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Nash bargaining and risk aversion ☆

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ABSTRACT

It is widely accepted among axiomatic bargaining theorists that if one bargainer is more risk averse than a second, the second will be a tougher bargaining opponent than the first against all opponents. We argue that this relationship between risk aversion and bargaining toughness is both highly fragile, and more nuanced than previously articulated. In the Nash and Kalai–Smorodinsky bargaining frameworks, we establish that when a bargainer is compared with a second who is "almost globally" more risk averse than the first, the supposedly immutable relationship between bargaining effectiveness and risk aversion evaporates. Specifically, we identify an upper-hemicontinuity failure of a correspondence which maps the power set of all lotteries to those utility pairs that satisfy our "almost global" comparative risk aversion relation on these subsets. We trace the consensus view that tougher bargainers are less risk-averse to an exclusive focus on precisely the point at which this correspondence implodes.

In the extensive literature on axiomatic bargaining theory, it is widely accepted that bargainers who are less risk-averse make tougher bargaining opponents.[1] This connection has been identified as "one of the results most frequently quoted in the bargaining literature" (Volij and Winter, 2002, p. 120). The first formal statements of the result are in the seminal works by Roth (1979) and Kihlstrom et al. (1981) (henceforth KRS). The theme is further developed in Roth and Rothblum (1982) (RR) and Safra et al. (1990) (SZZ). In particular, KRS's Theorems 1 and 2 relate, respectively, to the two best-known axiomatic bargaining models, developed by Nash and Kalai–Smorodinsky (KS). These theorems compare bargaining situations in which a given opponent with utility v bargains either against a benchmark player with utility u_0, or against another, globally more risk-averse player with utility u.

For brevity, we shall henceforth identify players with their utility functions: "*u bargains with v*" will serve as shorthand for "*the player with utility function u bargains against another with utility function v.*" We henceforth say that w is a *tougher*[2] ${}^{\text{Nash}}_{\text{KS}}$*-bargainer than w' against v* if ${}^{\text{Nash}}_{\text{KS}}$ bargaining with w yields v less utility than bargaining with w'. KRS's results establish that under both solution concepts, if u is more risk-averse than u_0, then u_0 is a tougher bargainer than u.

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[1] See, e.g., Kannai (1977), Sobel (1981), Osborne (1985), Thomson (1988) and Roth (1989).

[2] Throughout the paper, the relations "tougher than" and "more risk-averse than" will be irreflexive, i.e., the relevant inequalities will be strict rather than weak.

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In this paper, we argue that the widely cited relationship between comparative global risk aversion and comparative bargaining toughness is in fact both highly fragile and more nuanced than has previously been articulated. It is fragile in the sense that if the notion of "global" is relaxed ever so slightly, in a particular direction, the relationship evaporates. It can be refined by distinguishing between risks that matter for bargaining toughness and those that don't: in particular, in both frameworks, comparative aversion to risks involving a positive probability of negotiation breakdown — the worst possible outcome — plays a pivotal role. In the Nash framework, there is a link — in a sense to be made precise — between comparative bargaining toughness and comparative aversion to lotteries involving *bad* outcomes, but no relationship whatsoever if the lotteries involve only *good* outcomes. In the KS framework, we identify a necessary and sufficient condition for u to be a tougher bargainer than u_0 against v; it can never be satisfied if u is *globally* more risk averse than u_0, but *can* be satisfied if u is "almost globally" more risk averse than u_0, in the sense we define. Finally, we establish that given any benchmark bargainer u_0, there is, in both frameworks, a bargainer u who is strictly tougher than u_0 against every bargaining opponent, yet is more risk averse than u_0 with respect all lotteries *except* those that assign positive probability either to negotiation breakdown, or to comparably bad negotiated outcomes. Since u_0 is a strictly tougher bargainer against all opponents than any u who is *globally* more risk averse than u_0, our result reveals a discontinuity — more precisely, an upper hemicontinuity failure — in the relation that maps comparative risk aversion to bargaining toughness.

To make these ideas precise, we formalize our notion of "almost globally more risk averse than…" in the following way. The standard definition of comparative risk aversion is[3]: "u is more risk averse than u_0 if any risk that is undesirable for u is also undesirable for u_0." If the universe of possible lottery outcomes is X, then it is natural to assert that u is *almost globally* more risk averse than u_0 if for some subset $X' \subset X$ that almost coincides with X, any risk *involving only outcomes in* X' that is undesirable for u_0, is also undesirable for u. This comparison leaves open the possibility of some remaining risks — necessarily ones which assign positive probability to realizations in $X \backslash X'$ — that are undesirable to u_0 but acceptable to u. If we apply our notion of "almost" to a sequence of subsets $\{X^n\}$ that approach X, while excluding the very worst outcomes in X, the risks which remain exempt from comparison are concentrated in a vanishingly small subset of the universe of all possible risks. Yet for any n, the set of utility pairs $\langle u^n, u_0\rangle$ which satisfy our "almost global" comparison criterion is extremely large, relative to the set of pairs $\langle u, u_0\rangle$ such that u is globally more risk averse than u_0. These much larger sets, that meet our comparative criterion for each n, include pairs $\langle u^n, u_0\rangle$ such that u^n is a much tougher bargainer than u_0.

1. Preliminaries

In the classical formulation of axiomatic bargaining problems, two players, with utility functions u and v respectively, bargain over a set of possible *feasible outcomes*. If they fail to agree upon one of these, a *disagreement outcome*, D, is implemented. (In general, outcomes are points in $\mathbb{R}^2$, so utilities are multivariate. With one additional assumption, they can be represented by univariate functions. To reduce notation, we will use the same symbols for utilities and their univariate representations.) We assume that players' utilities are defined on the simplex $Z = \{\mathbf{z} \in \mathbb{R}^2_+ : z_i \geq 0; \sum_{i=1}^{2} z_i \leq 1\}$ and that $D = (0, 0)$. Throughout, we assume that u derives utility only from the first component of $\mathbf{z} \in Z$, while v derives utility only from the second. We will hold v constant and compare the "bargaining performance" of utility u against v relative to that of a benchmark utility u_0. Let $S = \{\big(u(\mathbf{z}), v(\mathbf{z})\big) : \mathbf{z} \in Z\} \subset \mathbb{R}^2$ denote the set of utility pairs that can be agreed upon, and let $\mathbf{d} = \big(u(D), v(D)\big) \in \mathbb{R}^2$ denote the utility pair that results from disagreement. Following Osborne and Rubinstein (1990, p. 10), we define a *bargaining problem* to be a pair $\langle S, \mathbf{d}\rangle$, where $S \subset \mathbb{R}^2$ is compact and convex and there exists $\mathbf{s} \in S$ such that $\mathbf{s} \gg \mathbf{d}$. Let $\mathcal{B}$ denote the set of all bargaining problems. A *bargaining solution* is a function $f : \mathcal{B} \to \mathbb{R}^2$ that assigns to each bargaining problem $\langle S, \mathbf{d}\rangle \in \mathcal{B}$ a unique element of S.

1.1. Bargainer utilities

In Sections 1–2, we impose two assumptions on bargainers' utility functions:

Assumption A1). *Utilities are von Neumann–Morgenstern, strictly concave and twice continuously differentiable on Z. For all $\mathbf{z} \in [0, 1]^2$, $\frac{\partial u(\mathbf{z})}{\partial z_1} > 0$ and $\frac{\partial v(\mathbf{z})}{\partial z_2} > 0$.*

An axiom invoked by both Nash and KS is that if either u or v is replaced by an affine transformation of itself, the bargaining outcome must remain unchanged.[4] We therefore assume, w.l.o.g.

Assumption A2). *Each player derives utility 0 from D and a maximum utility of 1 on Z.*

A second axiom imposed in both frameworks requires that the bargaining solution must lie on the Pareto frontier, which is the north-east boundary of Z, i.e., $\{(x, 1 - x) : x \in [0, 1]\}$. Invoking this axiom, we restrict our attention to this frontier,

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[3] Eeckhoudt et al. (2005, Defn. 1.4, p. 14). Eeckhoudt et al. add the caveat that u_0 and u must have the same level of income. In the context of bargaining theory, there is no natural notion of "income."

[4] For example, an affine transformation of u is a function $w = a + bu$, for some $(a, b) \in \mathbb{R} \times \mathbb{R}_{++}$.

denoted $P = [0, 1]$, and will henceforth treat all utility functions *as if* they were univariate functions defined on P, with the understanding that the scalar $x \in P$ represents the vector $(x, 1-x) \in Z$. Thus, $\frac{u(x)}{v(x)}$ will denote the utility that $\frac{u}{v}$ derives from the share $\frac{x}{1-x}$. We can now rewrite A1) as

$$\text{for all } x \in P, \quad u'(\cdot) \quad > \quad 0 \quad > \quad v'(\cdot). \tag{1}$$

To streamline our presentation, we restrict attention to a compact family of utility functions:

Assumption A3). *Bargainers' utilities are drawn from a set $\mathcal{U}$ of functions satisfying Assumptions A1)–A2) that is compact in the sup norm topology.*[5]

1.2. Risk aversion

The seminal papers that have considered comparative risk aversion in bargaining models compare players' risk aversion over the entire domain of their utility functions. This paper augments these comparisons with analogous ones on subsets of the utility domain. Following Eeckhoudt et al. (2005), we will say that bargainer u is *globally more risk-averse than* u_0 (abbreviated to GMRA) if any risk that is undesirable for u_0 is "even more" undesirable for u. (Throughout this paper the term "more" will denote a strict relation.) Analogously, we say that u is *strictly risk-averse than* u_0 *on* $X \subset P$ (abbreviated to MRA^X) if any risk *involving outcomes in* X that is undesirable for u_0 is even more undesirable for u.

The literature has identified three equivalent ways to formalize the concept of GMRA. All three can be extended immediately to MRA^X. First, for each $X \subset P$, we define u to be MRA^X than u_0 if[6] $u_{|X} = \phi(u_{0|X})$, where ϕ is an increasing, strictly concave function.

Prop. 1. *(Adapted from Eeckhoudt et al. (2005, Prop 1.5).) For $X \subset P$, the following three statements are equivalent:*

a) *u is MRA^X than u_0;*
b) $\forall x \in X$, $r^u(x) := -\frac{u''(x)}{u'(x)} > r^0(x) := -\frac{u_0''(x)}{u_0'(x)}$;
c) *for any uncertain event $\tilde{\mathbf{z}}$ with distribution μ whose support is contained in X, $u^{-1}(\mathbb{E}_\mu[\tilde{\mathbf{z}}]) > u_0^{-1}(\mathbb{E}_\mu[\tilde{\mathbf{z}}])$.*

$r^u(\cdot)$ and $r^0(\cdot)$ in b) are the Arrow–Pratt coefficients of absolute risk aversion for u and u_0, respectively (Pratt, 1964, p. 122); $u^{-1}(\mathbb{E}_\mu[\tilde{\mathbf{z}}])$ and $u_0^{-1}(\mathbb{E}_\mu[\tilde{\mathbf{z}}])$ in c) are the *certainty equivalents*[7] of the lottery $\tilde{\mathbf{z}}$, for u and u_0 respectively.

We focus primarily on subsets of P of the form $[\underline{x}, 1]$. For every $\underline{x} > 0$, the restriction MRA$^{[\underline{x},1]}$ is strictly weaker than GMRA; however, when $\underline{x} = 0$, the two relations coincide, since $[0, 1] = P$. The condition that u is MRA$^{[\underline{x},1]}$ than u_0 imposes several restrictions on the relationship between u and u_0, which are summarized in Lemma 1. Part e) of the lemma invokes some terminology: we say that u *intersects* u_0 *at* x if $u(x) = u_0(x)$. Further, we say that u *cuts* u_0 *from* $\substack{\text{below}\\\text{above}}$ at x if u intersects u_0 *at* x and if $u'(x) \gtreqless u_0'(x)$, with strict inequality if $x < 1$.

Lemma 1 *(Implications of MRA$^{[\underline{x},1]}$). For $\underline{x} \geq 0$, suppose that u is MRA$^{[\underline{x},1]}$. Then*

a) $\frac{d}{dz}\left(\frac{u'(\cdot)}{u_0'(\cdot)}\right) < 0$ *on* $[\underline{x}, 1]$;
b) *If $\exists y \in [\underline{x}, 1]$ s.t. $u'(y) \geq u_0'(y)$, then $u'(\cdot) > u_0'(\cdot)$ on $[\underline{x}, y)$;*
c) *If $\exists y \in [\underline{x}, 1)$ s.t. $u'(y) \leq u_0'(y)$, then $u(y) > u_0(y)$;*
d) *If $u(\underline{x}) = u_0(\underline{x})$, then $u(\cdot) > u_0(\cdot)$ on $(\underline{x}, 1)$;*
e) *if $u(y) < u_0(y)$ for some $y \in [\underline{x}, 1)$, then $\exists \tilde{y} \in (y, 1]$ s.t. u cuts u_0 from below at $\tilde{y}$.*

In words, if u is MRA$^{[\underline{x},1]}$ than u_0 then: b) if u is weakly steeper than u_0 at y, it is strictly steeper to the left of y; c) if u is weakly flatter than u_0 at y, then it lies above u_0 at y; d) if u and u_0 agree at $\underline{x}$, then u dominates u_0 on the interior of $[\underline{x}, 1]$. e) if u is dominated by u_0 at y, then when it intersects u_0 at some $\tilde{y} > y$ — which it must, by Assumption A2) — u's slope at $\tilde{y}$ will strictly exceed u_0's if $\tilde{y} < 1$, and weakly exceed u_0's if $\tilde{y} = 1$.

Since by Assumption A2), $u_0(0) = u(0) = 0$, an immediate implication of part d) of Lemma 1 is:

Remark 1. If u is GMRA than u_0 then $u(\cdot) > u_0(\cdot)$ on the interior of P.

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[5] For functions with domain S, a metric for the sup norm topology is: $\rho(f, g) = \sup_{s \in S} |f(s) - g(s)|$.
[6] Given $f : Y \to \mathbb{R}$, and $Y' \subset Y$, $f_{|Y'}$ denotes *the restriction of f to Y'*, i.e., the function $g : Y' \to \mathbb{R}$ such that for all $x \in Y'$, $g(x) = f(x)$.
[7] The certainty equivalent for u of a lottery $\tilde{\mathbf{z}}$ is a certain outcome y which yields u the same utility as $\tilde{\mathbf{z}}$.

2. Bargaining

We focus on the two best-known axiomatic bargaining solution concepts, $f^{\mathcal{N}}$ and $f^{\mathcal{KS}}$, formulated, respectively, by Nash (1950, 1953) and Kalai and Smorodinsky (1975). Nash imposed four axioms: Symmetry, Pareto Efficiency, Invariance to Equivalent Utility representations, and Independence of Irrelevant Alternatives (IIA).[8] He showed that these four axioms are satisfied by a unique element of P, which is the argmax, denoted $f^{\mathcal{N}}$, of the function $\mathcal{N}(\cdot|u,v)$ defined in (2) below. Having imposed Assumption A2), we can define the *Nash bargaining outcome* as[9]:

$$f^{\mathcal{N}}(u,v) \quad = \quad \text{argmax}_{x\in P}\, \mathcal{N}(x|u,v), \text{ where } \mathcal{N}(x|u,v) := \log(u(x)) + \log(v(x)) \tag{2}$$

$f^{\mathcal{N}}(u,v)$ is the unique x value that solves the first-order condition

$$\frac{\partial \mathcal{N}(x|u,v)}{\partial x} \quad = \quad 0 \quad = \quad \frac{u'(x)}{u(x)} \quad + \quad \frac{v'(x)}{v(x)}. \tag{3}$$

Because $\mathcal{N}(\cdot|u,v)$ is continuous and strictly concave, and because (from A2)) $\mathcal{N}(x|u,v)$ approaches $-\infty$ as x approaches either 0 or 1, a unique solution to (3) exists. Moreover,

$$\text{for all } u, v \text{ satisfying Assumptions A1)–A2)}, \quad f^{\mathcal{N}}(u,v) \quad \in \quad (0,1). \tag{4}$$

KS's axiomatic framework replaces Nash's IIA axiom with a monotonicity axiom. KS's concept is defined in terms of the disagreement utility vector — in our case, $(0,0)$ — and the vector — in our case, $(1,1)$ — representing the highest utility that each player can obtain from some negotiated agreement.

In words, the KS outcome is the intersection of the Pareto frontier of S with the 45° line (in utility space) through the origin. Formally,

$$f^{\mathcal{KS}}(u,v) \quad = \quad \mathcal{KS}(\cdot|u,v)^{-1}(0) \text{ where } \mathcal{KS}(x|u,v) := u(x) - v(x). \tag{5}$$

That is, $f^{\mathcal{KS}}(u,v)$ is the (unique) root of $\mathcal{KS}(\cdot|u,v)$. Once again, we have

$$\text{for all } u, v \text{ satisfying Assumptions A1)–A2)}, \quad f^{\mathcal{KS}}(u,v) \quad \in \quad (0,1). \tag{6}$$

Further, since the set $\mathcal{U}$ from which utilities are drawn is compact[10]:

$$\text{there exists } \bar{\epsilon} > 0 \text{ s.t. if } u, v \in \mathcal{U}, \text{ then for sc} \in \{\mathcal{KS}, \mathcal{N}\}\, f^{\text{sc}}(u,v) \in [\bar{\epsilon}, 1-\bar{\epsilon}]. \tag{7}$$

Now fix $v \in \mathcal{U}$. Let $\substack{x_0^{\mathcal{N}} \\ x^{\mathcal{N}}}$ denote the Nash outcome, and let $\substack{x_0^{\mathcal{KS}} \\ x^{\mathcal{KS}}}$ denote the KS outcome, when $\substack{u_0 \\ u}$ bargains against v. Since v prefers lower x-values to higher ones (see (1)), the classical result is that if u is GMRA than u_0, then $x_0^{\mathcal{N}} > x^{\mathcal{N}}$ and $x_0^{\mathcal{KS}} > x^{\mathcal{KS}}$. We will identify conditions under which these inequalities are reversed. We say that

$$u \textit{ is a tougher } \substack{\text{Nash} \\ \text{KS}}\textit{-bargainer than } u_0 \textit{ against } v \text{ if } \substack{x^{\mathcal{N}} > x_0^{\mathcal{N}} \\ x^{\mathcal{KS}} > x_0^{\mathcal{KS}}}. \tag{8}$$

Both comparisons in (8) are defined relative to specific solution concept and a specific bargaining partner. We also define a notion of "global relative toughness":

$$u \textit{ is a tougher bargaining opponent than } u_0 \text{ if}$$
$$\text{against every } v \in \mathcal{U}, \ u \textit{ is a tougher Nash- and KS-bargainer than } u_0. \tag{9}$$

Using Lemma 1, Lemmas 2 and 4 below identify conditions under which, for any $\underline{x} > 0$, u is MRA$^{[\underline{x},1]}$ than u_0 and a tougher KS- or Nash-bargainer than u_0 against v.

Lemma 2 *(Necessary and sufficient condition: KS). For $\underline{x} > 0$, if u is MRA$^{[\underline{x},1]}$ than u_0 and $\underline{x} < x_0^{\mathcal{KS}}$, then u is a tougher KS-bargainer than u_0 against v iff u cuts u_0 from below at some $x > x_0^{\mathcal{KS}}$.*

Lemma 2, combined with Remark 1, provides an alternative proof of KRS's result associating the GMRA relation to relative KS-bargaining toughness: for u to be KS-tougher than u_0 against v, u must cut u_0 from below; but from Remark 1, this cannot happen if u is GMRA than u_0. Conclude:

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[8] Apart from the Pareto and Invariance axioms, the others invoked by Nash and KS play no role in the present paper beyond implying the solution concepts defined by (2) and (5). Accordingly, we do not define them here. For a presentation and detailed discussion of each axiom, see Osborne and Rubinstein (1990).

[9] More generally, $\mathcal{N}(x|u,v) := \log(u(x) - u(D)) + \log(v(x) - v(D))$. Assumption A2) imposes $u(D) = v(D) = 0$.

[10] Clearly, $f^{\mathcal{N}}$ is a continuous function of u and v. Since $\mathcal{U}$ is compact, $f^{\mathcal{N}}$ attains a maximum and minimum on $\mathcal{U} \times \mathcal{U}$ (Weierstrass). From (4), both maximum and minimum belong to $(0,1)$. We can now define $\bar{\epsilon}^{\mathcal{N}} = \min_{u,v\in\mathcal{U}}(f^{\mathcal{N}}(u,v), 1 - f^{\mathcal{N}}(u,v))$. Define $\bar{\epsilon}^{\mathcal{KS}}$ analogously, and let $\bar{\epsilon} = \min(\bar{\epsilon}^{\mathcal{N}}, \bar{\epsilon}^{\mathcal{KS}})$.

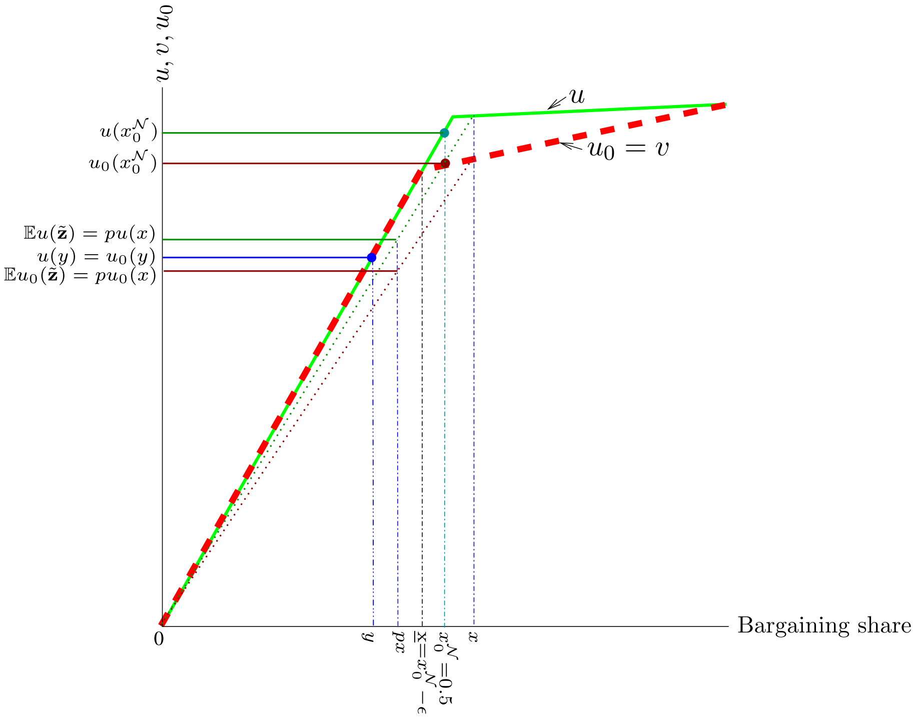

Fig. 1. $u(\cdot) > u_0(\cdot)$ and is tougher than u_0 against $v = u_0$.

Prop. 2. *(See KRS, Thm 2.) If u is GMRA than u_0, then u_0 is a tougher KS-bargaining opponent than u.*

Turning to Nash, Lemma 3 below provides a useful characterization of when u is a tougher Nash bargainer than u_0 against v, although the condition does not depend on u being MRA^X than u_0.

Lemma 3 *(Necessary and sufficient conditions: Nash). u is a tougher Nash-bargainer than u_0 against v iff $\frac{u'(x_0^{\mathcal{N}})}{u(x_0^{\mathcal{N}})} > \frac{u_0'(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})}$.*

Lemma 3 has an appealing intuitive interpretation: u is a tougher Nash-bargainer than u_0 against v iff, at the solution to the bargaining problem between u_0 and v, the elasticity[11] of u's utility w.r.t. u's negotiated share is greater than the corresponding elasticity for u_0.

The relationship between comparative risk aversion and toughness is less clearcut in Nash's framework than in KS's. Lemma 4 is a partial analog of Lemma 2, but it is not a characterization.

Lemma 4 *(Sufficient conditions: Nash). For $\underline{x} > 0$, if u is MRA$^{[\underline{x},1]}$ than u_0, and u cuts u_0 from below at some $x \geq x_0^{\mathcal{N}}$, then u is a tougher Nash-bargainer than u_0 against v.*

Since the inequality in the condition of Lemma 4 is weak — (u cuts u_0 from below at $x \geq x_0^{\mathcal{N}}$) — but the inequality that the condition implies is strict — ($x^{\mathcal{N}} > x_0^{\mathcal{N}}$) — it follows from continuity that the condition cannot also be necessary; there must exist an open neighborhood $\mathcal{X}$ containing $x_0^{\mathcal{N}}$ such that if u cuts u_0 from below at any $x \in \mathcal{X}$, then $x^{\mathcal{N}} > x_0^{\mathcal{N}}$. Indeed, in contrast to the KS formulation, it is not *necessary* for comparative Nash-toughness that u cuts u_0 from below at *any* point: as Fig. 1 illustrates, u can be Nash-tougher than u_0 against v, even when u lies everywhere above u_0. (For heuristic clarity, we have drawn u and u_0 in the figure as piecewise linear, so that neither function satisfies Assumption A1); obviously, there are smooth perturbations of both functions which do satisfy Assumption A1), while preserving the salient properties of the figure.) When u_0 bargains against $v = u_0$, by symmetry the Nash solution is $x_0^{\mathcal{N}} = 0.5$. But since u_0 kinks at $x_0^{\mathcal{N}} - \epsilon$, while u kinks just to the right of $x_0^{\mathcal{N}}$, we have $u'(x_0^{\mathcal{N}}) \gg u_0'(x_0^{\mathcal{N}})$, while $u(x_0^{\mathcal{N}}) \approx u_0(x_0^{\mathcal{N}})$. It follows from Lemma 3, therefore, that when $\epsilon > 0$ is sufficiently small, $x^{\mathcal{N}} > x_0^{\mathcal{N}}$. Note in this example that for $\underline{x} = x_0^{\mathcal{N}} - \epsilon$, u is MRA$^{[\underline{x},1]}$ than u_0. (Clearly, $u_{|[\underline{x},1]}$ is a concave transformation of $u_{0[\underline{x},1]}$.) However, u is not GMRA than u_0: for example, as the figure illustrates, there is a lottery $\tilde{\mathbf{z}}$, which realizes x with probability p and zero otherwise, and a sure outcome y, such that

[11] Given a function f of x, the elasticity of f w.r.t. x is defined as $\frac{x\,\partial f(x)/\partial x}{f(x)}$.

$$\mathbb{E}u(\tilde{\mathbf{z}}) = pu(x) > u(y) = u_0(y) > pu_0(x) = \mathbb{E}u_0(\tilde{\mathbf{z}})$$

so that u prefers $\tilde{\mathbf{z}}$ to the certain outcome y, but u_0 prefers y to $\tilde{\mathbf{z}}$.

Our next result, Proposition 3, is closely related to, but distinct from, the classical result that if u is GMRA than u_0, then u_0 will be a tougher Nash-bargaining opponent than u against v. To establish that u_0 is Nash-tougher than a specific opponent v, it suffices to assume only that u is more averse than u_0 with respect to risks involving outcomes that are less satisfactory to u than the solution obtained when u_0 Nash-bargains against v. In is in precisely this sense that what matters when it comes to Nash bargaining is relative aversiveness to risk over *bad* outcomes.

Prop. 3. *If u is MRA$^{[0,x_0^{\mathcal{N}}]}$than u_0, then u_0 is a tougher Nash bargainer than u against v.*

For completeness, we restate KRS's result for Nash-bargaining, which is the analog of Proposition 2 above: it can be rederived as an immediate corollary of Proposition 3

Prop. 4. *(See KRS, Thm. 1.) If u is GMRA than u_0, then u_0 is a tougher Nash-bargaining opponent than u.*

We now come to the main result of this paper: for any $\underline{x} > 0$ no implication can be drawn about u's bargaining toughness relative to u_0's from the fact that u is MRA$^{[\underline{x},1]}$ than u_0. Recalling that $\bar{\epsilon}$ was defined in (7):

Prop. 5. $\forall \underline{x} \in (0, \bar{\epsilon})$ *and $u_0 \in \text{int}(\mathcal{U})$, $\exists u \in \mathcal{U}$ who is MRA$^{[\underline{x},1]}$ than u_0 and a tougher bargaining opponent than u_0.*

Now let $\mathcal{R} : [0, 1) \twoheadrightarrow \mathcal{U} \times \mathcal{U}$ denote the correspondence mapping selected subsets of P to the utility pairs that satisfy our comparative risk aversion relation[12]:

$$\mathcal{R}(\underline{x}) = \{\langle u, u_0 \rangle \in \mathcal{U} \times \mathcal{U} : u \text{ is MRA}^{[\underline{x},1]} \text{ than } u_0\}. \tag{10}$$

For each n, MRA$^{[\underline{x},1]}$ is a strictly weaker comparative concept than GMRA, but in the limit, the distinction disappears, since $[0, 1] = P$. In the informal language of our introductory section, when n is very large, $\mathcal{R}(1/n)$ consists of utility pairs $\langle u^n, u_0^n \rangle$ such that u^n is "almost globally" more risk averse than u_0^n, while $\mathcal{R}(0)$ consists of pairs $\langle u, u_0 \rangle$ such that u is globally more risk averse than u_0, The disjunction between Proposition 5 and Propositions 2 & 4 is a consequence of the fact that $\mathcal{R}(\cdot)$ is not upper hemicontinuous[13] at $\underline{x} = 0$.

Proposition 5 establishes that for each $n > 1/\bar{\epsilon}$, there exists a pair $\langle u^n, u_0^n \rangle \in \mathcal{R}(1/n)$ such that u^n is a tougher bargaining opponent than u_0^n. Yet from Propositions 2 & 4, if u is GMRA than u_0, then u_0 is a tougher bargaining opponent than u. Since "tougher than" has been defined as a strict relation, and both the Nash and KS solution concepts are clearly continuous when their domains are endowed with the metric γ, defined in footnote 12, these propositions, taken together, necessarily imply that $\mathcal{R}(\cdot)$ is not upper hemicontinuous at zero.[14]

Fig. 2 above graphs a pair of utilities, $\langle \bar{u}, u_0 \rangle$, which illustrate two key messages of this paper. First, it demonstrates that $\bar{u}$ may be a tougher bargainer than u_0, even though $\bar{u}$ is more averse than u_0 to all risks except ones involving a positive probability of an extremely undesirable outcome. Second, it reveals the magnitude of the implosion (upper hemicontinuity failure) of the correspondence $\mathcal{R}(\cdot)$ at zero. Since the figure serves only as a heuristic example, we do not require that u_0 belongs to the compact set $\mathcal{U}$ specified in Assumption A3).[15] Moreover, in the figure, u_0 is drawn for clarity as piecewise linear, with a kink at $\underline{x}$; as in Fig. 1, it can obviously be smoothened and made strictly concave without changing any salient features of the example.

If $\underline{x} < \bar{\epsilon} \le \min\left\{ f^{\text{sc}}(\bar{u}, v) : v \in \mathcal{U}, \text{sc} \in \{\mathcal{KS}, \mathcal{N}\} \right\}$, then since $\bar{u}$ lies everywhere below u_0, $\bar{u}$ will be a tougher bargainer than u_0 against every opponent $v \in \mathcal{U}$ (Lemmas 2 and 4). On the other hand, u_0 is risk neutral, and hence less risk-averse than $\bar{u}$ with respect to all lotteries *except* ones that assign positive probabilities to outcomes on either side of $\underline{x}$. The figure illustrates one such exception: $\bar{u}$ prefers the illustrated certain outcome y to the lottery $\tilde{\mathbf{z}}$ realizing $x_1 < \underline{x} < x_2$ with equal probability, while u_0 prefers $\tilde{\mathbf{z}}$ to y. Indeed when $\underline{x} \approx 0$, u_0 is risk neutral with respect to *all* risks except ones involving a positive probability of either negotiation breakdown or negotiated outcomes that are barely preferred to breakdown. To summarize, the example challenges the conventional wisdom, since: a) $\bar{u}$ is more risk averse than u_0 with respect to

[12] Since $\mathcal{R}(\cdot)$ is defined on $\mathbb{R}$, we endow its domain with the Euclidean metric. We endow its co-domain with the metric $\gamma(\mathbf{f}, \mathbf{g}) = \max(\rho(f_1, g_1), \rho(f_2, g_2))$, where ρ is the sup norm metric (footnote 5).

[13] A correspondence $\psi : S \twoheadrightarrow T$ is *upper hemicontinuous* if for every $s \in S$ and every open neighborhood Ψ of $\psi(s)$ there is an open neighborhood $\mathcal{S}$ of s such that $\psi(s') \subset \Psi$ for every $s' \in \mathcal{S}$.

[14] While for Nash-bargaining, this implication is a little opaque, for KS-bargaining, it is transparent: from Remark 1, $\langle u, u_0 \rangle \in \mathcal{R}(0)$ implies $u(\cdot) > u_0(\cdot)$ on $(0, 1)$. Since from Assumption A2), $u \in \mathcal{U}$ implies $u(0) = 0$ and $u(1) = 1$, it follows that $\Psi := \{\langle u, u_0 \rangle \in \mathcal{U} \times \mathcal{U} : u(\cdot) > u_0(\cdot) \text{ on } (0, 1)\}$ is an open neighborhood of $\mathcal{R}(0)$. Now consider an arbitrary open neighborhood S of 0. For n sufficiently large, $1/n \in S$. From Proposition 5, there exists $\langle u^n, u_0^n \rangle \in \mathcal{R}(1/n)$ such that u^n is a tougher KS-bargaining opponent than u_0^n. From Lemma 2, u^n must cut u_0^n from below. Hence $\langle u^n, u_0^n \rangle \notin \Psi$.

[15] For *any* given compact set $\mathcal{U}$, if $\underline{x}$ is sufficiently small, u_0 will no longer belong to $\mathcal{U}$. However, neither of the heuristic messages conveyed by the example depend on this inclusion.

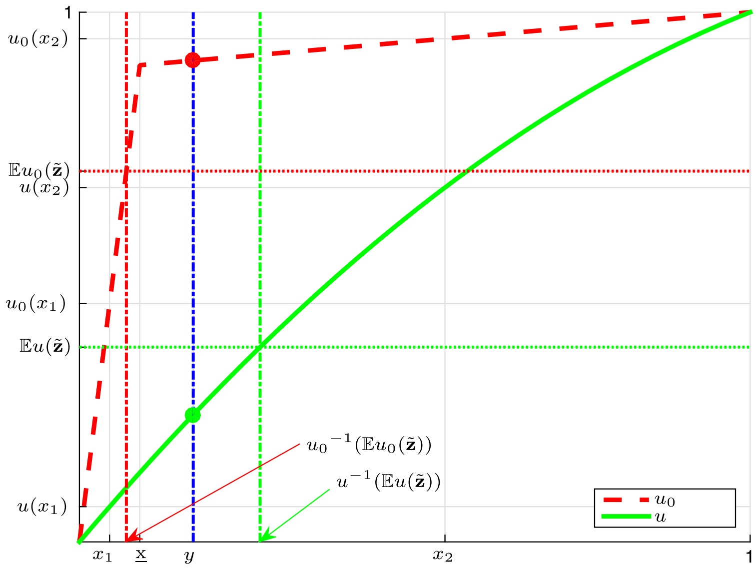

Fig. 2. u is more risk averse than u_0 w.r.t. almost all risks, but a much tougher bargainer.

"virtually all" risks *except* for ones effectively involve negotiation breakdown; but b) is (much) a tougher bargainer than u_0 against all opponents in $\mathcal{U}$, in both the Nash and KS frameworks. On the other hand, the properties that the example exhibits are intuitive: u_0 desperately wants to achieve an agreement, but cares very little about which particular agreement is obtained. Unsurprisingly, against almost all v's, the agreement predicted by both the Nash and KS frameworks will overwhelmingly favor v, delivering u_0 a share in a neighborhood of $\underline{x}$.

The second purpose of Fig. 2 is to illustrate the magnitude of the correspondence $\mathcal{R}(\cdot)$'s implosion at zero. For each $n \in \mathbb{N}$, there is a pair $\langle \bar{u}, u_0^n \rangle \in \mathcal{R}(1/n)$, with u_0^n having the same form as the function u_0 displayed in the figure, specifically $u_0^n(x) = \begin{cases} (n-1)x & \text{if } x < 1/n \\ (n+x-2)/(n-1) & \text{otherwise} \end{cases}$. Clearly, for much of its domain — for example, on $(\underline{x}, 1/2))$ — we have $\bar{u}(\cdot) \ll u_0^n(\cdot)$. On the other hand, from Remark 1, $\langle \bar{u}, u_0 \rangle \in \mathcal{R}(0)$ implies $\bar{u}(\cdot) \geq u_0(\cdot)$. Thus, for any n, $\mathcal{R}(0)$ is a *much* smaller set than $\mathcal{R}(1/n)$. Indeed, for each $\epsilon > 0$, there exists $\delta > 0$ and N such that for $n > N$, $u_0^n(\delta) > 1 - 0.5\epsilon$ while $\langle \bar{u}, u_0 \rangle \in \mathcal{R}(0)$ implies $u_0(\delta) < 0.5\epsilon$. Hence, in the metric on function pairs defined in footnote 12, $\gamma(\langle \bar{u}, u_0^n \rangle, \langle \bar{u}, u_0 \rangle) > 1 - \epsilon$. To summarize, we have established that there is a sequence $\{\langle \bar{u}, u_0^n \rangle\}$ such that for all n, $\langle \bar{u}^n, u_0 \rangle \in \mathcal{R}(1/n)$, and $\lim_n \{\langle \bar{u}^n, u_0 \rangle\}$ is of γ-distance 1 from any pair $\langle \bar{u}, u_0 \rangle \in \mathcal{R}(0)$. The significance of this implosion will be immediately apparent: for any n, the set $\mathcal{R}(1/n)$ is huge relative to $\mathcal{R}(0)$; there is room in the former set for a very diverse array of different kinds of bargainer pairs, including pairs such as $\langle \bar{u}, u_0^n \rangle$, where $\bar{u}(\cdot) \ll u_0^n(\cdot)$, and is much a tougher bargainer than u_0^n. But at the point of implosion, all bargaining pairs $\langle \bar{u}, u_0 \rangle$ are eliminated unless they satisfy $\bar{u}(\cdot) \geq u_0(\cdot)$.

3. Conclusion

The goal of this paper has been to challenge the virtual consensus among bargaining theorists that in axiomatic bargaining theory, comparative bargaining toughness and comparative global risk aversion are inextricably linked. We have argued that the relationship between these two comparisons is in fact quite fragile. To demonstrate this fragility, we weaken the concept of "globally more risk averse than," and compare the relative bargaining toughness of two agents, one of whom is more averse than the other to "almost all" but not all risks. We show that in this context, the almost globally more risk averse agent may be tougher than the other agent against all opponents. More abstractly, we argue that the consensus view regarding the relationship between risk aversion and bargaining toughness results from an exclusive focus on an implosion point of a correspondence that is not upper hemicontinuous.

The discontinuity we have identified can be interpreted in one of two ways. If one considers our notion of "almost globally more risk averse than..." to be closely comparable to "globally more risk averse than...," then one might conclude that the literature has placed excessive emphasis on results such as Propositions 2 & 4. Alternatively, one could take the view that "almost globally more risk averse than..." and "globally more risk averse than..." are really quite different concepts, because there is something qualitatively different about, on the one hand, risk aversion to lotteries that exclusively involve negotiated outcomes, and, on the other, lotteries which assign a positive probability to negotiation breakdown. An implication of this latter view would be that more attention should be devoted to developing a more nuanced understanding of this distinction.

Appendix A

Proof of Lemma 1. Part a): For $z \in [\underline{x}, 1]$,

$$\frac{d}{dz}\left(\frac{\bar{u}'(z)}{u_0'(z)}\right) = \frac{\bar{u}''(z)u_0'(z) - \bar{u}'(z)u_0''(z)}{u_0'(z)^2} = -\frac{\bar{u}'(z)u_0'(z)}{u_0'(z)^2}\left(\frac{-\bar{u}''(z)}{\bar{u}'(z)} - \frac{-u_0''(z)}{u_0'(z)}\right) < 0 \tag{11}$$

Part b) follows immediately from part a). Part a) also implies

$$\text{If } \exists y \in [\underline{x}, 1] \text{ s.t. } \bar{u}'(y) = u_0'(y) \text{ then } \bar{u}'(\cdot) < u_0'(\cdot) \text{ on } (y, 1]. \tag{12}$$

Part c): Note from b) that if $\bar{u}'(y) \le u_0'(y)$, then $\bar{u}'(\cdot) < u_0'(\cdot)$ on $[y, 1)$. From Assumption A2), $\bar{u}(1) = u_0(1)$. Hence,

$$\bar{u}(y) = \bar{u}(1) - \int_y^1 \bar{u}'(y)dy > u_0(1) - \int_y^1 u_0'(y)dy = u_0(y)$$

Part d): Since $\bar{u}(1) = u_0(1)$, $\bar{u}(\underline{x}) = u_0(\underline{x})$ implies $\int_{\underline{x}}^1(\bar{u}'(y) - u_0'(y))dy = 0$. Hence, $\exists \bar{y} \in [\underline{x}, 1)$ s.t. $\bar{u}'(\bar{y}) = u_0'(\bar{y})$. From part b), $\bar{u}'(\cdot) > u_0'(\cdot)$ on $[\underline{x}, \bar{y})$, so that $\bar{u}(\cdot) > u_0(\cdot)$ on $(\underline{x}, \bar{y})$. From (12), $\bar{u}'(\cdot) \le u_0'(\cdot)$ on $[\bar{y}, 1]$. From part c), $\bar{u}(\cdot) > u_0(\cdot)$ on $[\bar{y}, 1)$.
Part e): Assume that $\bar{u}(y) < u_0(y)$, for some $y \in [\underline{x}, 1)$. Since $\bar{u}(1) = u_0(1) = 1$, there exists at least one $y' \in (y, 1]$ such that $\bar{u}(y') = u_0(y')$. Let $\tilde{y}$ denote the smallest such number. If $\tilde{y} = 1$, then $\bar{u}'(\tilde{y}) \ge u_0'(\tilde{y})$ (since otherwise $\exists \epsilon > 0$ such that $\bar{u}'(\cdot) > u_0(\cdot)$ on $(1-\epsilon, 1)$ and hence some $y' < \tilde{y}$ such that $\bar{u}(y') = u_0(y')$). If $\tilde{y} < 1$, then from part c), $\bar{u}'(\tilde{y}) > u_0'(\tilde{y})$. In either case, $\bar{u}$ cuts u_0 from below at $\tilde{y}$.

Proof of Lemma 2.
Sufficiency: Fix $x > x_0^{\mathcal{KS}}$ s.t. $\bar{u}$ cuts u_0 from below at x. From part b) of Lemma 1, $\bar{u}'(\cdot) \ge u_0'(\cdot)$ on $[x_0^{\mathcal{KS}}, x)$, so that $\bar{u}(x_0^{\mathcal{KS}}) < u_0(x_0^{\mathcal{KS}})$, and hence, $\mathcal{KS}(x_0^{\mathcal{KS}}|\bar{u}, v) < \mathcal{KS}(x_0^{\mathcal{KS}}|u_0, v) = 0$. From (1) and (5), $\frac{\partial \mathcal{KS}(\cdot|\bar{u},v)}{\partial x} > 0$. Hence, $x^{\mathcal{KS}} > x_0^{\mathcal{KS}}$.
Necessity: Assume that $x^{\mathcal{KS}} > x_0^{\mathcal{KS}}$. In this case,

$$\bar{u}(x^{\mathcal{KS}}) = v(x^{\mathcal{KS}}) < v(x_0^{\mathcal{KS}}) = u_0(x_0^{\mathcal{KS}}) < u_0(x^{\mathcal{KS}})$$

The two inequalities follow from (1); the two equalities follow from (5). It now follows from part e) of Lemma 1 that $\bar{u}$ cuts u_0 from below at some $x \in (x_0^{\mathcal{KS}}, 1]$.

Proof of Lemma 3. Since $\mathcal{N}(\cdot|\bar{u}, v)$ is strictly concave, $f^{\mathcal{N}}(\bar{u}, v) = \mathcal{N}(\cdot|\bar{u}, v)^{-1}(0) > x_0^{\mathcal{N}}$ iff $\frac{\partial \mathcal{N}(x_0^{\mathcal{N}}|\bar{u},v)}{\partial x} > 0$. Since by definition $\frac{\partial \mathcal{N}(x_0^{\mathcal{N}}|\bar{u},v)}{\partial x} = \frac{\bar{u}_0'(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})} + \frac{v'(x_0^{\mathcal{N}})}{v(x_0^{\mathcal{N}})} := 0$, $\frac{\partial \mathcal{N}(x_0^{\mathcal{N}}|\bar{u},v)}{\partial x} > 0$ iff $\frac{\bar{u}'(x_0^{\mathcal{N}})}{\bar{u}(x_0^{\mathcal{N}})}\frac{\bar{u}'(x_0^{\mathcal{N}})}{\bar{u}(x_0^{\mathcal{N}})} + \frac{v'(x_0^{\mathcal{N}})}{v(x_0^{\mathcal{N}})} > \frac{\bar{u}_0'(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})}$.

Proof of Lemma 4. Fix $x \ge x_0^{\mathcal{N}}$ s.t. $\bar{u}$ cuts u_0 from below at x. By definition, $\bar{u}'(x) \ge u_0'(x)$. From part b) of Lemma 1, $\bar{u}'(\cdot) > u_0'(\cdot)$ on $[x_0^{\mathcal{N}}, x)$. Hence, $\bar{u}(x_0^{\mathcal{N}}) < u_0(x_0^{\mathcal{N}})$. From (3), $\frac{\partial \mathcal{N}(x_0^{\mathcal{N}}|\bar{u},v)}{\partial x} > \frac{\partial \mathcal{N}(x_0^{\mathcal{N}}|u_0,v)}{\partial x} = 0$. Since $\mathcal{N}(\cdot|\bar{u}, v)$ is strictly concave, $\frac{\partial^2 \mathcal{N}(x_0^{\mathcal{N}}|\bar{u},v)}{\partial x^2} < 0$. Hence, $x^{\mathcal{N}} > x_0^{\mathcal{N}}$.

Proof of Proposition 3. From (11), we have that $\frac{d}{dz}\left(\frac{\bar{u}'(\cdot)}{u_0'(\cdot)}\right) < 0$ on $\mathrm{MRA}^{[0,x_0^{\mathcal{N}}]}$. That is, there exists a strictly decreasing function $\alpha(\cdot)$ on $[0, 1]$ such that for all $x \in \mathrm{MRA}^{[0,x_0^{\mathcal{N}}]}$, $\bar{u}'(x) = \alpha(x)\bar{u}_0'(x)$. Therefore we can write $\frac{\bar{u}(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})} = \frac{\int_0^{x_0^{\mathcal{N}}} \bar{u}'(y)dy}{\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy} = \frac{\int_0^{x_0^{\mathcal{N}}} \alpha(y)\bar{u}_0'(y)dy}{\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy}$. From the first mean value theorem for integration, (Wikipedia, 2015), there exists $\hat{x} \in [0, x_0^{\mathcal{N}}]$ such that $\int_0^{x_0^{\mathcal{N}}} \alpha(y)\bar{u}_0'(y)dy = \alpha(\hat{x})\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy$. Since $\alpha(\cdot)$ is strictly decreasing, and $\bar{u}_0'(\cdot) > 0$, it follows that $\int_0^{x_0^{\mathcal{N}}} \alpha(y)\bar{u}_0'(y)dy > \alpha(x_0^{\mathcal{N}})\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy$, and hence that $\hat{x} < x_0^{\mathcal{N}}$. Hence $\frac{\bar{u}(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})} = \alpha(\hat{x})\frac{\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy}{\int_0^{x_0^{\mathcal{N}}} \bar{u}_0'(y)dy} = \alpha(\hat{x}) := \frac{\bar{u}'(\hat{x})}{\bar{u}_0'(\hat{x})} > \frac{\bar{u}'(x_0^{\mathcal{N}})}{\bar{u}_0'(x_0^{\mathcal{N}})}$. Hence $\frac{\bar{u}_0'(x_0^{\mathcal{N}})}{u_0(x_0^{\mathcal{N}})} > \frac{\bar{u}'(x_0^{\mathcal{N}})}{\bar{u}(x_0^{\mathcal{N}})}$. The result now follows from Lemma 3. □

Proof of Proposition 5. Fix $u_0 \in \mathrm{int}(\mathcal{U})$. Since $r^0(\cdot)$ (defined in part b) of Proposition 1) is continuous and takes values on the compact set P, it attains a maximum on P. Hence $\exists c \in \mathbb{R}_+$ s.t. $c > r^0(\cdot)$ on P. For each $\mathbf{z} \in Z$ s.t. $z_1 \in (\underline{x}/2, 1]$, let $\psi(\mathbf{z}) = \exp(\exp(-cz_1))$, and define $f(\mathbf{z}) = \log\Big(\psi\big((1,0)\big)\Big) - \log(\psi(\mathbf{z})) \le 0$. Let g be a smooth extension of f to Z satisfying $f(D) = 0$. For $\boldsymbol{\lambda} \in \mathbb{R}_+$, let $\bar{u}^{\boldsymbol{\lambda}} = u_0 + \boldsymbol{\lambda} g$ (so that $\bar{u}^0 = u_0$). Clearly, the set of functions satisfying Assumption A1) is open (in the sup norm topology) $\bar{u}^{\lambda}$ will also satisfy Assumption A1), if λ is sufficiently small. Moreover, since $u_0 \in \text{int}(\mathcal{U})$, $\bar{u}^{\lambda} \in \mathcal{U}$ if λ is sufficiently small. We now have, $\forall\lambda$.

$$\forall x \in [0,1], \quad \frac{\partial \bar{u}^{\lambda}(x)}{\partial \lambda} = f(x) < 0 \tag{13a}$$

and $\forall x \in [\underline{x}, 1)$:

$$f'(x) = \frac{c}{\exp(cx)} > 0; \quad f''(x) = \frac{-c^2}{\exp(cx)} < 0 \tag{13b}$$

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial \bar{u}^{\lambda}(x)}{\partial x}\right) = f'(x); \quad \frac{\partial}{\partial \lambda}\left(\frac{\partial^2 \bar{u}^{\lambda}(x)}{\partial x^2}\right) = f''(x) \tag{13c}$$

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial \bar{u}^{\lambda}(x)/\partial x}{\bar{u}^{\lambda}(x)}\right)\Bigg|_{\lambda=0} = \frac{f'(x)u_0(x) - f(x)\bar{u}_0'(x)}{u_0(x)^2} > 0 \quad (\text{since } f(x) < 0) \tag{13d}$$

$$\frac{\partial r^{\bar{u}^{\lambda}}(x)}{\partial \lambda} = \frac{\bar{u}_0'(x) f'(x)}{(\bar{u}_0'(x) + \lambda f'(x))^2}\left(c - r^0(x)\right) > 0 \tag{13e}$$

Inequality (13e), together with Proposition 1:b) implies that $\forall\lambda > 0$, $\bar{u}^{\lambda}$ is more risk averse than u_0 on $[\underline{x}, 1]$, and hence that $\langle \bar{u}^{\lambda}, u_0\rangle \in \mathcal{R}(\underline{x})$. Moreover, (13a) implies that for all λ, $\bar{u}^{\lambda}(\cdot) < u_0(\cdot)$. Necessarily, therefore, $\bar{u}^{\lambda}$ cuts u_0 from below at 1. Moreover, since $u_0, v \in \mathcal{U}$, we have from (7) that $\bar{\epsilon} \leq \min(x_0^{\mathcal{N}}, x_0^{\mathcal{KS}})$. Since by assumption, $\underline{x} < \bar{\epsilon}$, Lemmas 2 and 4 are applicable and the result now follows. □

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