsilon$ and $n_\eta = \gcd(n, q-\eta)$

4. Maximal 3-local subgroups

Let $G = E_6^{-\varepsilon}(q)$ with $3 \mid q-\varepsilon$. The first aim of this section is to classify the elementary abelian 3-subgroups of G. We show that their conjugacy classes correspond to the conjugacy classes of elementary abelian 3-subgroups in a subgroup $F_4(q)$ of G; this allows us to apply our results in [3,5]. Knowing the elementary abelian 3-subgroups of G (and their local structure), it is then straightforward to classify the maximal 3-local subgroups of G.

4.1. Elements of order 3 in $E_6^{-\varepsilon}(q)$. By [19, Table 4.7.3], the group G has exactly three conjugacy classes of order 3 element, called 3A, 3B, and 3C, with representatives z_{A}, z_{B}, and z_{C}, such that $\langle z_X \rangle = 3X$ is $3X$-pure for each $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, that is, z_X and z_X^{-1} are conjugate. The local structure is as follows; the notation is explained below:

$$C_G(3\mathrm{A}) = \langle (q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q), x_{\mathrm{A}} \rangle, \qquad N_G(3\mathrm{A}) = \langle (q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q), x_{\mathrm{A}}, \gamma_{\mathrm{A}} \rangle,$$
$$C_G(3\mathrm{B}) = \langle (q^2-1) \circ_{2_\varepsilon} \mathrm{Spin}_8^-(q), x_{\mathrm{B}} \rangle, \qquad N_G(3\mathrm{B}) = \langle (q^2-1) \circ_{2_\varepsilon} \mathrm{Spin}_8^-(q), x_{\mathrm{B}}, \gamma_{\mathrm{B}} \rangle$$
$$C_G(3\mathrm{C}) = \langle \mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3(q^2), x_{\mathrm{C}} \rangle, \qquad N_G(3\mathrm{C}) = \langle \mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3(q^2), x_{\mathrm{C}}, \gamma_{\mathrm{C}} \rangle,$$

where the actions of γ_X and x_X are as follows:

$\gamma_{\mathrm{A}} = \iota{:}1$	ι acts as $x \mapsto x^{-1}$ on $q-\varepsilon$
$\gamma_{\mathrm{B}} = (-\varepsilon\phi){:}\gamma$	γ acts as the order 2 field automorphism on $\mathrm{Spin}_8^-(q)$, cf. [19, Theorem 2.5.12(g)]; $-\varepsilon\phi$ acts as the field automorphism $x \mapsto x^{-\varepsilon q}$ on q^2-1
$\gamma_{\mathrm{C}} = \gamma{:}(-\varepsilon\phi)$	γ acts as the order 2 graph automorphism on $\mathrm{SL}_3^\varepsilon(q)$, so γ is inverse-transpose; ϕ acts as the field automorphism of level q on $\mathrm{SL}_3(q^2)$, that is, as φ_q as defined in [19, Theorem 1.15.4]; $-\phi$ acts on $\mathrm{SL}_3(q^2)$ as the product of the field automorphism of level q and a nontrivial graph automorphism of order 2
$x_{\mathrm{A}} = 1 : 2_\varepsilon$	if q is odd, then x_{A} acts as $\mathrm{diag}(1,1,1,1,1,\lambda) \in \mathrm{GL}_6^{-\varepsilon}(q)$ on $\mathrm{SL}_6^{-\varepsilon}(q)$ with $\lambda \in \mathrm{GF}(q^2)$ a non-square element
$x_{\mathrm{B}} = 1 : 2_\varepsilon$	if q is odd, then x_{B} induces the outer-diagonal automorphism of order 2 on $\mathrm{Spin}_8^-(q)$, cf. [19, Theorem 2.5.12(c)]; more precisely, x_{B} is induced by a $\mathrm{GF}(q)$-linear conformal endomorphism of the underlying space of $\mathrm{Spin}_8^-(q)$, corresponding to a non-square multiplier, cf. [16, p. 124].
$x_{\mathrm{C}} = x_1 : x_2$	each x_i acts as $o_i = \mathrm{diag}(1,1,\tau) \in \mathrm{GL}_3^\varepsilon(q)$ or $\mathrm{GL}_3(q^2)$ according as $i = 1$ or 2, where $\tau \in \mathrm{GF}(q^2)$ is an element of maximal 3-power order 3^a; define $\omega = \tau^{3^{a-1}}$.

Note that x_{A} does not necessarily have order 2, but $x_{\mathrm{A}}^2 \in (q-\varepsilon)\circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q)$; similarly for $x_{\mathrm{B}}, x_{\mathrm{C}}$ and γ_X. For example, $x_{\mathrm{C}}^3 \in \mathrm{SL}_3^\varepsilon(q)\circ_3 \mathrm{SL}_3(q^2)$, and each $o_i^3 = \mathrm{diag}(1,1,\tau^3)$ acts as the inner automorphism $\mathrm{diag}(\tau^{-1},\tau^{-1},\tau^2)$. By definition, $x_X^{\gamma_X}$ acts as x_X^{-1} for all $X \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$, and we can assume that $x_i^3 = \mathrm{diag}(\tau^{-1},\tau^{-1},\tau^2) \in \mathrm{SL}_3^\varepsilon(q)$ or $\mathrm{SL}_3(q^2)$ according as $i=1$ or 2, thus $\tilde{T} = \langle (q-\varepsilon)^2 \circ_3 (q^2-1)^2, x_{\mathrm{C}}\rangle \le C_G(3\mathrm{C})$ is semisimple and abelian of order $(q-\varepsilon)^4 \times (q+\varepsilon)^2$. Since

$$T = (q-\varepsilon)^2 \times (q^2-1)^2$$

is the unique maximal torus of G with the same order, $T =_G \tilde{T}$, cf. [13, Table II].

Remark 4.1. Let $L_1 = \mathrm{SL}_3^\varepsilon(q)$, $L_2 = \mathrm{SL}_3(q^2)$, $G_1 = \mathrm{GL}_3^\varepsilon(q) \ge L_1$ and $G_2 = \mathrm{GL}_3(q^2) \ge L_2$, so that $O^{r'}(C_G(3\mathrm{C})) = L_1 \circ_3 L_2$. Up to conjugacy, every $s \in C_G(3\mathrm{C}) \setminus L_1 \circ_3 L_2$ has the form $s = (t_1 : t_2)x_{\mathrm{C}}$ and each $t_i \in L_i$. We also write $s = s_1 : s_2$ with $s_i = t_i x_i$, where each s_i is defined as the action of an element $\tilde{s}_i \in G_i$ on L_i, namely $\tilde{s}_i = t_i o_i \in G_i$, so that $C_{L_i}(s_i) = C_{L_i}(\tilde{s}_i)$. Note that $\tilde{s}_i \in G_i$ is uniquely defined modulo $Z(G_i)$, and $Z(G_i) = q-\varepsilon$ or q^2-1 according as $i=1$ or 2. If s_i has order 3^j, then the order of $\tilde{s}_i$ is divisible by 3^j, and we can suppose that also $\tilde{s}_i$ is a 3-element. We consider an example which will be important later. Let 3^a be the largest 3-power dividing $q-\varepsilon$, that is, o_i has order 3^a. Let $y \in L_i$ be the permutation matrix corresponding to $(1,2,3)$. Now $\tilde{v} = y o_i \in G_i$ has order 3^{a+1}, and $\tilde{v}^3 \in Z(G_i)$. It follows from $3^{a+1} \nmid q^2-1$ that $\tilde{v}$ lies in a maximal torus of G_i of type $q^3-\varepsilon$ or q^6-1 according as $i=1$ or 2. All this implies that $v = (y:y)x_{\mathrm{C}} = v_1 : v_2$ with $v_i = yx_i$ has order 3 and $C_{L_i}(v_i) = C_{L_i}(\tilde{v}) = q^2+\varepsilon q+1$ or q^4+q^2+1 according as $i=1$ or 2.

Lemma 4.2. *The following groups are maximal 3-local:*

$$M_1 = N_G(3\mathrm{A}), \quad M_2 = N_G(3\mathrm{B}), \quad \textit{and} \quad M_3 = N_G(3\mathrm{C}).$$

If $E \le G$ such that $|E| = 3$ and $M = N_G(E)$ is maximal 3-local, then $M =_G M_i$ for some i.

Proof. First, consider M_1 and M_2, hence let $X \in \{\mathrm{A},\mathrm{B}\}$; then $O_3(N_G(3X))$ is abelian and $3X$ lies in $\mathcal{ER}_3(G)$, hence $M_i < G$ is maximal 3-local by Lemma 2.3. Now consider M_3. If $q > 2$, then $O_3(N_G(3\mathrm{C}))$ is abelian and 3C lies in $\mathcal{ER}_3(G)$, hence $M_3 < G$ is maximal 3-local by Lemma 2.3. If $q = 2$, then $R = O_3(M_3) = O_3(L_1) = 3_+^{1+2}$ where $L_1 = \mathrm{SU}_3(2) \le M_3$ as before. Since L_1 transitively permutes the four subgroups of R of order 9, the group $E = \langle z_{\mathrm{C}}\rangle$ is the only normal elementary abelian 3-subgroup of L_1, and hence of $N_G(E) = M_3$. Now M_3 is maximal 3-local by Lemma 2.3. □

4.2. A subgroup $F_4(q)$. It is well-known that $G = E_6^{-\varepsilon}(q)$ has a subgroup $H = F_4(q)$, see [18, Table 4.1]. Moreover, H contains a Sylow 3-subgroup D of G; recall that $|D| = 3^{4a+2}$ where 3^a is the largest 3-power dividing $q-\varepsilon$. The radical 3-subgroups of H have been determined in [3, 5]. Note that also H has three conjugacy classes $3\mathrm{A}'$, $3\mathrm{B}'$, and $3\mathrm{C}'$ of order 3-elements; in [3, 5] these classes are also called 3A, 3B, and 3C. We can assume that $3X' \subseteq 3X$ for $X = \mathrm{A},\mathrm{B},\mathrm{C}$: First, $t \in 3\mathrm{C}'$ if and only if $t \in O^{r'}(C_H(t))$ and $t \in 3\mathrm{C}$ if and only if $t \in O^{r'}(C_G(t))$, thus $3\mathrm{C}' \subseteq 3\mathrm{C}$. If q is even, then representatives of $3\mathrm{A}'$ and $3\mathrm{B}'$ have the same local structure in H, hence we can define $3\mathrm{A}'$ and $3\mathrm{B}'$ such that $3X' \subseteq 3X$ for all $X = \mathrm{A},\mathrm{B},\mathrm{C}$. If q is odd and $t \in 3\mathrm{B}'$, then $\mathrm{Spin}_7(q) \le C_H(t)$ (see [5, Table 2]), and $\mathrm{Spin}_7(q) \not\le C_G(z_{\mathrm{A}})$ implies $t \in 3\mathrm{B}$, hence also $3\mathrm{A}' \subseteq 3\mathrm{A}$.

Let $\tau \in \mathrm{GF}(q^2)$ be as in the definition of x_{C}. We can suppose $z_{\mathrm{C}} \in H$, and it is known (cf. [3, 5]) that

$$\tilde{M}_3 = N_H(\langle z_{\mathrm{C}}\rangle) = (\tilde{L}_1 \circ_3 \tilde{L}_2).\tilde{x}.\tilde{\gamma} \quad \text{with} \quad \tilde{L}_1, \tilde{L}_2 \cong \mathrm{SL}_3^\varepsilon(q),$$

where $\tilde{x} = \tilde{x}_1 : \tilde{x}_2$, both $\tilde{x}_i$ acting as $\mathrm{diag}(1,1,\tau)$ on $\mathrm{SL}_3^\varepsilon(q)$, and $\tilde{\gamma} = \tilde{\gamma}_1 : \tilde{\gamma}_2$, both $\tilde{\gamma}_i$ acting as inverse-transpose; moreover, we can assume that $\tilde{x}_i^3 = \mathrm{diag}(\tau^{-1},\tau^{-1},\tau^2) \in \tilde{L}_i$. Recall also that

$$M_3 = (L_1 \circ L_2).x.\gamma \quad \text{with} \quad L_1 \cong \mathrm{SL}_3^\varepsilon(q),\ L_2 \cong \mathrm{SL}_3(q^2),$$

where the action of γ depends on ε. Clearly, $\tilde{M}_3 \le M_3$, and we can assume that $\tilde{M}_3$ lies naturally in M_3:

Lemma 4.3. *If $q \ge 4$, then we can choose H such that $\tilde{M}_3 = (L_1 \circ_3 \tilde{L}_2).x.\gamma$ where $\tilde{L}_2 \le L_2 \cap \mathrm{GL}_3^\varepsilon(q)$.*

Proof. As above, write $\tilde{M}_3 = (\tilde{L}_1 \circ_3 \tilde{L}_2).\tilde{x}.\tilde{\gamma} \le M_3$, and let $L = L_1 \circ_3 L_2$ and $\tilde{L} = \tilde{L}_1 \circ_3 \tilde{L}_2$. Since both L_i and $\tilde{L}_i$ are perfect, we have $\tilde{L} = \tilde{L}_1 \circ_3 \tilde{L}_2 = C_H(z_{\mathrm{C}})' \le C_G(z_{\mathrm{C}})' = L_1 \circ_3 L_2 = L$. Let $g \in \tilde{L}_1$ be an non-central 3-element. The subgroup U generated by all $\tilde{L}$-conjugates of g is normal in $\tilde{L}_1$, and hence $U = \tilde{L}_1$ since $q \ge 4$.

Now write $g = g_1 : g_2$ where each $g_i \in L_i$ is a 3-element. It is well-known that the centraliser of a non-central 3-element in $\mathrm{SL}_3^\varepsilon(q)$ is isomorphic to $(q-\varepsilon)^2$, $\mathrm{GL}_2^\varepsilon(q)$, or $q^2+\varepsilon q+1$, cf. [5, Lemma 3.1(1)]; similarly for $\mathrm{SL}_3(q^2)$. Since $\tilde{L}_2 \leq C_L(g)$, this implies that one of g_1, g_2 is central, say $g_i = 1$; note that $Z(L) = 3$. The subgroup generated by L-conjugates of g is L_1 or L_2, so $U = \tilde{L}_1 \leq L_i$ for some i. If $i = 1$, then $\tilde{L}_1 = L_1$, and the same argument with $g \in \tilde{L}_2$ proves that $\tilde{L}_2 < L_2$. If $i = 2$, then we interchange $\tilde{L}_1$ and $\tilde{L}_2$, and get the same result.

Note that $\mathrm{SU}_3(q)$ has a unique 3-dimensional representation over $\mathrm{GF}(q^2)$. Thus, if $\varepsilon = -1$, then there is $g \in L_2 \leq C_G(z_{\mathrm{C}})$ such that $\tilde{L}_2^g \leq L_2 \cap \mathrm{GL}_3(q^2)$ preserves the Hermitian form $1_3 = \mathrm{diag}(1,1,1)$. Replacing H by H^g, we can assume that $\tilde{L}_1 = L_1$ and $\tilde{L}_2 \leq L_2$ satisfies the properties mentioned in the lemma. If $\varepsilon = 1$, then $3 \mid q-1$, and $\mathrm{SL}_3(q^2)$ has a unique conjugacy class of subgroups isomorphic to $\mathrm{GL}_3(q)$, see [23, Table 3.5.A]; note that if $3 \mid q-1$, then $c_5 = 1$ in [23, Table 3.5.A]. Since every subgroup $K = \mathrm{SL}_3(q)$ of $\mathrm{SL}_3(q^2)$ lies in a maximal subgroup of type $\mathrm{GL}_3(q)$, cf. [23, Table 3.5.A], we can assume, up to conjugacy, that $\tilde{L}_2 = L_2 \cap \mathrm{GL}_3(q)$.

Now $\tilde{L}_1 = L_1$ and $\tilde{L}_2 < L_2$ guarantee that γ_2 acts as inverse-transpose on $\tilde{L}_2$: If $\varepsilon = 1$, then γ_2 acts as the composition of the field automorphism $c \mapsto c^q$ and inverse-transpose on L_2, thus as inverse-transpose on $\tilde{L}_2 \leq \mathrm{GL}_3(q)$. If $\varepsilon = -1$, then γ_2 acts as the field automorphism $c \mapsto c^q$ on L_2; since $\tilde{L}_2 = \mathrm{SU}_3(q)$ preserves the Hermitian form 1_3, we have $1_3 = m^{\gamma_2} m^{\intercal}$ for all $m \in \tilde{L}_2$, that is, $m^{\gamma_2} = (m^{\intercal})^{-1}$. It remains to show that we can arrange $\tilde{x} = x$ and $\tilde{\gamma} = \gamma$. Recall that $x = x_1 : x_2$, each x_i acts as $\mathrm{diag}(1,1,\tau)$ on L_i, and $x_i^3 = \mathrm{diag}(\tau^{-1},\tau^{-1},\tau^2) \in L_i$. Note also that $\tilde{x} \in M_3$ is a 3-element not lying in $L_1 \circ_3 L_2$, thus $\tilde{x} = zx^j$ for some $z \in L_1 \circ_3 L_2$ and $j \in \{1,2\}$. If $j = 2$, then $z = \tilde{x}xx^{-3} \in L_1 \circ_3 L_2$ acts on each $\tilde{L}_i$ as $\mathrm{diag}(\tau,\tau,1)$, which is not possible. Hence $j = 1$ and z acts trivially on $\tilde{L}_1 \circ_3 \tilde{L}_2$; thus $z \in Z(L_1 \circ_3 L_2)$ has order 3 and, by replacing x by zx, we can suppose that $\tilde{x} = x$. Since $\tilde{\gamma}_1 \in M_3$ acts as inverse-transpose on $\tilde{L}_1$, we must have $\tilde{\gamma} = zx^j\gamma$ with $z = z_1 : z_2 \in L_1 \circ_3 L_2$ and $j \in \{0,1,2\}$. Since zx^j acts trivially on $\tilde{L}_1$, it follows that $j = 0$ and $z_1 \in Z(L_1)$. Now $\tilde{\gamma}_2 = z_2\gamma_2$ acts as inverse-transpose on $\tilde{L}_2$, hence $z_2 \in Z(L_1 \circ_3 L_2)$. Replacing γ by $z\gamma$ proves the assertion. □

4.3. Elementary abelian 3-subgroups. Let $T = (q-\varepsilon)^2 \times (q^2-1)^2$ be a maximal torus of $G = E_6^{-\varepsilon}(q)$ and define $E = \Omega_1(O_3(T)) = 3^4$. By [13], every maximal torus of G of type $(q-\varepsilon)^2 \times (q^2-1)^2$ is conjugate to T. Let $W = N_G(T)/T$, and let 3^a be the largest 3-power dividing $q-\varepsilon$.

Proposition 4.4. a) *If $\tilde{T} \leq G$ is a maximal torus with maximal order $|\Omega_1(O_3(\tilde{T}))|$, then $\tilde{T} =_G T$.*

b) *If $\tilde{E} = 3^f \leq G$ with $f \geq 4$, then $f = 4$ and $C_G(\tilde{E}) =_G T$.*

c) *$W = W(F_4)$ and $W(F_4) = 2_+^{1+4}.\Omega_4^+(2) = 2_+^{1+4}.(\mathrm{S}_3 \times \mathrm{S}_3)$.*

Proof. a) We may suppose that $\tilde{T} \leq L = C_G(3X)$ for some $X \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$. Thus $\tilde{T}_L = \tilde{T} \cap L = \tilde{T}$ is a maximal torus of L with maximal order $|\Omega_1(O_3(\tilde{T}_L))|$. Using the results of Section 4.1, a case distinction on $X \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$ and $\varepsilon \in \{\pm 1\}$ implies that $\tilde{T} = (q-\varepsilon)^2 \times (q^2-1)^2$, and, by uniqueness, $\tilde{T} =_G T$. For example, if $\varepsilon = 1$, then a maximal torus of $\mathrm{SU}_6(q)$ with largest 3-rank is $(q^2-1)^3/(q+1) = (q^2-1)^2 \times (q-1)$; if $\varepsilon = -1$, then a maximal torus of $\mathrm{SL}_6(q)$ with largest 3-rank is $(q^2-1) \times (q^2-1) \times (q+1)$; in both cases, if $\tilde{T} \leq C_G(3\mathrm{A})$, then indeed $\tilde{T} = (q-\varepsilon)^2 \times (q^2-1)^2$.

b) We can suppose that $\tilde{E} \leq C_G(3X)$ for some $X \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$; since $\tilde{E}$ must lie in some torus of $C_G(3X)$, it follows from part a) that $f = 4$. In particular, we can suppose that $\tilde{E} \leq \tilde{T} \leq C_G(\tilde{E})$ with $\tilde{T} =_G T$. Note that $C_G(\tilde{E}) = C_{C_G(3X)}(\tilde{E}) = \tilde{T}$, which proves the assertion.

c) The second assertion is known, see [5, Lemma 3.3]. The first assertion follows from [25, Proposition 25.3]; we give a direct proof here. For each $X \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$, the group $C_G(3X)$ contains a torus $\tilde{T} =_G T$, and $3X \leq Z(C_G(3X)) \leq C_G(\tilde{T}) = \tilde{T}$ shows that $T \cap 3X \neq \emptyset$. Suppose that $T \leq C_G(3X)$, hence $\mathrm{Out}_{N_G(3X)}(T)$ is isomorphic to $\langle\gamma_{\mathrm{A}}\rangle \times 2^3.\mathrm{S}_3$, $\langle\gamma_{\mathrm{B}}\rangle \times 2^3.\mathrm{S}_3$, or $\langle\gamma_{\mathrm{C}}\rangle \times \mathrm{S}_3 \times \mathrm{S}_3$, hence $2^7 \cdot 3^2$ divides $|W|$. Since each $1 \neq t \in E = \Omega_1(O_3(T))$ is of type 3A, 3B or 3C, it follows that $|\mathrm{Out}_{C_G(t)}(T)| \in \{2^4 \cdot 3, 2^2 \cdot 3^2\}$. Thus $W \leq \mathrm{Out}(E) = \mathrm{GL}_4(3)$ has order divisible by $2^7 \cdot 3^2$, has at least three orbits on the nontrivial elements of E, and each stabiliser of $t \in E \setminus \{1\}$ in W has order $2^4 \cdot 3$ or $2^2 \cdot 3^2$; note that $C_W(t) \cong \mathrm{Out}_{C_G(t)}(T)$. A direct computation shows that $\mathrm{GL}_4(3)$ has a unique such subgroup, namely, $W(F_4) = 2_+^{1+4}.\Omega_4^+(2)$; thus $W = W(F_4)$. □

By [5, Lemma 3.3], the group W is the unique index 2 subgroup of $2_+^{1+4}.\mathrm{SO}_4^+(2)$ with 25 conjugacy classes. Recall that $|G| = q^{36}(q^2-1)(q^5+\varepsilon)(q^6-1)(q^8-1)(q^9+\varepsilon)(q^{12}-1)$, thus a Sylow 3-subgroup of G has order 3^{4a+2}: Since $q^3-\varepsilon = (q-\varepsilon)^3 + 3\varepsilon(q-\varepsilon)^2 + 3(q-\varepsilon)$, the largest 3-power dividing $q^3-\varepsilon$ is 3^{a+1}; note that $3 \nmid q^i + \varepsilon$ for all i. Hence, $N_G(T) = T.W$ contains a Sylow 3-subgroup of G.

Lemma 4.5. *Let $H = F_4(q) \leq E_6^{-\varepsilon}(q) = G$. Up to conjugacy, the elementary abelian 3-subgroups of G are $E_1, \ldots, E_{15}$ as in Table V. If $E_i \leq H$, then $\mathrm{Out}_H(E) = \mathrm{Out}_G(E)$ with the local structure given in Table V.*

Proof. Let $E \leq G$ be an elementary abelian 3-group. Since H contains a Sylow 3-subgroup of G, we can assume that $E \leq H$. The elementary abelian 3-groups of H are classified (up to conjugacy) in [3, Table II] for q even, and in [5, Section 3] (in particular, in the proof of [5, Theorem 3.6]) for q odd; the group $3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14}$ is not explicitly mentioned in [5], but, in fact, the determination of the groups in [3, Table II] is independent of the parity of q. It follows that E is G-conjugate to one of the groups in Table V. Clearly, two different groups in the table are not G-conjugate; for the 3C-pure groups E this follows from the fact that all $\mathrm{Out}_G(E)$ are different, as shown below. The local structure (in H) of all groups is given in [3, Table II] and in the proof of [5, Theorem 3.6], depending on whether q is even or odd. The structure of $\mathrm{Out}_G(E)$ (and some centralisers) is given here; the structure of most $C_G(E)$ is determined in the subsequent sections.

Note that $\mathrm{C}_G(E)$ and $\mathrm{Out}_G(E)$ are known if E has order 3 or 3^4. For any elementary abelian E, we embed $\mathrm{Out}_H(E) \leq \mathrm{Out}_G(E)$. If E has order 3^f, then $\mathrm{Out}_G(E) \leq \mathrm{GL}_f(3)$. We now consider the different possibilities for E. Recall that we know $\mathrm{Out}_H(E)$ from Table V; it remains to prove that $\mathrm{Out}_H(E) = \mathrm{Out}_G(E)$.

• If $E = 3\mathrm{A}_6\mathrm{C}_2$, then $\mathrm{Out}_H(E) = \mathrm{S}_3.2 \leq \mathrm{Out}_G(E) \leq \mathrm{GL}_2(3)$. Up to conjugacy, $\mathrm{GL}_2(3)$ has a unique subgroup of order 12; this group is isomorphic to D_{12} and maximal in $\mathrm{GL}_2(3)$. Since E can be generated by an element of 3C and an element of 3A, which are non-conjugate, we must have $\mathrm{Out}_G(E) < \mathrm{GL}_2(3)$, thus $\mathrm{Out}_H(E) = \mathrm{Out}_G(E) = D_{12}$ follows.

• Since $E = 3\mathrm{A}_4\mathrm{B}_4$ is generated by elements in $E \cap 3\mathrm{A}$, we can consider $\mathrm{Out}_G(E) \leq \mathrm{S}_4$ (permuting the 4 elements in 3A), and also $\mathrm{Out}_G(E) \leq \mathrm{GL}_2(3)$; since $D_8 \leq \mathrm{Out}_G(E)$, this forces that $D_8 = \mathrm{Out}_G(E)$.

• If $E = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$, then we can write $E = \langle z_\mathrm{A}, s\rangle$ such that $N_G(E) = N_{N_G(\langle z_\mathrm{A}\rangle)}(E)$ and $N_G(E) = (q-\varepsilon).2 \circ_{2_\varepsilon} N_{\mathrm{SL}_6^{-\varepsilon}(q).2_\varepsilon}(\langle s\rangle)$. Clearly, $\mathrm{Out}_{\mathrm{SL}_6^{-\varepsilon}(q).2_\varepsilon}(\langle s\rangle)$ is cyclic, and $\mathrm{Out}_H(E) = 2\times 2$ implies $\mathrm{Out}_G(E) = \mathrm{Out}_H(E)$.

• If $E = (3\mathrm{C}^2)_1$, then $\mathrm{GL}_2(3) = \mathrm{Out}_H(E) \leq \mathrm{Out}_G(E) \leq \mathrm{GL}_2(3)$.

• Now let $E = (3\mathrm{C}^2)_2 = \langle z_\mathrm{C}, s\rangle$, hence $\mathrm{SL}_2(3) \leq \mathrm{Out}_G(E)$. It is shown in the proofs of [3, Proposition 3.9b)] and [5, Theorem 3.6, Case (3.2)], that $M = \mathrm{Out}_{N_H(\langle z_\mathrm{C}\rangle)}(E) = 3\times 2$, and we can consider $M \leq \mathrm{Out}_G(E)$. Suppose, for a contradiction, that $\mathrm{Out}_G(E) = \mathrm{GL}_2(3)$, so that there is $w \in N_G(E)$ with $z_\mathrm{C}^w = z_\mathrm{C}$ and $s^w = s^2$. In particular, $w \in M \leq \mathrm{GL}_2(3)$. A direct computation shows that $\mathrm{GL}_2(3)$ does not have a subgroup of order 6 containing a diagonal matrix $w = \mathrm{diag}(1,2)$; this proves that $\mathrm{Out}_G(E) = \mathrm{SL}_2(3)$. Since $\mathrm{Out}_G((3\mathrm{C}^2)_1) = \mathrm{GL}_2(3)$, the groups $(3\mathrm{C}^2)_1$ and $(3\mathrm{C}^2)_2$ are indeed not conjugate in G.

• We construct $E = 3\mathrm{B}_6\mathrm{C}_2$ as follows. Write $C_G(z_\mathrm{C}) = (L_1 \circ_3 L_2).3$, where $L_1 = \mathrm{SL}_3^\epsilon(q)$ and $L_2 = \mathrm{SL}_3(q^2)$, and choose $y \in \mathrm{SL}_3^\epsilon(q) = L_2 \cap H$ of order 3 such that $C_{\mathrm{SL}_3^\epsilon(q)}(y) = (q-\epsilon)^2$, thus $C_{L_2}(y) = (q^2-1)^2$ and $C_G(\langle z_\mathrm{C}, y\rangle) = ((q^2-1)^2 \circ_3 L_1).3$. It follows from Table V that $\langle z_\mathrm{C}, y\rangle = 3\mathrm{B}_6\mathrm{C}_2$, hence $C_G(E)$ is determined. The proof of $\mathrm{Out}_G(3\mathrm{B}_6\mathrm{C}_2) = D_{12}$ is similar to that of $\mathrm{Out}_G(3\mathrm{A}_6\mathrm{C}_2) = D_{12}$.

• Similarly, let $E = (3\mathrm{C}^3)_1$; it follows from the proof of [5, Theorem 3.6, Case (8.1)] and [3, Proposition 3.12] that $(3\mathrm{C}^2)_2 \not\leq_G E$, thus $(3\mathrm{C}^2)_1 \leq_G E$. Since $\mathrm{SL}_3(3) = \mathrm{Out}_H(E) \leq \mathrm{Out}_G(E)$, suppose, for a contradiction, that $\mathrm{Out}_G(E) = \mathrm{GL}_3(3)$. Then $-1_3 \in \mathrm{Out}_G(E)$, and there are two elements in $\mathrm{Out}_G((3\mathrm{C}^2)_1)$ acting as $\mathrm{diag}(-1,-1)$ on $(3\mathrm{C}^2)_1$, namely, $\mathrm{diag}(-1,-1,-1), \mathrm{diag}(-1,-1,1) \in \mathrm{Out}_G(E)$, a contradiction to $\mathrm{Out}_G((3\mathrm{C}^2)_1) = \mathrm{GL}_2(3)$. This proves $\mathrm{Out}_G(E) = \mathrm{SL}_3(3)$.

• Let $E = (3\mathrm{C}^3)_2$; by the proofs of [5, Theorem 3.6, Case (8.1)] and [3, Proposition 3.12a)], the group E contains both $(3\mathrm{C}^2)_1$ and $(3\mathrm{C}^2)_2$. We know that $3^2.\mathrm{SL}_2(3) \leq \mathrm{Out}_G(E) \leq \mathrm{GL}_3(3)$, and $\mathrm{Out}_G(E) \leq \mathrm{SL}_3(3)$ follows as in the previous case. The same arguments as in the proof of [3, Proposition 3.12a)] show that $\mathrm{Out}_G(E) = 3^2.\mathrm{SL}_2(3)$; these arguments are solely in $\mathrm{GL}_3(3)$.

• Let $E = 3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14}$. If q is even, then the proof of [3, Proposition 3.11, Case (2)] shows $\tilde{E} = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4 \le E$, and $E = \langle \tilde{E}, t\rangle$ for a certain $t \in \mathrm{GL}_2^{\varepsilon}(q) \times \mathrm{SL}_2(q) \le C_G(\tilde{E})$; in particular, since $C_H(E) = (q-\varepsilon)^4 \le C_G(E)$, it follows that $C_G(E) = T = (q-\varepsilon)^2 \times (q^2-1)^2$. If $w \in N_G(E)$, then w normalises $C_G(E) = T$, thus $N_G(E) \le N_G(T) = T.W$. The computation in the proof of [3, Corollary 3.13] shows that $N_G(E)/C_G(E) = N_W(E) = 2 \times \mathrm{S}_3 \times \mathrm{S}_3$. If q is odd, then the proof of [5, Theorem 3.6, Case (5)] shows that $C_H(\langle 3\mathrm{A}_6\mathrm{C}_2, z_o\rangle) = (q-\epsilon)^4$ for a certain $z_o \in H$; the group $\langle 3\mathrm{A}_6\mathrm{C}_2, z_o\rangle$ has order 3^3 and it is not of type $3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$ and $3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$ (as they have different centralisers), thus $\langle 3\mathrm{A}_6\mathrm{C}_2, z_o\rangle = 3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14}$. It follows from the definition of z_o that $E = 3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14} \le C_G(3\mathrm{A}_6\mathrm{C}_2) = ((q-\epsilon)^2 \circ_3 \mathrm{SL}_3(q^2)).3$ and $C_G(3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14}) = T = (q-\epsilon)^2 \times (q^2-1)^2$. Hence $N_G(E) \le N_G(T) = T.W$ and $\mathrm{Out}_G(E) = N_W(E)$. As for q even, $N_W(E) = 2\times\mathrm{S}_3\times\mathrm{S}_3$, so $\mathrm{Out}_G(E) = 2\times\mathrm{S}_3\times\mathrm{S}_3$.

• The groups $E = 3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$ and $E = 3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$ are considered in Lemmas 4.9 and 4.10. □

	E	$\mathrm{Out}_K(E)$	$C_H(E)$	$C_G(E)$
E_1	3A	2	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{Sp}_6(q).2_\varepsilon)$	$((q-\varepsilon)\circ_{2_\varepsilon}\mathrm{SL}_6^{-\varepsilon}(q)).2_\varepsilon$
E_2	3B	2	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{Spin}_7(q).2_\varepsilon)$	$((q^2-1)\circ_{2_\varepsilon}\mathrm{Spin}_8^-(q)).2_\varepsilon$
E_3	3C	2	$(\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{SL}_3^\varepsilon(q)).3$	$(\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{SL}_3(q^2)).3$
E_4	$3\mathrm{A}_4\mathrm{B}_4$	D_8	$(q-\varepsilon)^2\circ_{2_\varepsilon}(\mathrm{Sp}_4(q).2^*)$	$[((q^2-1)\times(q-\varepsilon))\circ_{4_{-\varepsilon}}\mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon}$
E_5	$3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$	2×2	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{GL}_2^\varepsilon(q)\times\mathrm{SL}_2(q)).2_\varepsilon$	$[((q^2-1)\times(q-\varepsilon))\circ_{(2_\varepsilon)^2}(\mathrm{SL}_2(q)\times\mathrm{SL}_2(q^2))].(2_\varepsilon)^2$
E_6	$3\mathrm{A}_6\mathrm{C}_2$	D_{12}	$((q-\varepsilon)^2\circ_3\mathrm{SL}_3^\varepsilon(q).3)$	$((q-\varepsilon)^2\circ_3\mathrm{SL}_3(q^2)).3$
E_7	$3\mathrm{B}_6\mathrm{C}_2$	D_{12}	$((q-\varepsilon)^2\circ_3\mathrm{SL}_3^\varepsilon(q).3)$	$((q^2-1)^2\circ_3\mathrm{SL}_3^\epsilon(q)).3$
E_8	$(3\mathrm{C}^2)_1$	$\mathrm{GL}_2(3)$	$(q-\varepsilon)^4.3$	$((q-\varepsilon)^2\times(q^2-1)^2).3$
E_9	$(3\mathrm{C}^2)_2$	$\mathrm{SL}_2(3)$	$(q^2+\varepsilon q+1)^2.3$	$((q^2+\varepsilon q+1)\times(q^4+q^2+1)).3$
E_{10}	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	$2^3.\mathrm{S}_3$	$(q-\varepsilon)^3\circ_{2_\varepsilon}(\mathrm{SL}_2(q).2_\varepsilon)$	$[((q^2-1)\times(q-\varepsilon)^2)\circ_{2_\varepsilon}\mathrm{SL}_2(q^2)].2_\varepsilon$
E_{11}	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	$2^3.\mathrm{S}_3$	$(q-\varepsilon)^3\circ_{2_\varepsilon}(\mathrm{SL}_2(q).2_\varepsilon)$	$[((q^2-1)^2\times(q-\varepsilon))\circ_{2_\varepsilon}\mathrm{SL}_2(q)].2_\varepsilon$
E_{12}	$3\mathrm{A}_6\mathrm{B}_6\mathrm{C}_{14}$	$2\times\mathrm{S}_3\times\mathrm{S}_3$	$(q-\varepsilon)^4$	$(q-\varepsilon)^2\times(q^2-1)^2$
E_{13}	$(3\mathrm{C}^3)_1$	$\mathrm{SL}_3(3)$	3^3	3^3
E_{14}	$(3\mathrm{C}^3)_2$	$3^2.\mathrm{SL}_2(3)$	3^3	3^3
E_{15}	$3\mathrm{A}_{24}\mathrm{B}_{24}\mathrm{C}_{32}$	W	$(q-\varepsilon)^4$	$(q-\varepsilon)^2\times(q^2-1)^2$

TABLE V. Elementary abelian 3-subgroups of $G = E_6^{-\varepsilon}(q)$ lying in $H = F_4(q) \le G$, with $K \in \{H, G\}$ and $n_\eta = \gcd(n, q-\eta)$

In the remainder of this section, we determine $C_G(E)$ for the groups E in Table V; this completes the proof of Lemma 4.5. We start with $E \le G$ elementary abelian of order 3^f with $z_{\mathrm{A}} \in E$. By Proposition 4.4, if $f \ge 4$, then $f = 4$ and $E = \Omega_1(O_3(T))$; if $f = 1$, then $E = 3\mathrm{A}$. Thus, suppose $f \in \{2,3\}$ in the following, and write $E = \langle z_A, s, t\rangle$ with $t = 1$ if $f = 2$. Clearly,

$$E \le C_G(z_{\mathrm{A}}) = (q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{SL}_6^{-\varepsilon}(q).2_\varepsilon),$$

and $s, t \in \mathrm{SL}_6^{-\varepsilon}(q)$. If $\varepsilon = -1$, then $3 \nmid q-1$, hence the minimal polynomial of $s \in \mathrm{SL}_6(q)$ has irreducible factors $f_1 = x-1$ and $f_2 = x^2+x+1$; note that every root of f_2 has order 3. Thus, the characteristic polynomial of s lies in $\{f_1^4 f_2, f_1^2 f_2^2, f_2^3\}$. If $\varepsilon = 1$, then $s \in \mathrm{SU}_6(q)$ and there are two elements of order 3 in $\mathrm{GF}(q^2)$, say ρ and ρ^2. Since $\rho^{-q} \ne \rho$, the characteristic polynomial of s can be factorised into powers of $f_1 = x-1$ and $f_3 = (x-\rho)(x-\rho^2)$, and thus lies in $\{f_1^4 f_3, f_1^2 f_3^2, f_3^3\}$. This shows that, independent of ε, there are three conjugacy classes of order 3 elements in $\mathrm{SL}_6^{-\varepsilon}(q)$ with representatives $t_1, t_2, t_3 \in \mathrm{SL}_6^{-\varepsilon}(q)$ such that, via base change (possibly over $\mathrm{GF}(q^2)$), we have

$$\begin{aligned} t_1 &= \mathrm{diag}(1,1,1,1,\omega,\omega^{-1}),\\ t_2 &= \mathrm{diag}(1,1,\omega,\omega^{-1},\omega,\omega^{-1}),\ \text{and} \end{aligned}$$

$$t_3 \quad = \quad \mathrm{diag}(\omega, \omega^{-1}, \omega, \omega^{-1}, \omega, \omega^{-1}),$$

where $\omega \in \mathrm{GF}(q^2)$ has order 3, cf. [15, p. 112] . It is also shown in [15, Proposition (1A)] that

$$(4.1) \qquad C_{\mathrm{GL}_6^{-\varepsilon}(q)}(t_i) \quad = \quad \begin{cases} \mathrm{GL}_1(q^2) \times \mathrm{GL}_4^{-\varepsilon}(q) & \text{if } i = 1, \\ \mathrm{GL}_2(q^2) \times \mathrm{GL}_2^{-\varepsilon}(q) & \text{if } i = 2, \\ \mathrm{GL}_3(q^2) & \text{if } i = 3. \end{cases}$$

We need the following lemma; for a set M of matrices write $\det(M) = \{\det(m) \mid m \in M\}$.

Lemma 4.6. *If $i \in \{1, 2\}$ and $\mathrm{GL}_i(q^2) \le C_{\mathrm{GL}_6^{-\varepsilon}(q)}(t_i)$ as in (4.1), then $\det(\mathrm{GL}_i(q^2)) = \det(\mathrm{GL}_6^{-\varepsilon}(q))$.*

Proof. It is sufficient to consider the case $i = 1$, because $\mathrm{GL}_1(q^2)$ can be considered as a subgroup of $\mathrm{GL}_2(q^2)$.

First, let $\varepsilon = -1$, and let $z = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ so that $C_{\mathrm{GL}_2(q)}(z) = q^2 - 1$. Note that $\mathrm{diag}(z, 1, 1, 1, 1)$ and $\mathrm{diag}(z, z, 1, 1)$ are conjugate in $\mathrm{GL}_6(q)$ to t_1 and t_2, respectively. The elements of $K = C_{\mathrm{GL}_2(q)}(z)$ are $g = \left(\begin{smallmatrix} a & b \\ -b & a+b \end{smallmatrix}\right)$, with $\det(g) = a^2 + ab + b^2$. If $a \in \mathrm{GF}(q)$ is a primitive element and $b = 0$, then $\det(g) = a^2$; hence $\det(K) = \mathrm{GF}(q)^\times$ if q is even. Now let q be odd and write $b = ac$ with $c \in \mathrm{GF}(q)$, thus $\det(g) = a^2(1 + c + c^2)$. Note that $1 + c + c^2 = d^2$ with $d \in \mathrm{GF}(q)$ has at most two solutions for c and, since $1 + x + x^2$ is irreducible over $\mathrm{GF}(q)$, it follows that $M = \{1 + c + c^2 \mid c \in \mathrm{GF}(q)\}$ has cardinality at least $(q + 1)/2$, and $0 \notin M$. On the other hand, $N = \{d^2 \mid d \in \mathrm{GF}(q)\}$ has cardinality $(q + 1)/2$, and $0 \in N$. This implies that there is $c \in \mathrm{GF}(q)$ such that $a^2(1 + c + c^2)$ is not a square in $\mathrm{GF}(q)$; this proves the lemma for $\varepsilon = -1$.

Now suppose $\varepsilon = 1$, and let $\rho, \rho^2 \in \mathrm{GF}(q^2)$ be the two elements of order 3, thus $z = \mathrm{diag}(\rho, \rho^2)$, and $\mathrm{diag}(z, 1, 1, 1, 1)$ and $\mathrm{diag}(z, z, 1, 1)$ are conjugate in $\mathrm{GU}_6(q)$ to t_1 and t_2, respectively. We suppose the hermitian form preserved by $\mathrm{GU}_2(q)$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now $K = C_{\mathrm{GU}_2(q)}(z)$ consists of matrices of the form $g = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a = b^{-q}$, thus $\det(g) = b^{-(q-1)}$. In particular, if $b \in \mathrm{GF}(q^2)$ has order $q^2 - 1$, then $\det(g)$ has order $q + 1$. Since every $h \in \mathrm{GU}_6(q)$ has a determinant of order dividing $q + 1$, the assertion follows. □

Together with (4.1), this leads to the following lemma; afterwards we consider $E = \langle z_{\mathrm{A}}, s, t\rangle$ with $t = 1$.

Lemma 4.7. *We have*

$$(4.2) \qquad C_{\mathrm{SL}_6^{-\varepsilon}(q)}(t_i) \quad = \quad \begin{cases} [(q-\varepsilon) \times (q+\varepsilon) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon} & \textit{if } i = 1, \\ [(q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2) \times (q+\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_2(q)].(2_\varepsilon)^2 & \textit{if } i = 2, \\ ((q-\varepsilon) \circ_3 \mathrm{SL}_3(q^2)).3 & \textit{if } i = 3. \end{cases}$$

Lemma 4.8. *If $E = \langle z_A, s\rangle$, and s is conjugate to t_i in $\mathrm{SL}_6^{-\varepsilon}(q)$, then*

$$C = C_G(E) = C_{C_G(z_{\mathrm{A}})}(s) = (q-\varepsilon) \circ_{2_\varepsilon} (C_{\mathrm{SL}_6^{-\varepsilon}(q)}(s).2_\varepsilon),$$

that is,

$$(4.3) \qquad C \quad = \quad \begin{cases} [((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon} & \textit{if } i = 1, \\ [((q^2-1) \times (q-\varepsilon)) \circ_{(2_\varepsilon)^2} (\mathrm{SL}_2(q) \times \mathrm{SL}_2(q^2))].(2_\varepsilon)^2 & \textit{if } i = 2, \\ ((q-\varepsilon)^2 \circ_3 \mathrm{SL}_3(q^2)).3 & \textit{if } i = 3. \end{cases}$$

Depending on whether $i = 1, 2, 3$, we have $E = 3\mathrm{A}_4\mathrm{B}_4$, $E = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$, and $E = 3\mathrm{A}_6\mathrm{C}_2$, respectively. The local structure is given in Table V.

Proof. Recall that $C_G(z_{\mathrm{A}}) = ((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q)).x_{\mathrm{A}} = (q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_6^{-\varepsilon}(q).2_\varepsilon)$ where $x_{\mathrm{A}} = 1 : 2_\varepsilon$; if 2_ε is nontrivial, then it acts by conjugation by $\mathrm{diag}(1, 1, 1, 1, 1, \lambda)$ on $\mathrm{SL}_6^{-\varepsilon}(q)$.

If $i = 1$, then there is a conjugate of s' of s in $\mathrm{SL}_6^{-\varepsilon}(q)$ such that x_{A} commutes with $C_{\mathrm{SL}_6^{-\varepsilon}(q)}(s')$; thus, there is a conjugate of x_{A} which commutes with $C_{\mathrm{SL}_6^{-\varepsilon}(q)}(s)$. It follows that

$$C_G(E) = C_{C_G(z_{\mathrm{A}})}(s) = (q-\varepsilon) \circ_{2_\varepsilon} [((q-\varepsilon) \times (q+\varepsilon) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)).4_{-\varepsilon}].2_\varepsilon.$$

Similarly, if $i=2$, then

$$C_G(E) = C_{C_G(z_{\mathrm{A}})}(s) = (q-\varepsilon)\circ_{2_\varepsilon} [((q-\varepsilon)\circ_{2_\varepsilon} \mathrm{SL}_2(q^2)\times (q+\varepsilon)\circ_{2_\varepsilon} \mathrm{SL}_2(q)).(2_\varepsilon)^2].2_\varepsilon.$$

Now suppose s is conjugate to t_3. If q is even, then $2_\varepsilon = 1$, and

$$C_G(E) = (q-\varepsilon)\times [((q-\varepsilon)\circ_3 \mathrm{SL}_3(q^2)).3] = ((q-\varepsilon)^2\circ_3 \mathrm{SL}_3(q^2)).3.$$

Let q be odd in the remainder of this paragraph. If $\varepsilon=1$, then there is a conjugate of s which is diagonal, hence commuting with x_{A}; this shows that if $\varepsilon = 1$, then a conjugate of x_{A} lies in $C_{C_G(z_{\mathrm{A}})}(s)$. Now let $\varepsilon=-1$. We can assume that $t_3 \leq T_1 \leq T_2$, where $T_1 = (q+1)\times(q^2-1)^2 \leq C_{\mathrm{SL}_6(q)}(t_3)$ and $T_2 = (q^2-1)^3 \leq C_{\mathrm{GL}_6(q)}(t_3)$ are maximal tori. Note that there exists a 2-element $e \in T_2 \setminus T_1 Z(\mathrm{GL}_6(q))$ with $e^2\in T_1$ which induces a nontrivial outer-diagonal automorphism $\bar{e}$ on $\mathrm{SL}_6(q)$, thus $C_{\mathrm{SL}_6(q).e}(t_3) = C_{\mathrm{SL}_6(q)}(t_3).2$. By [19, Theorem 2.5.12], the group of outer diagonal automorphisms of $\mathrm{SL}_6(q)$ is of order $\gcd(6,q-1)=2$; recall that $3\mid q+1$. This proves that also the action of x_{A} centralises t_3, and

$$C_G(E) = (q-\varepsilon)\circ_{2_\varepsilon} [((q-\varepsilon)\circ_3 \mathrm{SL}_3(q^2)).3].2_\varepsilon.$$

The structure of E follows from $C_H(E)\leq C_G(E)$ and the structure of $C_H(E)$ given in Table V, where $H = F_4(q)\leq G$. Thus, if $i=1$, then $E = 3\mathrm{A}_4\mathrm{B}_4$ with $C_H(E) = (q-\varepsilon)^2\times \mathrm{Sp}_4(q)\leq C_G(E)$. Recall that $4_{-\varepsilon}$ and 2_ε act as outer-diagonal automorphisms on $\mathrm{SL}_4^{-\varepsilon}(q)$, and $4_{-\varepsilon}$ has a unique subgroup acting as 2_ε on $\mathrm{SL}_4^{-\varepsilon}(q)$, hence there is $u\in 4_{-\varepsilon}$ and a generator $v\in 2_\varepsilon$ such that uv has order 2_ε and acts trivially on $\mathrm{SL}_4^{-\varepsilon}(q)$; this yields

$$\begin{aligned} C_G(E) &= (q-\varepsilon)\circ_{2_\varepsilon} [((q-\varepsilon)\times(q+\varepsilon)\circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)).4_{-\varepsilon}].2_\varepsilon \\ &= (([(q-\varepsilon)\circ_{2_\varepsilon}((q-\varepsilon)\times(q+\varepsilon))].2_\varepsilon)\circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)).4_{-\varepsilon}. \end{aligned}$$

Let $U = [(q-\varepsilon)\circ_{2_\varepsilon}((q-\varepsilon)\times(q+\varepsilon))].2_\varepsilon \leq C_G(E)$ and choose a maximal torus $V = (q-\varepsilon)\times(q^2-1)\leq \mathrm{SL}_4^{-\varepsilon}(q)$ such that $(U\circ_{4_{-\varepsilon}} V).4_{-\varepsilon}\leq C_G(E)$ is self-centralising and abelian, hence isomorphic to the maximal torus $(q^2-1)^2\times(q-\varepsilon)^2$; in particular, $U = (q-\varepsilon)\times(q^2-1)$. In the same way, if $i=2$, then

$$\begin{aligned} C_G(E) &= (q-\varepsilon)\circ_{2_\varepsilon}[((q-\varepsilon)\circ_{2_\varepsilon}\mathrm{SL}_2(q^2)\times(q+\varepsilon)\circ_{2_\varepsilon}\mathrm{SL}_2(q)).(2_\varepsilon)^2].2_\varepsilon \\ &= [(([(q-\varepsilon)\circ_{2_\varepsilon}((q-\varepsilon)\times(q+\varepsilon))].2_\varepsilon)\circ_{(2_\varepsilon)^2}(\mathrm{SL}_2(q)\times\mathrm{SL}_2(q^2))].(2_\varepsilon)^2 \\ &= ((q^2-1)\times(q-\varepsilon))\circ_{(2_\varepsilon)^2}(\mathrm{SL}_2(q)\times\mathrm{SL}_2(q^2)).(2_\varepsilon)^2. \end{aligned}$$

Analogously, if $i=3$, then there is $u\in((q-\varepsilon)\circ_3\mathrm{SL}_3(q^2)).3\leq C_G(E)$ acting as 2_ε, hence

$$\begin{aligned} C_G(E) &= (q-\varepsilon)\circ_{2_\varepsilon}[((q-\varepsilon)\circ_3\mathrm{SL}_3(q^2)).3].2_\varepsilon \\ &= ([((q-\varepsilon)\circ_{2_\varepsilon}(q-\varepsilon)).2_\varepsilon]\circ_3\mathrm{SL}_3(q^2)).3 \\ &= ((q-\varepsilon)^2\circ_3\mathrm{SL}_3(q^2)).3. \end{aligned}$$

□

Lemma 4.9. *If* $E = 3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$, *then* $C_G(E) = [((q^2-1)^2\times(q-\varepsilon))\circ_{2_\varepsilon}\mathrm{SL}_2(q)].2_\varepsilon$ *and* $\mathrm{Out}_G(E) = 2^3.\mathrm{S}_3$.

Proof. By the proofs of [3, Proposition 3.11, Cases (1),(1')] and [5, Theorem 3.6, Case (4)], we have $\tilde{E} = 3\mathrm{A}_4\mathrm{B}_4\leq E$, and $E=\langle\tilde{E},t\rangle$ with $t\in\mathrm{Sp}_4(q)\leq\mathrm{SL}_4^{-\varepsilon}(q) = O^{r'}(C_G(\tilde{E}))$ of type 3A; in particular, t is conjugate in $\mathrm{Sp}_4(q)$ to $\mathrm{diag}(\omega,\omega^{-1},1,1)$. Analogously to (4.2), we obtain $C_{\mathrm{SL}_4^{-\varepsilon}(q)}(t) = ([(q-\varepsilon)\times(q+\varepsilon)]\circ_{2_\varepsilon}\mathrm{SL}_2(q)).2_\varepsilon$, and $C_G(E)\leq C_G(\tilde{E}) = (((q^2-1)\times(q-\varepsilon))\circ_{4_{-\varepsilon}}\mathrm{SL}_4^{-\varepsilon}(q)).4_{-\varepsilon}$ yields

$$C_G(E) = (((q^2-1)\times(q-\varepsilon))\circ_{4_{-\varepsilon}}(((q-\varepsilon)\times(q+\varepsilon)\circ_{2_\varepsilon}\mathrm{SL}_2(q)).2_\varepsilon)).4_{-\varepsilon}.$$

An argument as before shows that the abelian and self-centralising subgroup

$$(((q^2-1)\times(q-\varepsilon))\circ_{4_{-\varepsilon}}(((q-\varepsilon)\times(q+\varepsilon)\circ_{2_\varepsilon}(q-\varepsilon)).2_\varepsilon)).4_{-\varepsilon}\leq C_G(E)$$

is isomorphic to $(q^2-1)^2\times(q-\varepsilon)^2$; this proves that $C_G(E)$ has the structure as in the lemma.

Note that $K = O^{r'}(C_G(E)) = \mathrm{SL}_2(q)$, and there is $y \in K \cap 3\mathrm{A}$ with $E \leq C_G(y)$. Note that $\mathrm{SL}_2(q)$ has a unique class of order 3 elements, and we can assume that y is defined over the prime field $\mathrm{GF}(r)$ of $\mathrm{GF}(q)$, so that it is centralised by field automorphisms of $\mathrm{SL}_2(q)$. It follows that $E = \Omega_1(O_3(\tilde{T}))$ where

$$E \leq \tilde{T} = (q-\varepsilon) \times (q^2-1)^2 \leq M \quad \text{and} \quad M = \mathrm{SL}_6^{-\varepsilon}(q) = O^{r'}(C_G(y))$$

with $\mathrm{Out}_M(\tilde{T}) = 2^3.\mathrm{S}_3$ (cf. [9]). In particular, $C_M(E)$ is a Levi subgroup of M, and it follows that $\tilde{T} = C_M(E)$, $N_M(E) \leq N_M(\tilde{T})$, and one can deduce that

$$N_G(E) \cap C_G(y) = \tilde{T}.2^3.\mathrm{S}_3 \times (q-\varepsilon),$$

where $(q-\varepsilon) \leq K$. We can assume that $E \leq H = F_4(q) \leq G$, which proves that $\mathrm{Out}_H(E) = 2^3.\mathrm{S}_3 \leq \mathrm{Out}_G(E)$; we show that $\mathrm{Out}_G(E) = \mathrm{Out}_H(E)$. Recall that K and $\tilde{T}$ lie in $C_G(E)$. If $w \in N_G(E) \setminus C_G(E)$, then w acts on K. If w acts as an inner automorphism on K, then there is $w' \in K$ such that $\tilde{w} = ww' \in N_G(E)$ acts trivially on K, hence $\tilde{w} \in N_G(E) \cap C_G(y)$, and $\tilde{w}C_G(E) = wC_G(E) \in (N_G(E) \cap C_G(y))C_G(E)/C_G(E) = (2^3.\mathrm{S}_3)C_G(E)/C_G(E) \leq \mathrm{Out}_G(E)$. If w acts on K as an outer automorphism, then it acts as a field automorphism, thus $w \in C_G(y)$. Again, $w \in N_G(E) \cap C_G(y)$, and $wC_G(E) \in (N_G(E) \cap C_G(y))C_G(E)/C_G(E) = (2^3.\mathrm{S}_3)C_G(E)/C_G(E) \leq \mathrm{Out}_G(E)$. In conclusion, we have proved that $\mathrm{Out}_G(E) = 2^3.\mathrm{S}_3$. □

Lemma 4.10. *If $E = 3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$, then $C_G(E) = (((q^2-1) \times (q-\varepsilon)^2) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)).2_\varepsilon$ and $\mathrm{Out}_G(E) = 2^3.\mathrm{S}_3$.*

Proof. By the proofs of [3, Proposition 3.11, Cases (1),(1')] and [5, Theorem 3.6, Case (4)], we have $\tilde{E} = 3\mathrm{A}_4\mathrm{B}_4 \leq E$, and $E = \langle \tilde{E}, t\rangle$ with $t \in \mathrm{Sp}_4(q) \leq \mathrm{SL}_4^{-\varepsilon}(q) = O^{r'}(C_G(\tilde{E}))$ of type 3B; in particular, t is conjugate in $\mathrm{Sp}_4(q)$ to $\mathrm{diag}(\omega, \omega^{-1}, \omega, \omega^{-1})$. Analogously to (4.2), we obtain $C_{\mathrm{SL}_4^{-\varepsilon}(q)}(t) = ((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)).2_\varepsilon$ and $C_G(E) \leq C_G(\tilde{E}) = (((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)).4_{-\varepsilon}$ yields

$$C_G(E) = (((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} (((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)).2_\varepsilon)).4_{-\varepsilon}.$$

Now we proceed as in Lemma 4.9: First, it follows that $C_G(E)$ has the form as in the lemma. Let $K = O^{r'}(C_G(E)) = \mathrm{SL}_2(q^2)$ and choose $y \in K$ of order 3; note that K has a unique class of order 3 elements, thus we can assume that y is fixed under field automorphisms of K. By construction, $y \in 3\mathrm{B}$; recall that $\mathrm{SL}_2(q^2) \leq \mathrm{SL}_4^{-\varepsilon}(q)$. Let $M = \mathrm{Spin}_8^-(q) = O^{r'}(C_G(y))$ and $\tilde{T} = (q^2-1) \times (q-\varepsilon)^2 \leq M$ such that $E = \Omega_1(O_3(\tilde{T}))$. We have $\mathrm{Out}_M(\tilde{T}) = 2^3.\mathrm{S}_3$ (cf. [9]) and $\tilde{T} = C_M(E)$, so $N_M(E) \leq N_M(\tilde{T})$, and one can deduce that

$$N_G(E) \cap C_G(y) = \tilde{T}.2^3.\mathrm{S}_3 \times (q^2-1)$$

where $(q^2-1) \leq K$. The proof continues as in Lemma 4.9. □

Lemma 4.11. *If $E = (3\mathrm{C}^2)_1$, then $C_G(E) = T.3$ where $T = (q-\varepsilon)^2 \times (q^2-1)^2$.*

Proof. First, let $q \geq 4$, so we can choose $H = F_4(q) \leq G$ as in Lemma 4.3; suppose $E \leq C_H(z_\mathrm{C}) = (\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)).x_\mathrm{C}$. The proofs of [3, Proposition 3.9a)] and [5, Theorem 3.6, Case (3.1)] show that we can assume $E = \langle z_\mathrm{C}, s\rangle$ where $s = \mathrm{diag}(1, \omega, \omega^2) : \mathrm{diag}(1, \omega, \omega^2) \in \mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q) \leq C_G(z_\mathrm{C})$. As shown in [3,5],

$$C_H(E) = \langle (q-\varepsilon)^2 \circ_3 (q-\varepsilon)^2, x_\mathrm{C}\rangle.y = (q-\varepsilon)^4.3$$

for some $y \in N_H(T) = T.W$. More precisely, if $T = (T_1 \circ_3 T_2).x_\mathrm{C}$ where $T_1 = (q-\varepsilon)^2 \leq \mathrm{SL}_3^\varepsilon(q)$ and $T_2 = (q^2-1)^2 \leq \mathrm{SL}_3(q^2)$ consist of diagonal matrices, then we can choose $y = v : v^{-1} \in W$ where each v acts on T_i by permuting the diagonal entries according to the permutation $(1,2,3)$. A direct computation now shows that $C_G(E) = \langle (q-\varepsilon)^2 \circ_3 (q^2-1)^2, x_\mathrm{C}\rangle.y = T.3$. For $q = 2$ the same follows from a direct computation. □

Lemma 4.12. *If $E = (3\mathrm{C}^2)_2$, then $C_G(E) = ((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3$.*

Proof. For $q = 2$ this follows by a direct computation; now let $q \geq 4$ and let $H = F_4(q) \leq G$ as in Lemma 4.3, so we can suppose $E \leq C_H(z_\mathrm{C}) = (\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)).x_\mathrm{C}$. Recall that $x_\mathrm{C} = x_1 : x_2$, and 3^a is the maximal 3-power dividing $q-\varepsilon$. By the proofs of [3, Proposition 3.9a)] and [5, Theorem 3.6, Case (3.1)], we can assume $E = \langle z_\mathrm{C}, s\rangle$ where $s = s_1x_1 : s_2x_2 \in (\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)).x_\mathrm{C} \leq C_G(z_\mathrm{C})$, and each s_ix_i has order 3^{a+1}, acting as an element in a torus of $\mathrm{GL}_3^\varepsilon(q)$ of type $q^3-\varepsilon$. We can suppose this torus has diagonal form; let v be the permutation matrix which permutes these corresponding basis elements as a 3-cycle. In particular, $C_{\mathrm{SL}_3^\varepsilon(q)}(s_i) = q^2+\varepsilon q+1$, and

$\tilde{T} = \langle (q^2+\varepsilon q+1)\circ_3(q^2+\varepsilon q+1), s\rangle = (q^2+\varepsilon q+1)^2$ is a maximal torus of H; finally, $C_H(E) = \tilde{T}.u$ with $u = v{:}v^{-1}$. A similar computation shows that $\hat{T} = \langle (q^2+\varepsilon q+1)\circ_3(q^4+q^2+1), s\rangle = (q^2+\varepsilon q+1)\times(q^4+q^2+1) \leq C_G(E)$ is a maximal torus of G, cf. [13]. Since $C_H(E) \leq C_G(E)$, the assertion follows. □

Lemma 4.13. *If $E = (3\mathrm{C}^3)_1$ or $E = (3\mathrm{C}^3)_2$, then $C_G(E) = E$.*

Proof. First let $E = (3\mathrm{C}^3)_2$. Recall that $(3\mathrm{C}^2)_1, (3\mathrm{C}^2)_2 \leq E$ and $E = C_H(E) \leq C_G(E)$, cf. the proof of Lemma 4.5, hence $C_G(E) \leq C_G((3\mathrm{C}^2)_1) \cap C_G((3\mathrm{C}^2)_2)$, which proves the assertion.

Now let $E = (3\mathrm{C}^3)_1 \leq C_G(z_\mathrm{C}) = (\mathrm{SL}_3^\varepsilon(q)\circ_3 \mathrm{SL}_3(q^2)).3$, and let $T = ((q-\varepsilon)^2\circ_3(q^2-1)^2).3 \leq C_G(z_\mathrm{C})$ as usual. By the construction in [3, Proposition 3.12] and [5, Theorem 3.6, Case (8.1)], we can define $\tilde{E} = \langle z_\mathrm{C}, z_1, z_2\rangle =_G (3\mathrm{C}^3)_2$ for some $z_1 \in T$ and $z_2 \in T.3\backslash T$ such that $F_1 = \langle z_\mathrm{C}, z_1\rangle =_G (3\mathrm{C}^2)_1$ and $F_2 = \langle z_\mathrm{C}, z_2\rangle =_G (3\mathrm{C}^2)_2$; moreover $E = (3\mathrm{C}^3)_1 = \langle z_\mathrm{C}, z_1, tz_2\rangle$ for some $t \in T$. The actions of tz_2 and z_2 on T coincide, thus

$$C_T(tz_2) = C_T(F_2) \leq C_G(F_2) = ((q^2+\epsilon q+1)\times(q^4+q^2+1)).3,$$

and hence $C_T(tz_2) = C_G(F_2)\cap T$. Since E is abelian, we have $F_1 \leq C_T(tz_2)$, and $C_T(tz_2) = F_1$ follows. Write $C = C_G(E)$; clearly, $C \leq C_G(F_1) = T.3$, and it follows that $C/(C\cap T) = CT/T \leq T.3/T = 3$, thus $|C| \leq 3|C\cap T|$. But $C\cap T = C_T(tz_2) = F_1$, so $C = E$. □

4.3.1. *Controlling 3-fusion.* An interesting consequence of Lemma 4.5 is that the subgroup $H = F_4(q)$ of $G = E_6^\varepsilon(q)$ controls 3-fusion: Suppose $z_\mathrm{C} \in H \leq G$, and fix a Sylow 3-subgroup $D \leq H$ of G with $z_\mathrm{C} \in D$, see Section 4.2. Let $A, B \leq D$ be elementary abelian, and let $f_g\colon A \to B$ be a group homomorphism defined by conjugation by $g \in G$, that is, $f_g(a) = a^g$ for all $a \in A$. By the proof of Lemma 4.5, there exists $h \in H$ with $f_g(A) = A^g = A^h$, hence $g = nh$ for some $n \in N_G(A)$. By Table V, we have $\mathrm{Out}_G(A) = \mathrm{Out}_H(A)$, thus there exists $m \in N_H(A)$ such that $k = mh \in H$ satisfies $a^g = a^k$ for all $a \in A$. Hence, $f_g = f_k$, where $f_k\colon A \to B$, $a \mapsto a^k$. For $K \in \{G, H\}$ write

$$\mathrm{Hom}_K(A,B) = \{f\colon A\to B \mid \exists k \in K\ \forall a \in A\colon f(a) = a^k\},$$

thus we have shown that $\mathrm{Hom}_G(A,B) = \mathrm{Hom}_H(A,B)$ for all elementary abelian $A, B \leq D$. It is proved in [6, Theorem B] that this holds for all $A, B \leq D$; the next corollary proves Theorem C.

Corollary 4.14. *We use the previous notation. If $A, B \leq D$, then $\mathrm{Hom}_H(A,B) = \mathrm{Hom}_G(A,B)$. Moreover, if $1 \neq A \leq D$ and $A^g \leq D$ with $g \in G$, then $g = ch$ for some $c \in C_G(A)$ and $h \in H$.*

Proof. In the notation of [6], let $\mathcal{F}$ and $\mathcal{G}$ be the Frobenius categories of H and G, respectively, on D, thus $\mathrm{Hom}_\mathcal{F}(A,B) = \mathrm{Hom}_H(A,B)$ and $\mathrm{Hom}_\mathcal{G}(A,B) = \mathrm{Hom}_G(A,B)$. As shown above, $\mathrm{Hom}_H(A,B) = \mathrm{Hom}_G(A,B)$ for all elementary abelian $A, B \leq D$, and [6, Theorem B] proves that $\mathcal{F} = \mathcal{G}$. This implies the assertion. □

4.4. Maximal 3-local subgroups. Using the results of the previous sections, it is straightforward to classify the maximal 3-local subgroups of $G = E_6^{-\varepsilon}(q)$ with $3 \mid q-\varepsilon$.

Theorem 4.15. *Up to conjugacy, the maximal 3-local subgroups of $G = E_6^{-\varepsilon}(q)$ with $3 \mid q-\varepsilon$ are $M_1, \ldots, M_9$ as given in Table VI; the group M_7 is only defined if $q > 2$.*

Proof. Throughout this proof, let E_i be defined as in Table V. Recall that every maximal 3-local $M \leq G$ is conjugate to some $N_G(E_i)$. By Lemma 4.2, the group $M_i = N_G(E_i)$ with $i \in \{1,2,3\}$ is maximal 3-local; it is easy to see that $M_9 = N_G(E_{15}) = T.W$ is maximal 3-local. Note that if $i \in \{5,6,7\}$, then $N_G(E_i) \leq N_G(E_j)$ for some $j \in \{1,2,3\}$. Note that $C_G(E_{12}) = T$, hence $N_G(E_{12}) \leq T.W = N_G(E_{15})$. It remains to consider $N_G(E_i)$ with $i \in \{4,8,9,10,11,13,14\}$.

Recall that $M_4 = N_G(E_4) = C_G(E_4).D_8$ and $D_8 \leq \mathrm{GL}_2(3)$ acts irreducibly; thus $N_G(E_4) \not\leq N_G(E_i)$ for $i \in \{1,2\}$, and $N_G(E_4)$ is maximal 3-local by Lemma 2.3. Note that $T \trianglelefteq N_G(E_8)$, thus $N_G(E_8) \leq N_G(E_{15}) = T.W$. We have $E_9 = O_3(N_G(E_9))$, hence $E_9 \in \mathcal{ER}_3(G)$, and $M_5 = N_G(E_9)$ is maximal 3-local by Lemma 2.3; recall that $N_G(E_9) = C_G(E_9).\mathrm{SL}_2(3)$, and $\mathrm{SL}_2(3)$ acts irreducibly.

If $i \in \{10, 11\}$, then $N_G(E_i) = C_G(E_i).2^3.\mathrm{S}_3$, and $2^3.\mathrm{S}_3 \le \mathrm{GL}_3(3)$. A computation shows that, up to conjugacy, there is a unique $2^3.\mathrm{S}_3 \le \mathrm{GL}_3(3)$, and this group is irreducible with $O_3(2^3.\mathrm{S}_3) = 1$. By Lemma 2.3, this shows that $M_6 = N_G(E_{10})$ is maximal 3-local. If $q = 2$, then $O_3(N_G(E_{11})) =_G T$, thus $N_G(E_{11}) \le_G N_G(E_{15})$; if $q > 2$, then $O_3(N_G(E_{11})) = E_{11}$, and $M_7 = N_G(E_{11})$ is maximal 3-local by Lemma 2.3.

The assertion on E_{13} and E_{14} follows as in the proof of [2, Proposition 3.12]: Firstly, $E_{13} = O_3(N_G(E_{13}))$, and $N_G(E_{13}) = C_G(E_{13}).\mathrm{SL}_3(3)$ with $\mathrm{SL}_3(3)$ acting transitively on $E_{13} \setminus \{1\}$; thus, $M_8 = N_G(E_{13})$ is maximal 3-local by Lemma 2.3. Secondly, $N_G(E_{14}) = C_G(E_{14}).3^2.\mathrm{SL}_2(3)$, and, as shown in [3, 5], the group $3^2.\mathrm{SL}_2(3)$ stabilises a subgroup of $Y < E_{14}$ of order 9. Note that $|N_G(E_{14})| = 2^3 \cdot 3^6$ and $|N_G((3\mathrm{C}^2)_2)|_3 = 3^4$, thus $N_G(E_{14}) \le N_G(Y)$ implies $Y = (3\mathrm{C}^2)_1$; in particular, $N_G(E_{14}) < N_G(Y) =_G N_G(E_8) \le_G N_G(E_{15})$. □

	$N_G(E)$	E	$C_G(E)$	condition
M_1	$((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q)).2_\varepsilon.2$	3A	$((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q)).2_\varepsilon$	–
M_2	$((q^2-1) \circ_{2_\varepsilon} \mathrm{Spin}_8^-(q)).2_\varepsilon.2$	3B	$((q^2-1) \circ_{2_\varepsilon} \mathrm{Spin}_8^-(q)).2_\varepsilon$	–
M_3	$(\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3(q^2)).3.2$	3C	$(\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3(q^2)).3$	–
M_4	$[((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon}.D_8$	$3\mathrm{A}_4\mathrm{B}_4$	$[((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon}$	–
M_5	$((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3.\mathrm{SL}_2(3)$	$(3\mathrm{C}^2)_2$	$((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3$	–
M_6	$[((q^2-1) \times (q-\varepsilon)^2) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)].2_\varepsilon.2^3.\mathrm{S}_3$	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	$[((q^2-1) \times (q-\varepsilon)^2) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)].2_\varepsilon$	–
M_7	$[((q^2-1)^2 \times (q-\varepsilon)) \circ_{2_\varepsilon} \mathrm{SL}_2(q)].2_\varepsilon.2^3.\mathrm{S}_3$	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	$[((q^2-1)^2 \times (q-\varepsilon)) \circ_{2_\varepsilon} \mathrm{SL}_2(q)].2_\varepsilon$	$q > 2$
M_8	$3^3.\mathrm{SL}_3(3)$	$(3\mathrm{C}^3)_1$	3^3	–
M_9	$((q-\varepsilon)^2 \times (q^2-1)^2).W$	$3\mathrm{A}_{24}\mathrm{B}_{24}\mathrm{C}_{32}$	$(q-\varepsilon)^2 \times (q^2-1)^2$	–

Table VI. Maximal 3-local subgroups of $G = E_6^{-\varepsilon}(q)$ with $3 \mid q - \varepsilon$

We end this section with an easy observation, cf. Theorem 4.15 and Proposition 4.4.

Corollary 4.16. *If $T = (q^2-1)^2 \times (q-\varepsilon)^2 \le G$ is a maximal torus, then $N_G(T) = N_G(O_3(T)) = N_G(\Omega_1(O_3(T)))$ and $C_G(T) = C_G(O_3(T)) = C_G(\Omega_1(O_3(T)))$.*

5. Radical 3-subgroups

The aim of this section is to classify the radical 3-subgroups of $G = E_6^{-\varepsilon}(q)$ with $3 \mid q - \varepsilon$. Our strategy is to consider $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_i}(R)$, where M_i is a maximal 3-local subgroup as in Table VI. Due to Lemma 4.5 and Theorem 4.15, our approach is similar to [3, 5], with some necessary technical modifications, see also Remark 5.5 below. Recall that 3^a is the largest 3-power dividing $q - \varepsilon$, and fix a maximal torus

$$T = (q^2-1)^2 \times (q-\varepsilon)^2 \le G.$$

5.1. Radical subgroups with $N_G(R) = N_{M_1}(R)$ or $N_G(R) = N_{M_2}(R)$. The following lemma is analogous to [5, Lemma 4.1] and [3, Lemma 4.2].

Lemma 5.1. *If $R \in \mathcal{R}_3(G)$ with $N_G(R) \le_G M_i$ for some $i \in \{1, 2\}$, then $R =_G O_3(M_i)$, or $R =_G O_3(N_G(3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4)) = 3^a \times 3^a$ with $q > 2$, or $N_G(R) \le_G M_j$ for some $j \ge 3$, hence $R \in \mathcal{R}_3(M_j)$.*

Proof. Suppose $N_G(R) \le M_i$ and $R \ne O_3(M_i)$; recall that $M_i = (K_1 \circ_{2_\varepsilon} K_2).2_\varepsilon.2$ with $K_1 \in \{q-\varepsilon, q^2-1\}$ and $K_2 \in \{\mathrm{SL}_6^{-\varepsilon}(q), \mathrm{Spin}_8^-(q)\}$, hence $R = R_1 \times R_2$ with $R_1 = 3^a \le K_1$ and $1 \ne R_2 \in \mathcal{R}_3(K_2)$. Let $Y = \Omega_1(Z(R))$, which is elementary abelian of rank $t \ge 2$; write $Y = \langle s_1, \ldots, s_t \rangle$ with $s_1 \in R_1$ and $s_2, \ldots, s_t \in R_2$. Since $K_2 \le C_G(s_1)$, we have $s_1 \in 3\mathrm{A} \cup 3\mathrm{B}$; in particular Y is not 3C-pure and $N_G(R) \le N_G(Y)$.

If $t = 2$, then $Y \in_G \{E_4, \ldots, E_7\}$ as in Table V, hence $N_G(R) \le N_G(Y) \le_G M$ for some $M \in \{M_3, M_4, N_G(3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4)\}$. We consider $Y = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$ and deduce from Table V that $R = O_3(N_G(R)) =$

$O_3(N_{N_G(Y)}(R)) = O_3(N_{C_G(Y)}(R))$; hence R is radical in $C_G(Y)$, and $O_3(C_G(Y)) \le R$ by Lemma 2.2. If $q = 2$, then $O_3(C_G(Y)) = 3 \times 3 \times 3 \le Z(R)$, a contradiction to $t = 2$; thus $q > 2$, in which case $O_3(C_G(Y)) = 3^a \times 3^a$. It follows that $R = O_3(C_G(Y)) = 3^a \times 3^a$ and $q > 2$. If $t = 3$, then $Y \in_G \{E_{10}, E_{11}, E_{12}\}$; in any case, $N_G(R) \le N_G(Y) \le M_j$ with $j \in \{6, 7, 9\}$. If $t \ge 4$, then $t = 4$ and $C_G(Y) =_G T$ by Proposition 4.4; thus $N_G(R) \le N_G(Y) \le N_G(T) = M_9$.

Note that $O = O_3(N_G(E))$ with $E = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$ is indeed radical in G if and only if $q > 2$: we have just seen that $O \le C_G(E)$. If $q > 2$, then $E = \Omega_1(O)$, $N_G(O) \le N_G(E)$, and $O \in \mathcal{R}_3(G)$ follows from $O_3(N_G(O)) = O_3(N_{N_G(E)}(O)) = O_3(N_G(E)) = O$. If $q = 2$, then $O = 3^3$ is one of E_{10}, E_{11}, E_{12} as in Table V. If $i \in \{11, 12\}$, then $O < O_3(N_G(E_i))$, hence O is not radical. If $O = E_{10}$, then $N_G(E) \le N_G(O)$ means $N_G(3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4) \le_G N_G(3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8)$, which is impossible by an order argument, see Table V. □

5.2. Radical 3-subgroups of M_3. We determine, up to conjugacy, the radical 3-subgroups of

$$M = M_3 = (\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3(q^2)).x.\gamma$$

where $x = x_\mathrm{C} = x_1 : x_2$ and $\gamma = \gamma_\mathrm{C} = \gamma_1 : \gamma_2$. Recall that γ_1 acts as inverse-transpose on $\mathrm{SL}_3^\varepsilon(q)$, and γ_2 acts on $\mathrm{SL}_3(q^2)$ as the field automorphism $c \mapsto c^q$ if $\varepsilon = -1$, and as inverse-transpose composed with the field automorphism $c \mapsto c^q$ if $\varepsilon = 1$. Moreover, x_i acts as $o_i = \mathrm{diag}(1, 1, \tau)$ where $\tau \in \mathrm{GF}(q^2)$ has maximal 3-power order 3^a; recall that $\omega = \tau^{3^{a-1}}$. We define $L_1 = \mathrm{SL}_3^\varepsilon(q)$ and $L_2 = \mathrm{SL}_3(q^2)$, and $J_i = L_j.x_j.\gamma_j$, so that

$$L = L_1 \circ_3 L_2 \le M \le J_1 \circ_3 J_2 = J.$$

Note that if $Q \in \mathcal{R}_3(L)$, then $Q = Q_1 \circ_3 Q_2$ and each $Q_i \in \mathcal{R}_3(L_i)$; conversely, if $Q_i \in \mathcal{R}_3(L_i)$ for $i = 1, 2$, then $Q_1 \circ_3 Q_2 \in \mathcal{R}_3(L)$. The next lemma is [3, Corollary 4.3] (proved in [5, Lemma 2.2]) and holds for all q.

Lemma 5.2. *If $Q \in \mathcal{R}_3(L)$, then there is $R \in \mathcal{R}_3(M)$ with $Q = R \cap L$, and the following holds.*

a) *If $Q \notin \mathcal{R}_3(M)$ or $R = Q$, then $R = O_3(N_M(Q))$.*

b) *If $Q \in \mathcal{R}_3(M)$ and $R \ne Q$, then $R/Q = 3$ and $R = \langle Q, w\rangle$ for some $w \in N_M(Q) \setminus N_L(Q)$.*

Conversely, every $R \in \mathcal{R}_3(M)$ has this form for some $Q \in \mathcal{R}_3(L)$.

The next lemma merges [3, Lemma 4.4] and [5, Lemma 3.1(2)].

Lemma 5.3. *Let $T_1 = (q - \varepsilon)^2 \le L_1 = \mathrm{SL}_3^\varepsilon(q)$ be a maximal torus, $D_1 \le L_1$ a Sylow 3-subgroup, and $Z_1 = Z(L_1) = 3$. Let $F_1 = 3^a = O_3(\mathrm{GL}_2^\varepsilon(q))$ for some $\mathrm{GL}_2^\varepsilon(q) \le L_1$. Denote by $\mathcal{R}_3(L_1)/L_1$ a set of L_1-conjugacy classes of radical 3-subgroups of L_1.*

a) *If $q = 2$, then $D_1 = 3_+^{1+2}$ and $\mathcal{R}_3(L_1)/L_1 = \{D_1\}$ with $C_{L_1}(D_1) = Z_1$ and $N_{L_1}(D_1) = L_1$, $\mathrm{Out}_{L_1}(D_1) = Q_8$.*

b) *If $q = 4$, then $D_1 = 3_+^{1+2}$ and $\mathcal{R}_3(L_1)/L_1 = \{Z_1,\ D_1\}$ with $C_{L_1}(D_1) = Z_1$ and $N_{L_1}(D_1) = D_1.Q_8$.*

c) *If $q \ge 5$ and $a = 1$, then $D_1 = 3_+^{1+2}$ and $\mathcal{R}_3(L_1)/L_1 = \{Z_1,\ O_3(T_1),\ D_1\}$ with $C_{L_1}(D_1) = Z_1$, $N_{L_1}(D_1) = D_1.Q_8$, and $C_{L_1}(O_3(T_1)) = T_1$, $N_{L_1}(O_3(T_1)) = T_1.\mathrm{S}_3$.*

d) *If $q \ge 5$ and $a \ge 2$, then $D_1 = (3^a \times 3^a).3$ and*

$$\mathcal{R}_3(L_1)/L_1 = \begin{cases} \{Z_1,\ F_1,\ K_{1,1},\ K_{1,2},\ K_{1,3},\ D_1\} & \text{if } q - \varepsilon = 3^a, \\ \{Z_1,\ F_1,\ K_{1,1},\ K_{1,2},\ K_{1,3},\ O_3(T_1),\ D_1\} & \text{if } q - \varepsilon \ne 3^a, \end{cases}$$

where each $K_{1,j} \cong 3_+^{1+2}$ satisfies $C_{L_1}(K_{1,j}) = 3$ and $N_{L_1}(K_{1,j}) = K_{1,j}.\mathrm{SL}_2(3)$; every 3-element in $J_1 \setminus L_1$ acts fixed-point freely on $\{K_{1,1}, K_{1,2}, K_{1,3}\}$, and $\mathrm{Out}_{J_1}(K_{1,1}) = \mathrm{SL}_2(3).2$ and $\mathrm{Out}_{J_1}(K_{1,j}) = \mathrm{SL}_2(3)$ if $j \ne 1$. In particular, there is $u \in N_{J_1}(K_{1,1})$ with $(K_{1,2})^u = K_{1,3}$. As before, $C_{L_1}(O_3(T_1)) = T_1$ and $N_{L_1}(O_3(T_1)) = T_1.\mathrm{S}_3$. The Sylow subgroup satisfies $C_{L_1}(D_1) = Z_1$ and $N_{L_1}(D_1) = D_1.2$. We have $F_1 = 3^a = O_3(\mathrm{GL}_2^\varepsilon(q))$, and we can assume that F_1 is generated by an element $\mathrm{diag}(s, s, t)$ with $s \ne t$, so that $\mathrm{GL}_2^\varepsilon(q)$ is centralised by x_1. In addition, $C_{L_1}(F_1) = \mathrm{GL}_2^\varepsilon(q) = N_{L_1}(F_1)$.

In all cases, we can assume that $D_1 = \langle O_3(T_1), v_1\rangle$ where $O_3(T_1)$ consists of diagonal matrices and v_1 is the permutation matrix corresponding to $(1,2,3)$ with respect to this basis. Thus, D_1 is normalised by x_1 and γ_1; note that $v_1^{x_1} = \mathrm{diag}(1,\tau,\tau^{-1})v_1$ and $v_1^{\gamma_1} = v_1$.

The same results holds for $L_2 = \mathrm{SL}_3(q^2)$ if one replaces q and ε by q^2 and 1, respectively; $Z_2, D_2, T_2, F_2, K_{2,1}$, $K_{2,2}, K_{2,3}$ and the element v_2 are defined analogously. We have $\Omega_1(O_3(T_1)) = 3\mathrm{A}_6\mathrm{C}_2$ and $\Omega_1(O_3(T_2)) = 3\mathrm{B}_6\mathrm{C}_2$.

An important consequence is the following corollary.

Corollary 5.4. *Let $A = \mathrm{SL}_3^\varepsilon(q) \le \mathrm{SL}_3(q^2) = B$ such that $A \le \mathrm{GL}_3^\varepsilon(q)$. If $q \ge 5$, then there exists a set of B-conjugacy representatives $\mathcal{R}_3(B)/B$ which is also a set of A-conjugacy representatives $\mathcal{R}_3(A)/A$.*

Proof. Since A contains a Sylow 3-subgroup of B, we can choose $\mathcal{R}_3(B)/B$ as a subset of A. Note that 3^a is the largest 3-power dividing $q-\varepsilon$ and q^2-1, and let $\mathcal{R}_3(B)/B$ be given as in Lemma 5.3. The assertion follows if each $R \in \mathcal{R}_3(B)/B$ is radical in A. Again, we use Lemma 5.3: if $R = Z(B)$, then $R = Z(A)$ is radical; if $R = D \le A$ is a Sylow 3-subgroup of B, then R is a Sylow 3-subgroup of A, hence radical in A. If $R = O_3(S) \le A$ for a maximal torus $S = (q^2-1)^2 \le B$, then $R = O_3(\hat S)$ for a maximal torus $\hat S = (q-\varepsilon)^2 \le S$ of A, thus R is radical in A. Let $R = O_3(P) \le B$ with $P = \mathrm{GL}_2(q^2) \le B$. Note that R lies in a maximal torus $(q^2)^2 - 1$ of P, thus in a maximal torus $U = (q^2)^2-1$ of B. Clearly, $O_3(U) = O_3(P) = 3^a$, which is unique up to conjugacy in B since $U \le B$ is. In particular, we can assume $R = O_3(\mathrm{GL}_2^\varepsilon(q))$ where $\mathrm{GL}_2^\varepsilon(q) \le P \cap A$, and R is radical in A by Lemma 5.3; note that, as just shown, $O_3(\mathrm{GL}_2^\varepsilon(q)) \le A$ is unique up to conjugacy in A. Similarly, the assertion of the lemma follows if we show that A has three conjugacy classes of subgroups isomorphic to 3_+^{1+2}. We establish this in the remainder of this proof.

The group generated by $\mathrm{diag}(1,\omega,\omega^2)$ and the permutation matrix defined by $(1,2,3)$ is isomorphic to 3_+^{1+2} and lies in A. Now let $K \le A$ with $K = \langle s,t\rangle \cong 3_+^{1+2}$ and $s^t = sz$ for some $1 \ne z \in Z(K)$. Note that, up to conjugacy, $s = \mathrm{diag}(1,\omega,\omega^2)$. Using that $z \in C_A(s)$ is diagonal, $z \notin \{s, s^{-1}\}$, and $|z| = |t| = 3$, a direct computation starting from $zt = tz$ and $st = tsz$ shows that (modulo replacing t by t^2 if necessary)

$$z = \mathrm{diag}(\omega,\omega,\omega) \quad \text{and} \quad t = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & j \\ (ij)^{-1} & 0 & 0 \end{pmatrix}$$

for some i, j in the underlying field of A. Thus, there are $(q-\varepsilon)^2$ possibilities for t. The centraliser of such a t in $C_A(s) = (q-\varepsilon)^2$ has order 3, generated by $\mathrm{diag}(\omega,\omega,\omega)$. This proves that, up to conjugacy in $C_G(A)$, there are three possibilities for t. This implies that A has three classes of 3_+^{1+2}, and proves the lemma. ☐

In the notation of Lemma 5.2, write $w \in N_M(Q) \setminus N_L(Q)$ as $w = w_1 : w_2$ with $w_i \in J_i \setminus L_i$. Note that

$$\begin{aligned} C_{L_1}(w_1) &\in \{q^2+\varepsilon q+1, (q-\varepsilon)^2, \mathrm{GL}_2^\varepsilon(q)\}, \\ C_{L_2}(w_2) &\in \{q^4+q^2+1, (q^2-1)^2, \mathrm{GL}_2(q^2)\}, \end{aligned}$$

and $|q^4+q^2+1|_3 = |q^2+\varepsilon q+1|_3 = 3$. As in [5, Lemma 4.2] and [3, Proposition 4.5], we consider each $Q_i \in \mathcal{R}_3(L_i)$ and $C_{L_i}(w_i)$, and determine the possibilities for $R \in \mathcal{R}_3(M)$ with $R \cap L = Q_1 \circ_3 Q_2 \in \mathcal{R}_3(L)$.

Remark 5.5. Let $H = F_4(q) \le G$ and $z_{\mathrm{C}} \in H$; by Lemma 4.3, we can assume that $\tilde M_3 = N_H(z_{\mathrm{C}}) = (L_1 \circ \tilde L_2).x.\gamma$ with $\tilde L_2 = \mathrm{SL}_3^\varepsilon(q) < L_2$. Since $\tilde M_3$ contains a Sylow 3-subgroup of M_3, and H controls 3-fusion in G, it is tempting to determine $\mathcal{R}_3(M_3)$ from the groups $\mathcal{R}_3(\tilde M_3)$ given in [3,5]. However, we did not find a *short* argument, and eventually it seems to us that it is more efficient to mimic the classification of $\mathcal{R}_3(\tilde M_3)$. (In particular, since any correspondence between $\mathcal{R}_3(\tilde M_3)$ and $\mathcal{R}_3(M_3)$ would still require to determine the new local structure in G.) Due to Corollary 5.4, our proof follows closely the lines of [3]. In many cases, the arguments are exactly the same (with the obvious modifications due to replacing $\tilde L_2$ by L_2); we therefore often refer to the proof of [3, Proposition 4.5] instead of copying it verbatim. Note that Corollary 4.14b) implies that $N_G(R) = C_G(R)N_H(R) = C_G(R).\mathrm{Out}_H(R)$ for every 3-subgroup $R \le H$.

Proposition 5.6. *We use the previous notation. Up to conjugacy, the radical 3-group of M_3 are $R_1, \ldots, R_{45}$ as in Table VII. The groups $R_1, \ldots, R_{45}$ are ordered with respect to their occurrence in the proof; moreover, the rows of Table VII are partitioned according to the 8 cases in the proof. The column "char" displays $\Omega_1(Z(R_i))$*

or $\Omega_1(Z([R_i,R_i]))$. The last column (labelled "conditions") lists the conditions under which the corresponding group is defined; if the conditions in row i are not satisfied, then the group R_i is not defined. The groups $R_1,\ldots,R_{45}$ are all radical in G, and pairwise non-conjugate in G.

Proof. Let $M = M_3$ and write $Q \in \mathcal{R}_3(L)$ as $Q = Q_1 \circ_3 Q_2$ with $Q_i \in \mathcal{R}_3(L_i)$ for $i \in \{1,2\}$. By the above comments, every $R \in \mathcal{R}_3(M)$ satisfies $Q = R \cap L$ for some $Q \in \mathcal{R}_3(L)$. By Lemma 5.2, there are two possibilities. Firstly, $R = O_3(N_M(Q))$, and either $R = Q$ or $Q \notin \mathcal{R}_3(M)$. Secondly, $R = \langle Q, w\rangle$ for some $w \in N_M(Q) \setminus N_L(Q)$, hence $R/Q = 3$, $Q = O_3(N_M(Q))$, and $w = w_1 : w_2$ with $w_i \in J_i \setminus L_i$. Since w is a 3-element, we can assume that $w = ex$ for some $e \in L$; recall $x = x_{\mathrm{C}}$. Note that if $R \in \mathcal{R}_3(M)$, then $C_M(R) = C_G(R)$ since $C_G(R) \le C_G(z_{\mathrm{C}}) \le M$. We now make a case distinction on the possibilities for Q_i; we refer to [3, Proposition 4.5] for details of the proof, cf. Remark 5.5.

Case (1) Suppose both $Q_i = Z(L_i) = 3$, so $q \ge 4$ and $Q = Z(L) = 3$.

Clearly, $R = R_1 = Q = O_3(M) \in \mathcal{R}_3(M)$. The proof of [3, Proposition 4.5, Case (1)] shows that the following is possible. Firstly, $R = R_2 = (3\mathrm{C}^2)_2$ with $N_M(R) = C_M(R).(3 \times 2)$, cf. the proof of Lemma 4.5. Secondly, $R = R_3 = 3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$, and $N_M(R) = C_G(R).\gamma$ follows from $N_M(\langle w\rangle)/C_M(w) \le 2$. Thirdly, $R = R_4 = 9$ or $R = R_5 = 9$ is possible, depending on whether $a = 1$ or $a \ge 2$.

Case (2) Suppose both $Q_i = D_i$, thus $Q = D_1 \circ_3 D_2$ with $C_M(Q) = 3$. The argument in the proof of [3, Theorem 4.8, Case (2)] proves that either $R = R_6 = Q.x$, or $a = 1$ and $R = R_7 = Q$; in both cases, $N_G(R) \le N_G(Z(Q)) = M$, hence $N_G(R) = N_M(R)$.

Case (3) Suppose $Q_1 = Z(L_1) = 3$, so $q \ge 4$, and $Q_2 = D_2$, so that $Q = 3 \circ_3 D_2$ and $C_M(Q) = L_1 \circ_3 3$.

Note that $\mathrm{Out}_M(Q) = \mathrm{Out}_{L_2}(D_2).\mathrm{S}_3$, thus $Q = O_3(N_M(Q))$, and $Q = R_8$ or $Q = R_9$, depending on whether $a = 1$ or $a \ge 2$. We might also have $R = \langle Q, w\rangle$ for some $w \in N_M(Q) \setminus N_L(Q)$. The argument in the proof of [3, Theorem 4.8, Case (3)] proves that $R = R_{10} = D_2.3$, or $a = 1$ and $R = R_{11} = D_2.3$.

Similarly, if $Q_1 = D_1$ and $Q_2 = Z(L_2) = 3$, then we get $R_{12},\ldots,R_{15}$; there is no restriction on q.

Case (4) Suppose both $Q_i = O_3(T_i) = 3^a \times 3^a$, hence $q \ge 5$ and $q - \varepsilon \ne 3^a$. Recall that $T = (T_1 \circ_3 T_2).x$, and it follows that $N_M(Q) = T.(\mathrm{S}_3 \times \mathrm{S}_3).\gamma$, thus $R = R_{16} = O_3(N_M(Q)) = O_3(T)$.

Case (5) Suppose $Q_1 = O_3(T_1) = 3^a \times 3^a$ and $Q_2 = D_2$, so $q \ge 5$ and $q - \varepsilon \ne 3^a$. The proof of [3, Theorem 4.8, Case (5)] shows that either $a \ge 2$ and $R = R_{17} = Q.3$, or $a = 1$ and $R = R_{17} = Q.3$ or $R = R_{18} = Q$. Similarly, if $Q_2 = O_3(T_2) = 3^a \times 3^a$ and $Q_1 = D_1$, then $q \ge 4$, and either $a \ge 2$ and $R = R_{19} = Q.3$, or $a = 1$ and $R = R_{19} = Q.3$ or $R = R_{20} = Q$.

Case (6) Suppose $Q_1 = O_3(T_1) = 3^a \times 3^a$ and $Q_2 = Z(L_2) = 3$, so $q \ge 5$ and $q - \varepsilon \ne 3^a$.

The proof of [3, Proposition 4.5] shows that the following is possible. First, $\Omega_1(Q) = 3\mathrm{A}_6\mathrm{C}_2$, hence $R = R_{21} = Q$. We might also have $R = \langle Q, w\rangle$, which yields $R = R_{22} = 3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$ and $a = 1$, or $R = R_{23} = Q.3 = 3^a \times 3^{a+1}$. (By the construction in [3], the group $R = R_{23}$ is abelian and $C_M(R) = (T_1 \circ_3 (q^4+q^2+1)).3$, hence $R = 3^a \times 3^{a+1}$ or $R = (3^a)^2 \times 3$; the latter is not possible since $(q^4+q^2+1) \not\le C_G(3^3)$, cf. Table V.)

If $Q_1 = Z(L_1) = 3$ and $Q_2 = O_3(T_2)$, then $q \ge 4$; we get the groups R_{24}, R_{25}, and R_{26}.

Case (7) Suppose $Q_1 = F_1 = 3^a$, hence $q \ge 5$ and $a \ge 2$. Running over all possibilities for Q_2, the proof of [3, Proposition 4.5, Case (7)] shows that the following is possible. If $Q_2 = Z(L) = 3$, then $R = R_{27} = Q = O_3(N_M(Q))$, or $R = R_{28} = Q.3 = 3^{a+1}$. If $Q_2 = F_2 = 3^a$, then $R = R_{29} = O_3(N_M(Q)) = Q.x$. If $Q_2 = K_{2,i} = 3_+^{1+2}$, then $R = R_{30} = Q$. If $Q_2 = O_2(T_2) = 3^a \times 3^a$, then $R = R_{31} = Q.3$. If $Q_2 = D_2$, then $R = R_{32} = Q.x$.

If $Q_2 = F_2 = 3^a$, then an analogous argument yields $R_{33},\ldots,R_{37}$.

Case (8) Suppose $Q_1 = K_{1,i} = 3_+^{1+2}$, hence $q \ge 5$ and $a \ge 2$; we may suppose $i = 1$ since every 3-element of $J_1 \setminus L_1$ acts transitively on $\{K_{1,1}, K_{1,2}, K_{1,3}\}$. Running over all possibilities for Q_2, the proof of [3, Proposition 4.5, Case (8)] shows that the following is possible. If $Q_2 = Z(L) = 3$, then $R = R_{38} = Q$. If $Q_2 = K_{2,1} = 3_+^{1+2}$, then $R = R_{39} = Q = 3_+^{1+4}$. If $Q_2 = K_{2,i} = 3_+^{1+2}$ with $i \ne 1$, then we can assume $i = 2$, and $R = R_{40} = 3_+^{1+4}$. If $Q_2 = O_3(T_2)$, then $R = R_{41} = Q$. If $Q_2 = D_2$, then $R = R_{42} = Q$.

If $Q_2 = K_{2,1}$, then we get the groups $R_{43},\ldots,R_{45}$; note that $K_{1,1} \circ_3 K_{2,2}$ and $K_{1,2} \circ_3 K_{2,1}$ are conjugate.

Note that if $N_G(R_i) = N_M(R_i)$, then R_i is radical in G; for the groups with $N_G(R_i) \neq N_M(R_i)$ it can be deduced from the normaliser structure given in Table VII that $R_i \in \mathcal{R}_3(G)$. Now consider $R_i, R_j \in \mathcal{R}_3(M)$ such that $R_i \cong R_j$ and both groups have isomorphic local structure. If $R_i^g = R_j$ for some $g \in G$, then it follows from Table VII that g normalises $\langle z_{\mathrm{C}} \rangle \leq R_i \cap R_j$. This implies $g \in M$, thus $R_i =_M R_j$ and $i = j$. □

5.3. The remaining subgroups. We now complete the classification of the 3-radical subgroups of G.

Theorem 5.7. *Up to conjugacy, the radical 3-subgroups of* $G = E_6^{-\varepsilon}(q)$ *with* $3 \mid q - \varepsilon$ *are the groups* $R_1, \ldots, R_{45}$ *in Table VII and the groups* $R_{46}, \ldots, R_{55}$ *in Table VIII. As before, the column labelled "char" displays* $\Omega_1(Z(R_i))$ *or* $\Omega_1(Z([R_i, R_i]))$*. The last column (labelled "conditions") lists the conditions under which the corresponding group is defined; if the conditions in row* i *are not satisfied, then the group* R_i *is not defined.*

Proof. We can suppose that each $R \in \mathcal{R}_3(G)$ lies in some maximal 3-local M_i as defined in Table VI. By Lemma 5.1, we can assume that either R is radical in M_3, hence given in Table VII, or $R \in \mathcal{R}_3(M_i)$ with $i \geq 4$, or $R = R_{46} = O_3(M_1)$, $R = R_{47} = O_3(M_2)$, or $R = O_3(N_G(3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4)) = 3^a \times 3^a$. In the latter case, $\Omega_1(R) =_G \Omega_1(R_j)$ with $j \in \{3, 29\}$, depending on whether $a = 1$ or $a \geq 2$, thus $R \leq_G N_G(R_j)$ and $R =_G R_j$.

In the following, we consider $R \in \mathcal{R}_3(G)$ with $R \in \mathcal{R}_3(M_i)$ and $N_G(R) = N_{M_i}(R)$ for some $i \geq 4$. We determine these groups in the remainder of this proof.

Case (M_4) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_4}(R)$, where $M_4 = [((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon}.D_8$. First, we have $R_{48} = O_4 = O_3(M_4) = 3^a \times 3^a$. Now suppose $O_4 < R$. Then we must have $R = O_4 \times P$ for some $P \leq \mathrm{SL}_4^{-\varepsilon}(q)$, and $N_G(R) \leq N_G(\Omega_1(Z(R)))$. Note that $\Omega_1(Z(R)) = 3^3$ or $\Omega_1(Z(R)) = 3^4$, cf. Proposition 4.4. Thus, $N_G(R) \leq M_j$ for some $j \geq 6$; we ignore this case here and consider M_j with $j \geq 6$ in detail below.

Case (M_5) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_5}(R)$, where $M_5 = ((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3.\mathrm{SL}_2(3)$. Let $O_5 = O_3(M_5) = (3\mathrm{C}^2)_2$. First, we have $R = R_{49} = O_5$. If $O_5 < R$, then $C_G(R) < C_G(O_5)$, and either $R =_G (3\mathrm{C}^3)_2$ with $N_G(R) = R.3^2.\mathrm{SL}_2(3)$, a contradiction to $N_G(R) \leq M_5$, or $R = \langle O_5, u \rangle$ for some 3-element $u \in M_5 \setminus C_G(O_5)$. In this case, R is non-abelian of order 27; there are only 5 isomorphism types of groups of order 27, and $Z(R) = 3$ follows since R is non-abelian. Since $u \notin C_G(O_5)$, we must have $Z(R) =_G 3\mathrm{C}$, thus $N_G(R) \leq_G M_3$ and we can ignore this case.

Case (M_6) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_6}(R)$, where $M_6 = [((q^2-1) \times (q-\varepsilon)^2) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)].2_\varepsilon.2^3.\mathrm{S}_3$. Let $O_6 = O_3(M_6) = (3^a)^3$ and note that $\Omega_1(O_6) = 3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$ since $M_6 = N_G(\Omega_1(O_6))$. If $q \geq 5$ and $a = 1$, then $O_6 = 3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8 =_G R_{22}$. If $q \geq 5$ and $q - \varepsilon \neq 3^a$, then $O_6 =_G R_{36}$; note that $N_G(R_{36}) =_G M_6$. If $q \in \{2, 4\}$, or $q \geq 5$ with $a \geq 2$ and $q - \varepsilon = 3^a$, then $R = R_{50} = O_6$.

Now suppose $O_6 < R$. If $R = U_6$ is a Sylow 3-subgroup of M_6, then $|N_G(R)|_3 > |N_{M_6}(R)|_3$ by p-group theory, hence $N_G(R) \neq N_{M_6}(R)$; note that U_6 is not a Sylow 3-subgroup of G. Hence, we consider $O_6 < R < U_6$. If R contains a nontrivial element $u \in \mathrm{SL}_2(q^2) = O^{r'}(C_G(O_6))$, then $R = O_3(N_{M_6}(R))$ and $R < U_6$ imply $R = (3^a)^4$; recall that S_3 acts trivially on $\mathrm{SL}_2(q^2)$, see the proof of Lemma 4.10. Thus $\Omega_1(R) = 3^4 = \Omega_1(O_3(T))$ by Proposition 4.4, and $N_G(R) \leq N_G(\Omega_1(O_3(T))) = N_G(T) = M_9$. We ignore this case since we consider M_9 in detail. It remains to consider $R = \langle O_6, u \rangle = (3^a)^3.u$ with $u \in M_6 \setminus C_G(O_6)$. Let $Q_1 = (q^2-1) \times (q-\varepsilon)^2 \leq M_6$ and $Q_2 = (q^2-1) \leq O^{r'}(C_G(O_6))$ such that $\tilde{T} = C_{M_6}(Q_1 \circ_{2_\varepsilon} Q_2) =_G T$. Thus, up to conjugacy, we can suppose $T = (T_1 \circ_3 T_2).x_{\mathrm{C}} \leq M_3 \cap M_6$ such that $O_6 \leq T$ and $u \in N_G(T) \setminus T$; recall $u \in N_G(Q_1) \cap C_G(Q_2)$. Up to conjugacy, W has three subgroups of order 3, namely, $H_1 = \langle 1 : v_2 \rangle$, $H_2 = \langle v_1 : 1 \rangle$, and $H_3 = \langle v_1 : v_2^{-1} \rangle$, where each v_i acts on T_i by permuting the diagonal entries cyclically. Since u commutes with $Q_2 = (q^2-1) \leq T_2$, we can assume that $\langle u \rangle = H_2$, hence $O_3(Z(R)) = O_3(T_2) = 3\mathrm{B}_6\mathrm{C}_2$. Thus $N_G(R) \leq M_3$, and we ignore this case.

Case (M_7) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_7}(R)$, where $M_7 = [((q^2-1)^2 \times (q-\varepsilon)) \circ_{2_\varepsilon} \mathrm{SL}_2(q)].2_\varepsilon.2^3.\mathrm{S}_3$; recall that $q \geq 4$ so that M_7 is indeed maximal 3-local. If $q = 4$, then $O_3(M_7) = R_{25}$; the analogous argument as for Case (M_6) shows that the only new group is $R = R_{51} = O_3(M_7)$ with $a \geq 2$ and $q - \varepsilon = 3^a$.

Case (M_8) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_8}(R)$, where $M_8 = 3^3.\mathrm{SL}_3(3)$ and $O_8 = O_3(M_8) = (3\mathrm{C}^3)_1$. We have $R = R_{52} = O_8$, so let $O_8 < R$ in the following. By the Borel-Tits Theorem, $N_{M_8}(R) = 3^3.P$ for some parabolic $P \leq \mathrm{SL}_3(3)$, and a direct computation proves $P \in_G \{3_+^{1+4}, 3^{2+3}, U_8\}$ where U_8 is a 3-Sylow subgroup of M_8. Note that U_8 is a Sylow 3-subgroup of G if and only if $a = 1$. Thus, if $a \geq 2$, then $|N_G(U_8)|_3 > |N_{M_8}(U_8)|_3$,

	R	char	$C_{M_3}(R) = C_G(R)$	$N_{M_3}(R)$	$N_G(R)$	conditions		
R_1	3	3C	$(L_1 \circ_3 L_2).3$	M_3	M_3	$q \geq 4$	–	–
R_2	3^2	$(3\mathrm{C}^2)_2$	$((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3$	$C_{M_3}(R).(3 \times 2)$	$C_G(R).\mathrm{SL}_2(3)$	$q \geq 4$	–	–
R_3	3^2	$3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 \mathrm{GL}_2(q^2)).3$	$C_{M_3}(R).2$	$C_G(R).2^2$	$q \geq 4$	$a=1$	–
R_4	9	3C	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 (q^4+q^2+1)).3$	$C_{M_3}(R).6$	$N_{M_3}(R)$	$q \geq 4$	$a=1$	–
R_5	9	3C	$((q^2+\varepsilon q+1) \circ_3 \mathrm{GL}_2(q^2)).3$	$C_{M_3}(R).6$	$N_{M_3}(R)$	$q \geq 4$	$a=1$	–
R_6	$(D_1 \circ_3 D_2).3$	3C	3	$R.2^3$	$N_{M_3}(R)$	–	–	–
R_7	$D_1 \circ_3 D_2$	3C	3	$(D_1.Q_8 \circ_3 D_2.Q_8).\mathrm{S}_3$	$N_{M_3}(R)$	–	$a=1$	–
R_8	D_2	3C	L_1	$(L_1 \circ_3 D_2.Q_8).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	$a=1$	–
R_9	D_2	3C	L_1	$(L_1 \circ_3 D_2.2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	$a \geq 2$	–
R_{10}	$D_2.3$	$3\mathrm{B}_6\mathrm{C}_2$	$q^2+\varepsilon q+1$	$((q^2+\varepsilon q+1).3 \circ_3 D_2.2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	–	–
R_{11}	$D_2.3$	$3\mathrm{B}_6\mathrm{C}_2$	$\mathrm{GL}_2^\varepsilon(q)$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 D_2.2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	$a=1$	–
R_{12}	D_1	3C	L_2	$(D_1.Q_8 \circ_3 L_2).\mathrm{S}_3$	$N_{M_3}(R)$	–	$a=1$	–
R_{13}	D_1	3C	L_2	$(D_1.2 \circ_3 L_2).\mathrm{S}_3$	$N_{M_3}(R)$	–	$a \geq 2$	–
R_{14}	$D_1.3$	$3\mathrm{A}_6\mathrm{C}_2$	q^4+q^2+1	$(D_1.2 \circ_3 (q^4+q^2+1)).\mathrm{S}_3$	$N_{M_3}(R)$	–	–	–
R_{15}	$D_1.3$	$3\mathrm{A}_6\mathrm{C}_2$	$\mathrm{GL}_2(q^2)$	$(D_1.2 \circ_3 \mathrm{GL}_2(q^2)).\mathrm{S}_3$	$N_{M_3}(R)$	–	$a=1$	–
R_{16}	$(3^a)^4$	$3\mathrm{A}_{24}\mathrm{B}_{24}\mathrm{C}_{32}$	T	$T.(\mathrm{S}_3 \times \mathrm{S}_3).2$	$T.W$	$q \geq 5$	–	$q-\varepsilon \neq 3^a$
R_{17}	$(O_3(T_1) \circ_3 D_2).3$	$3\mathrm{B}_6\mathrm{C}_2$	T_1	$(T_1 \circ_3 D_2).(\mathrm{S}_3 \times 2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	–	$q-\varepsilon \neq 3^a$
R_{18}	$O_3(T_1) \circ_3 D_2$	$3\mathrm{B}_6\mathrm{C}_2$	T_1	$(T_1 \circ_3 D_2).(\mathrm{S}_3 \times Q_8).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	$a=1$	–
R_{19}	$(D_1 \circ_3 O_3(T_2)).3$	$3\mathrm{A}_6\mathrm{C}_2$	T_2	$(D_1 \circ_3 T_2).(2 \times \mathrm{S}_3).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	–	–
R_{20}	$D_1 \circ_3 O_3(T_2)$	$3\mathrm{A}_6\mathrm{C}_2$	T_2	$(D_1 \circ_3 T_2).(Q_8 \times \mathrm{S}_3).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	$a=1$	–
R_{21}	$O_3(T_1)$	$3\mathrm{A}_6\mathrm{C}_2$	$(T_1 \circ_3 L_2).3$	$(T_1.\mathrm{S}_3 \circ_3 L_2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	–	$q-\varepsilon \neq 3^a$
R_{22}	$(O_3(T_1) \circ_3 3^a).3$	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	$(T_1 \circ_3 \mathrm{GL}_2(q^2)).3$	$(T_1.\mathrm{S}_3 \circ_3 \mathrm{GL}_2(q^2)).\mathrm{S}_3$	$C_G(R).2^3.\mathrm{S}_3$	$q \geq 5$	$a=1$	–
R_{23}	$3^a \times 3^{a+1}$	$3\mathrm{A}_6\mathrm{C}_2$	$(O_3(T_1) \circ_3 (q^4+q^2+1)).3$	$(T_1.\mathrm{S}_3 \circ_3 (q^4+q^2+1).3).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	–	$q-\varepsilon \neq 3^a$
R_{24}	$O_3(T_2)$	$3\mathrm{B}_6\mathrm{C}_2$	$(L_1 \circ_3 T_2).3$	$(L_1 \circ_3 T_2.\mathrm{S}_3).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	–	–
R_{25}	$(3^a \circ_3 O_3(T_2)).3$	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 T_2).3$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 T_2.\mathrm{S}_3).\mathrm{S}_3$	$C_G(R).2^3.\mathrm{S}_3$	$q \geq 4$	$a=1$	–
R_{26}	$3^{a+1} \times 3^a$	$3\mathrm{B}_6\mathrm{C}_2$	$((q^2+\varepsilon q+1) \circ_3 O_3(T_2)).3$	$((q^2+\varepsilon q+1).3 \circ_3 T_2.\mathrm{S}_3).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 4$	–	–
R_{27}	F_1	3C	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 L_2).3$	$C_M(R).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{28}	3^{a+1}	3C	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 (q^4+q^2+1)).3$	$C_M(R).6$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{29}	$(F_1 \circ_3 F_2).3$	$3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 \mathrm{GL}_2(q^2)).3$	$C_{M_3}(R).2$	$C_{M_3}(R).2^2$	$q \geq 5$	$a \geq 2$	–
R_{30}	$F_1 \circ_3 3_+^{1+2}$	3C	$\mathrm{GL}_2^\varepsilon(q)$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{31}	$(F_1 \circ_3 O_3(T_2)).3$	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 T_2).3$	$C_G(R).2.\mathrm{S}_3$	$C_G(R).2^3.\mathrm{S}_3$	$q \geq 5$	$a \geq 2$	–
R_{32}	$(F_1 \circ_3 D_2).3$	$3\mathrm{B}_6\mathrm{C}_2$	$\mathrm{GL}_2^\varepsilon(q)$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 D_2.2).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{33}	F_2	3C	$(L_1 \circ_3 \mathrm{GL}_2(q^2)).3$	$C_M(R).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{34}	3^{a+1}	3C	$((q^2+\varepsilon q+1) \circ_3 \mathrm{GL}_2(q^2)).3$	$C_M(R).6$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{35}	$3_+^{1+2} \circ_3 F_2$	3C	$\mathrm{GL}_2(q^2)$	$(3_+^{1+2}.\mathrm{SL}_2(3) \circ_3 \mathrm{GL}_2(q^2)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{36}	$(O_3(T_1) \circ_3 F_2).3$	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	$(T_1 \circ_3 \mathrm{GL}_2(q^2)).3$	$C_G(R).2.\mathrm{S}_3$	$C_G(R).2^3.\mathrm{S}_3$	$q \geq 5$	$a \geq 2$	$q-\varepsilon \neq 3^a$
R_{37}	$(D_1 \circ_3 F_2).3$	$3\mathrm{A}_6\mathrm{C}_2$	$\mathrm{GL}_2(q^2)$	$(D_1.2 \circ_3 \mathrm{GL}_2(q^2)).\mathrm{S}_3$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{38}	3_+^{1+2}	3C	L_2	$(3_+^{1+2}.\mathrm{SL}_2(3) \circ_3 L_2).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{39}	3_+^{1+4}	3C	3	$3_+^{1+4}.(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{40}	3_+^{1+4}	3C	3	$3_+^{1+4}.(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3))$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{41}	$3_+^{1+2} \circ_3 O_3(T_2)$	$3\mathrm{B}_6\mathrm{C}_2$	T_2	$(3_+^{1+2}.\mathrm{SL}_2(3) \circ_3 T_2.\mathrm{S}_3).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{42}	$3_+^{1+2} \circ_3 D_2$	3C	3	$(3_+^{1+2}.\mathrm{SL}_2(3) \circ_3 D_2.2).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{43}	3_+^{1+2}	3C	L_1	$(L_1 \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–
R_{44}	$O_3(T_1) \circ_3 3_+^{1+2}$	$3\mathrm{A}_6\mathrm{C}_2$	T_1	$(T_1.\mathrm{S}_3 \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	$q-\varepsilon \neq 3^a$
R_{45}	$D_1 \circ_3 3_+^{1+2}$	3C	3	$(D_1.2 \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).2$	$N_{M_3}(R)$	$q \geq 5$	$a \geq 2$	–

TABLE VII. Radical 3-subgroups of $M_3 = N_G(z_C)$ with $G = E_6^{-\varepsilon}(q)$, L_i, T_i, F_i, and D_i as in Lemma 5.3; as before, 3^a is the largest 3-power dividing $q - \varepsilon$

hence $N_G(U_8) \neq N_{M_8}(U_8)$; if $a=1$, then $U_8 =_G R_6$. If $R = 3_+^{1+4}$, then $N_G(R) \leq_G M_3$, so we can ignore this case. If $R = 3^{2+3}$, then $N_G(R) \leq N_G(Z(R))$ with $Z(R) \leq O_8 = (3\mathrm{C}^3)_1$, hence $Z(R) = (3\mathrm{C}^2)_1$ by the proof of Lemma 4.13. We assume $R \in \mathcal{R}_3(G)$, thus R is radical in $N_G(Z(R)) = T.3.\mathrm{GL}_2(3)$, and $O_3(N_G(Z(R))) = O_3(T) \leq R$ by Lemma 2.2. This forces $a=1$, and $R = R_{53} = (3^a)^4.3$. Since $C_G(R) \leq C_G(O_8) = O_8 \leq R$, we have $C_G(R) = Z(R) = (3\mathrm{C}^2)_1$. Finally, $N_G(R) = N_{N_G(Z(R))}(R) = R.\mathrm{GL}_2(3)$. Conversely, note that R_{53} (hence $a=1$) is a Sylow 3-subgroup of $T.3 \leq N_G(Z(R)) = T.3.\mathrm{GL}_2(3)$, thus R_{53} is indeed radical in G.

Case (M_9) Suppose $R \in \mathcal{R}_3(G)$ with $N_G(R) = N_{M_9}(R)$, where $M_9 = T.W$ with $W = 2_+^{1+4}.(\mathrm{S}_3 \times \mathrm{S}_3)$. As before, write $T = (T_1 \circ_3 T_2).x_{\mathrm{C}} \leq M_3$ with each $T_i \leq L_i$ diagonal. We have $O_9 = O_3(M_9) = O_3(T) =_G R_{16}$ if $q \geq 5$ and $q-\varepsilon \neq 3^a$, and $R = R_{54} = O_9$ if $q - \varepsilon = 3^a$; note that the Sylow 3-subgroup of M_9 is $U_9 =_G R_6$. Now let $R \notin_G \{O_9, U_9\}$, so that $R = O_9.P$ for some $P \leq W$ of order 3, cf. Lemma 2.2. As shown above, W has three subgroups of order 3, namely, $H_1 = \langle 1 : v_2\rangle$, $H_2 = \langle v_1 : 1\rangle$, and $H_3 = \langle v_1 : v_2^{-1}\rangle$. If $R = O_3(T).H_i$ with $i \in \{1,2\}$, then $Z(R) = O_3(T_i)$ and $\Omega_1(O_3(T_i)) \in \{3\mathrm{A}_6\mathrm{C}_2, 3\mathrm{B}_6\mathrm{C}_2\}$, hence $N_G(R) \leq N_G(\Omega_1(Z(R))) \leq M_3$, and we can ignore this case. Now let $R = O_3(T).H_3$. The group H_3 is radical in W with $N_W(H_3) = H_3.\mathrm{GL}_2(3)$. As shown in the proof of Lemma 4.11, there exists $E = (3\mathrm{C}^2)_1 \leq T$ with $C_G(E) = T.H_3$ and $N_G(E) = T.H_3.\mathrm{GL}_2(3)$; thus we can consider $R = O_9.H_3$ as a Sylow 3-subgroup of $C_G(E)$. This yields $E \leq \Omega_1(Z(R))$. Since $R \not\leq C_G(\Omega_1(O_9)) = T$, we have $\Omega_1(Z(R)) \neq \Omega_1(O_9)$. Similarly, $R \not\leq C_G(3^3)$ for every $3^3 \leq T$ by an order argument, see Table V. This proves that $\Omega_1(Z(R)) = E = (3\mathrm{C}^2)_1$, and $N_G(R) = N_{N_G(E)}(R) = R.\mathrm{GL}_2(3)$; in particular, R is indeed radical in G. Note that $C_G(R) \leq C_G(O_9) = T$, hence $C_G(R) = C_T(R) = C_T(H_3)$. Since $|C_T(H_3)| = 3^2$, it follows that $C_G(R) = E$; thus $R = R_{55}$ if $a \geq 2$, and $R = R_{53}$ if $a = 1$. □

Corollary 5.8. *Up to conjugacy, the radical 3-subgroups of $G = E_6(2)$ are the groups R_i given in Tables VII and VIII, where* $i \in \{6, 7, 12, 14, 15, 46, 47, 48, 49, 50, 52, 53, 54\}$.

R		char	$C_G(R)$	$N_G(R)$	conditions
R_{46}	3^a	3A	$((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{-\varepsilon}(q)).2_\varepsilon$	$C_G(R).2$	–
R_{47}	3^a	3B	$((q^2-1) \circ_{2_\varepsilon} \mathrm{Spin}_8^-(q)).2_\varepsilon$	$C_G(R).2$	–
R_{48}	$3^a \times 3^a$	$3\mathrm{A}_4\mathrm{B}_4$	$[((q^2-1) \times (q-\varepsilon)) \circ_{4_{-\varepsilon}} \mathrm{SL}_4^{-\varepsilon}(q)].4_{-\varepsilon}$	$C_G(R).D_8$	–
R_{49}	3^2	$(3\mathrm{C}^2)_2$	$((q^2+\varepsilon q+1) \times (q^4+q^2+1)).3$	$C_G(R).\mathrm{SL}_2(3)$	–
R_{50}	$3^a \times 3^a \times 3^a$	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	$[((q^2-1) \times (q-\varepsilon)^2) \circ_{2_\varepsilon} \mathrm{SL}_2(q^2)].2_\varepsilon$	$C_G(R).2^3.\mathrm{S}_3$	$q \in \{2,4\}$, or $a \geq 2$ and $q - \varepsilon = 3^a$
R_{51}	$3^a \times 3^a \times 3^a$	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	$[((q^2-1)^2 \times (q-\varepsilon)) \circ_{2_\varepsilon} \mathrm{SL}_2(q)].2_\varepsilon$	$C_G(R).2^3.\mathrm{S}_3$	$a \geq 2$ and $q - \varepsilon = 3^a$
R_{52}	3^3	$(3\mathrm{C}^3)_1$	3^3	$3^3.\mathrm{SL}_3(3)$	–
R_{53}	$(3^a)^4.3$	$(3\mathrm{C}^2)_1$	3^2	$R.\mathrm{GL}_2(3)$	$a = 1$
R_{54}	$(3^a)^4$	$3\mathrm{A}_{24}\mathrm{B}_{24}\mathrm{C}_{32}$	T	$T.W$	$q - \varepsilon = 3^a$
R_{55}	$(3^a)^4.3$	$(3\mathrm{C}^2)_1$	3^2	$R.\mathrm{GL}_2(3)$	$a \geq 2$

Table VIII. Radical 3-subgroups of $G = E_6^{-\varepsilon}(q)$ which are not radical in M_3; as before, $n_\eta = \gcd(n, q-\eta)$ and 3^a is the largest 3-power dividing $q - \varepsilon$

6. Essential rank

Let G be a finite group with nontrivial Sylow p-subgroup $D \leq G$. The objects of the Frobenius category $\mathcal{F} = \mathcal{F}_D(G)$ are all subgroups of D, its morphisms are $\mathrm{Hom}_{\mathcal{F}_D(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$, the group homomorphisms $Q \to R$ induced by conjugation by an element of G. The Frobenius category is a fusion system as defined in [24]. The essential rank of a fusion system $\mathcal{F}$ is defined as the number of $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-essential subgroups, see [20] for the precise definition and more details. For a Frobenius category $\mathcal{F}$, the essential rank is precisely the number of G-classes of p-essential subgroups of G, cf. [1]; here a subgroup $U \leq G$ is p-essential if, first, $Z(U)$ is a Sylow p-subgroup of $C_G(U)$, and, second, $\mathrm{Out}_G(U)$ has a strongly p-embedded subgroup, that is, a proper subgroup $V \le \mathrm{Out}_G(U)$ with $p \mid |V|$ and $p \nmid |V \cap V^g|$ for all $g \in \mathrm{Out}_G(U) \setminus V$. Every p-essential subgroup is p-radical.

Lemma 6.1. *Let H be a group with a cyclic Sylow p-subgroup P, and let $Z = \Omega_1(P)$. Then either $N_H(Z) = H$ or $N_H(Z)$ is a strongly p-embedded subgroup of H.*

Proof. Clearly, Z is the unique subgroup of $N = N_H(Z)$ of order p. Let $g \in H$ with $|N \cap N^g|$ divisible by p; then $Z \le N \cap N^g$, whence $Z = Z^g$, and so $g \in N$. □

Proof of Theorem B. We have to identify the 3-essential groups R_i in Tables VII and VIII, and the 5-essential groups R_i in Tables II and III. First, consider $p = 5$. The only groups in Table II with $Z(R)$ a Sylow 5-subgroup of $C_G(R)$ are R_{16} and R_{18}. We have $N_G(R_{16})/C_G(R_{16}) = W(E_6)$, which has a strongly 5-embedded subgroup by Lemma 6.1; note that $O_5(W(E_6)) = 1$ and $W(E_6)$ has 5-rank 1. Moreover, R_{18} is not 5-essential since $N_G(R_{18})/C_G(R_{18})$ is cyclic. The only groups in Table III with $Z(R)$ a Sylow 5-subgroup of $C_G(R)$ are R_{19} and R_{21}. However, only $N_G(R_{21})/R_{21}C_G(R_{21}) = \mathrm{SL}_2(5)$ has a strongly 5-embedded subgroup. Thus, up to conjugacy, the 5-essential subgroups of G are $\{R_{16}, R_{21}\}$ as given in Tables II and III; this proves the assertion for $p = 5$. We proceed analogously for $p = 3$. The only 3-essential subgroups in Table VII are R_{42} and R_{45}, which exist only for $q \ge 5$ and $a \ge 2$, and R_7, which exists only for $a = 1$. The only 3-essential subgroups in Table VIII are R_{53} if $a = 1$, and R_{55} if $a \ge 2$. This proves the theorem. □

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