

Asymptotic enumeration of Eulerian circuits in graphs with strong mixing properties

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Abstract. We prove an asymptotic formula for the number of Eulerian circuits in graphs with strong mixing properties and with all vertices having even degrees. This number is determined up to a multiplicative error of the form $O(n^{-1/2+\varepsilon})$, where n is the number of vertices.

Keywords: Eulerian circuit, asymptotic analysis, Gaussian integral, algebraic connectivity, Laplacian matrix.

§ 1. Introduction

Let G be a simple connected undirected graph all of whose vertices have an even degree. An *Eulerian circuit* in G is a closed path that uses all edges of G precisely once (see, for example, [1]). Eulerian circuits are regarded as *equivalent* if one of them is a cyclic permutation of the other. Clearly, the size of such an equivalence class is equal to the number of edges of G . Let $EC(G)$ be the number of equivalence classes of Eulerian cycles in G .

The problem of counting the exact number of Eulerian circuits in a simple graph (a graph without loops or multiple edges) is complete for the class $\#P$, that is, the existence of a polynomial algorithm for this problem implies that there is a polynomial algorithm for every problem in $\#P$ and, in particular, that the classes P and NP are equivalent (see [2]). In other words, the problem of counting the exact number of Eulerian cycles is difficult from the viewpoint of complexity theory. We also note that, in contrast to many other hard problems of counting on graphs (see, for example, [3] and [4]), there are still no effective algorithms (not even approximate or probabilistic ones) for counting the number of Eulerian circuits in the general case. Such algorithms are known only for special classes of graphs of low density (see [5], [6]).

An exact expression for the number of Eulerian circuits in the complete graph K_n with odd n is unknown. (It is easily seen that $EC(K_n) = 0$ when n is an even number.) Only the following asymptotic formula (see [7]) has been obtained for odd integers n as $n \rightarrow \infty$:

$$\begin{aligned} EC(K_n) &= 2^{\frac{(n-1)^2}{2}} \pi^{-\frac{n-1}{2}} n^{\frac{n-2}{2}} \left(\left(\frac{n-1}{2} - 1 \right)! \right)^n (1 + O(n^{-\frac{1}{2}+\varepsilon})) \\ &= 2^{\frac{n+1}{2}} \pi^{\frac{1}{2}} e^{-\frac{n^2}{2} + \frac{11}{12}n} n^{\frac{(n-2)(n+1)}{2}} (1 + O(n^{-\frac{1}{2}+\varepsilon})) \end{aligned} \quad (1.1)$$

for every fixed $\varepsilon > 0$.

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The analytic approach of [7], based on representing the result as a multidimensional integral and estimating it as the dimension tends to infinity, was generalized in [8] to find the asymptotic behaviour of the number of Eulerian circuits in simple graphs of large algebraic connectivity. We refer to such graphs as graphs with strong mixing properties.

The mixing properties of a graph can be described by various classical parameters: the algebraic connectivity, the Cheeger constant (isoperimetric number), or the spectral gap between 1 and the second largest eigenvalue of the transition probability matrix for a random walk on the graph. We note that each of these parameters gives rise to an equivalent definition of the class of graphs with strong mixing properties (see [9] for more details).

It was also shown in [9] that, for every fixed $p > 0$, a random graph in Hilbert’s model $G(n, p)$ (each edge of an n -vertex graph occurs independently with probability p) possesses strong mixing properties with probability close to 1 as $n \rightarrow \infty$ (up to a quantity which is exponentially small with respect to n).

In this paper we continue the studies of [7]–[9]. We prove an asymptotic formula for the number of Eulerian circuits in graphs with strong mixing properties. The main result is presented in detail in §2 of the present paper.

The estimate of the number of Eulerian circuits was actually reduced in [8] to an estimate of an n -dimensional integral of Gaussian type. We partly repeat this reduction in §§3, 8 of the present paper. Moreover, in §§4, 6, 7 we estimate integrals of this type. The main result is proved in §5.

An orientation of the edges of a graph is called an *Eulerian orientation* if, at every vertex, the number of incoming edges is equal to the number of outgoing edges. We note that the asymptotic behaviour of the number of Eulerian orientations for graphs with strong mixing properties was determined in [10]. It would appear that the results of [10] and the estimates in the present paper enable one to prove the asymptotic formula for the number of Eulerian orientations that was stated without proof in [9]. We plan to develop results in this direction in a subsequent paper.

§ 2. Main result

Let G be an undirected simple graph with vertex set $VG = \{v_1, v_2, \dots, v_n\}$ and edge set EG . We define an $n \times n$ matrix Q by

$$Q_{jk} = \begin{cases} -1 & \text{if } \{v_j, v_k\} \in EG, \\ d_j & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where $n = |VG|$ and d_j is the degree of $v_j \in VG$. The matrix $Q = Q(G)$ is called the *Laplacian matrix* of G . Its eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are non-negative real numbers, and the number of zero eigenvalues is equal to the number of connected components of G . In particular, $\lambda_1 = 0$. The number $\lambda_2 = \lambda_2(G)$ is called the *algebraic connectivity* of G . Moreover, the following inequalities hold:

$$2 \min_j d_j - n + 2 \leq \lambda_2 \leq \frac{n}{n-1} \min_j d_j. \tag{2.2}$$

Additional information on the spectral properties of the Laplacian matrix can be found, for example, in [11] and [12].

An acyclic connected subgraph of G containing all the vertices of G is called a *spanning tree* of the graph G . By a theorem of Kirchhoff [13] (the matrix tree theorem) we have

$$t(G) = \frac{1}{n} \lambda_2 \lambda_3 \cdots \lambda_n = \det M_{11}, \quad (2.3)$$

where $t(G)$ is the number of spanning trees of G , and the matrix M_{11} is obtained from Q by deleting the first row and the first column.

We say that G is a γ -mixing graph, $\gamma > 0$, if

$$\lambda_2 = \lambda_2(G) \geq \gamma |VG|, \quad (2.4)$$

where λ_2 is the algebraic connectivity of the graph.

The following theorem is the main result of this paper.

Theorem 2.1. *Let G be an undirected simple graph with n vertices v_1, v_2, \dots, v_n , all of which have an even degree. Suppose that G is γ -mixing for some $\gamma > 0$. Then*

$$EC(G) = (1 + \delta(G)) e^{K_{ec}} \left(2^{|EG| - \frac{n-1}{2}} \pi^{-\frac{n-1}{2}} \sqrt{t(G)} \prod_{j=1}^n \left(\frac{d_j}{2} - 1 \right) \right), \quad (2.5)$$

$$K_{ec} = -\frac{1}{4} \sum_{\{v_j, v_k\} \in EG} \left(\frac{1}{d_j + 1} - \frac{1}{d_k + 1} \right)^2,$$

where EG is the set of edges of G , d_j is the degree of v_j , $t(G)$ is the number of spanning trees of G and, for every $\varepsilon > 0$,

$$|\delta(G)| \leq C n^{-\frac{1}{2} + \varepsilon}. \quad (2.6)$$

Here the constant $C > 0$ depends only on γ and ε .

A proof of Theorem 2.1 is given in §5. It is based on the results presented in §§3, 4.

Remark 2.1. In the case of a complete graph we have

$$\lambda_2(K_n) = n, \quad EK_n = \frac{n(n-1)}{2}, \quad t(K_n) = n^{n-2}, \quad K_{ec} = 0. \quad (2.7)$$

We find that the result of Theorem 2.1 is equivalent to (1.1) in the case of a complete graph.

§ 3. Reduction to an integral

A *directed rooted tree* with root v is a connected directed graph T such that $v \in VT$ has no outgoing edges and every other vertex has exactly one outgoing edge. In other words, T is a tree all of whose edges are directed towards v .

Let G be a connected undirected simple graph with n vertices v_1, v_2, \dots, v_n , all having an even degree. Note that for every spanning tree T of G and for every vertex $v_r \in VG$ there is a unique orientation of the edges of T making it a directed rooted tree with root v_r . We write \mathcal{T}_r for the set of directed rooted spanning trees with root v_r that can be obtained in this way.

We recall that (see [8], § 4, formulae (4.6), (4.7))

$$EC(G) = \prod_{j=1}^n \left(\frac{d_j}{2} - 1 \right)! \cdot 2^{|EG|-n+1} \pi^{-n} S, \tag{3.1}$$

where, for every $r \in \mathbb{N}$, $r \leq n$,

$$S = \int_{U_n(\pi/2)} \prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \tan \Delta_{jk}) d\vec{\xi}. \tag{3.2}$$

Here $\Delta_{jk} = \xi_j - \xi_k$ and

$$U_n(\rho) = \{ (\xi_1, \xi_2, \dots, \xi_n)^T \in \mathbb{R}^n : |\xi_j| \leq \rho, j = 1, \dots, n \}. \tag{3.3}$$

Consider the integration domain giving the main contribution to the value of the integral (3.2). In what follows we fix a constant ε such that $0 < \varepsilon \ll 1$. We define

$$V_0 = \left\{ \vec{\xi} \in U_n \left(\frac{\pi}{2} \right) : |\xi_j - \xi_k|_\pi \leq n^{-\frac{1}{2} + \varepsilon}, 1 \leq j, k \leq n \right\}, \tag{3.4}$$

$$|\xi_j - \xi_k|_\pi = \min_{l \in \mathbb{Z}} |\xi_j - \xi_k + \pi l|,$$

and let S_0 be the contribution to S of the domain $\vec{\xi} \in V_0$:

$$S_0 = \frac{1}{n} \sum_{r=1}^n \int_{U_n(\pi/2)} \prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \operatorname{tg} \Delta_{jk}) d\vec{\xi}. \tag{3.5}$$

In this section we use the standard notation $f = O(g)$ as $n \rightarrow \infty$. It means that there are $c, n_0 > 0$ such that $|f| \leq c|g|$ for all $n \geq n_0$.

Under the hypotheses of Theorem 2.1 there is a $c > 0$ (depending only on γ) such that

$$S = (1 + O(\exp(-cn^{2\varepsilon}))) S_0 \tag{3.6}$$

as $n \rightarrow \infty$ (see Theorem 6.3 in [8] for a proof of (3.6)).

We put

$$W = \widehat{Q}^{-1} = (Q + J)^{-1}, \tag{3.7}$$

where Q is the Laplacian matrix and J is the matrix all of whose entries are equal to 1. We also define $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ by

$$\alpha_j = W_{jj}, \tag{3.8}$$

and put

$$R(\vec{\xi}) = \operatorname{tr}(\Lambda(\vec{\xi})W\Lambda(\vec{\xi})W), \tag{3.9}$$

where $\operatorname{tr}(\cdot)$ is the trace of a matrix and $\Lambda(\vec{\xi})$ is the diagonal matrix whose diagonal entries are equal to the corresponding components of the vector $Q\vec{\xi}$.

The sum over \mathcal{T}_r in the integrand of (3.2) can be written as a determinant by the following theorem from [14], which generalizes the aforesaid Kirchhoff matrix tree theorem.

Theorem 3.1. *Let w_{jk} ($1 \leq j, k \leq n, j \neq k$) be arbitrary. We define an $n \times n$ matrix A by*

$$A_{jk} = \begin{cases} -w_{jk} & \text{if } j \neq k, \\ \sum_{r \neq j} w_{jr} & \text{if } k = j, \end{cases} \tag{3.10}$$

where the sum is taken over $1 \leq r \leq n$ and $r \neq j$. For every r such that $1 \leq r \leq n$, let M_r be the principal minor of A obtained by removing the r -th column and r -th row. Then

$$\det M_r = \sum_T \prod_{(v_j, v_k) \in ET} w_{jk}, \tag{3.11}$$

where the sum is taken over all directed rooted trees T with vertex set $VT = \{v_1, v_2, \dots, v_n\}$ and root v_r .

Using formulae (3.2) and (3.6), Theorem 3.1 and the Taylor series expansion of $\cos \Delta_{jk}$ and $\tan \Delta_{jk}$ in the domain V_0 , one can obtain the following proposition.

Proposition 3.1. *Suppose that the hypotheses of Theorem 2.1 hold. Then, as $n \rightarrow \infty$,*

$$S_0 = (1 + O(n^{-\frac{1}{2} + 6\varepsilon})) 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} n^{-1} \det \widehat{Q} \text{Int}, \tag{3.12}$$

$$\text{Int} = \int_{U_n(n^{-1/2+\varepsilon})} \exp\left(i \vec{\xi}^T Q \vec{\alpha} - \frac{1}{2} \vec{\xi}^T \widehat{Q} \vec{\xi} - \frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + \frac{1}{2} R(\vec{\xi})\right) d\vec{\xi}, \tag{3.13}$$

where \widehat{Q} , $\vec{\alpha}$ and $R(\vec{\xi})$ are defined in (3.7), (3.8) and (3.9) respectively.

Proposition 3.1 will be proved in § 8. Actually, this proof was implicitly given in [8], Lemma 5.3.

Thus we can see that to prove Theorem 2.1 it remains only to estimate the integral Int in (3.12).

§ 4. Asymptotic estimates of integrals

We fix constants $a, b, \varepsilon > 0$. In this section we use the notation $f = O(g)$ meaning that $|f| \leq c|g|$ for some $c > 0$ depending only on a, b and ε .

For every real $p \geq 1$ and any vector $\vec{x} \in \mathbb{R}^n$ we put

$$\|\vec{x}\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}}. \tag{4.1}$$

When $p = \infty$, we put

$$\|\vec{x}\|_\infty = \max_j |x_j|. \tag{4.2}$$

The matrix norm corresponding to the p -norm of vectors is given by

$$\|A\|_p = \sup_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p}. \tag{4.3}$$

If A is the matrix of a self-adjoint operator (a symmetric matrix), then the norm $\|A\|_2$ is equal to the maximum modulus of eigenvalues of A . Therefore for $p \geq 1$ we have

$$\|A\|_p \geq \|A\|_2. \tag{4.4}$$

For invertible matrices we define the condition number

$$\mu_p(A) = \|A\|_p \|A^{-1}\|_p \geq \|AA^{-1}\|_p = 1. \tag{4.5}$$

Let I be the $n \times n$ identity matrix and let $A = I + X$ be a matrix satisfying the following conditions:

- (i) A is a positive definite symmetric matrix,
- (ii) $|X_{jk}| \leq \frac{a}{n}$, $X_{jj} = 0$, $\|A^{-1}\|_2 \leq b$.

Note that

$$\|A^{-1}\|_2^{-1} \leq \|A\|_2 \leq \|A\|_\infty = \|A\|_1 = \max_j \sum_{k=1}^n |A_{jk}| = O(1). \tag{4.6}$$

We recall (from Lemma 3.2 in [10]) that, under the assumptions (i), (ii),

$$\mu_\infty(A) = \mu_1(A) = O(\mu_2(A)). \tag{4.7}$$

Using (4.4), (4.6), (4.7), we obtain the following lemma.

Lemma 4.1. *Suppose that A satisfies the conditions (i), (ii). Then*

$$\|A^{-1}\|_\infty = \|A^{-1}\|_1 = O(1), \tag{4.8}$$

$$|X'_{jk}| = O(n^{-1}), \tag{4.9}$$

where

$$X' = A^{-1} - I = A^{-1}(I - A) = -A^{-1}X. \tag{4.10}$$

We use the following notation:

$$\langle g \rangle_{F,\Omega} = \int_{\Omega} g(\vec{\theta}) e^{F(\vec{\theta})} d\vec{\theta}, \tag{4.11}$$

where g, F are some functions defined on \mathbb{R}^n . For $r > 0$ we put

$$\langle g \rangle_{F,r} = \langle g \rangle_{F,U_n(rn^\varepsilon)}, \tag{4.12}$$

where

$$U_n(\rho) = \{(\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n : |\theta_j| \leq \rho, j = 1, \dots, n\}. \tag{4.13}$$

We take a function F of the following form:

$$F(\vec{\theta}) = -\vec{\theta}^T A \vec{\theta} + H(\vec{\theta}), \tag{4.14}$$

where A satisfies (i), (ii) and H satisfies the following conditions, which will be used later:

$$H(\vec{\theta}) \leq c_1 \frac{\vec{\theta}^T A \vec{\theta}}{n}, \tag{4.15}$$

$$\left\| \frac{\partial H(\vec{\theta})}{\partial \vec{\theta}} \right\|_\infty \leq c_2 \frac{\|\vec{\theta}\|_\infty^3 + \|\vec{\theta}\|_\infty}{n}. \tag{4.16}$$

In the case when $H \equiv 0$, we use the following notation:

$$\langle g \rangle_\Omega = \langle g \rangle_{F,\Omega}, \quad \langle g \rangle_r = \langle g \rangle_{F,r}, \quad \langle g \rangle = \langle g \rangle_{+\infty}. \tag{4.17}$$

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $U_n(r_1 n^\varepsilon) \subset \Omega \subset U_n(r_2 n^\varepsilon)$ for some $r_2 > r_1 > 0$. Suppose that A satisfies (i), (ii) and the assumptions (4.15), (4.16) hold for some $c_1, c_2 > 0$. Then*

$$\langle 1 \rangle_\Omega = (1 + O(\exp(-c_3 n^{2\varepsilon}))) \langle 1 \rangle, \tag{4.18}$$

$$\langle 1 \rangle_{F,\Omega} = O(\langle 1 \rangle), \tag{4.19}$$

$$\langle \theta_k^2 \rangle_{F,\Omega} = \frac{1}{2} \langle 1 \rangle_{F,\Omega} + O(n^{-1+4\varepsilon}) \langle 1 \rangle, \tag{4.20}$$

$$\langle \theta_k^4 \rangle_{F,\Omega} = \frac{3}{4} \langle 1 \rangle_{F,\Omega} + O(n^{-1+7\varepsilon}) \langle 1 \rangle \tag{4.21}$$

and, for $k \neq l$,

$$\langle \theta_k \theta_l \rangle_{F,\Omega} = O(n^{-1+5\varepsilon}) \langle 1 \rangle, \tag{4.22}$$

$$\langle \theta_k \theta_l^3 \rangle_{F,\Omega} = O(n^{-1+7\varepsilon}) \langle 1 \rangle, \tag{4.23}$$

$$\langle \theta_k^2 \theta_l^2 \rangle_{F,\Omega} = \frac{1}{4} \langle 1 \rangle_{F,\Omega} + O(n^{-1+7\varepsilon}) \langle 1 \rangle, \tag{4.24}$$

where F is defined in (4.14) and $c_3 = c_3(r_1, r_2, c_1, c_2, a, b, \varepsilon) > 0$.

Moreover, for every $\vec{p} = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ with $\|\vec{p}\|_\infty = O(n^{-1/2})$ we have

$$\begin{aligned} \langle \theta_k e^{i\vec{\theta}^T \vec{p} - ip_k \theta_k} \rangle_{F,\Omega} &= \frac{i}{2} \sum_{j \neq k, j \leq n} p_j (A^{-1})_{jk} \langle e^{i\vec{\theta}^T \vec{p} - ip_k \theta_k} \rangle_{(F - \frac{1}{2} p_k^2 \theta_k^2), \Omega} \\ &\quad + O(n^{-1+5\varepsilon}) \langle 1 \rangle, \end{aligned} \tag{4.25}$$

where $(A^{-1})_{jk}$ is the (j, k) -th entry of the matrix A^{-1} .

A proof of Lemma 4.2 will be given in § 6.

§ 5. Proof of Theorem 2.1

The Laplacian matrix Q of G (as defined in (2.1)) has an eigenvector $(1, 1, \dots, 1)^T$ with eigenvalue $\lambda_1 = 0$. We put $\widehat{Q} = Q + J$, where J is the matrix all of whose entries are equal to 1. Note that Q and \widehat{Q} have the same sets of eigenvalues and eigenvectors except for the eigenvalue corresponding to the eigenvector $(1, 1, \dots, 1)^T$ (this eigenvalue is equal to 0 for Q and n for \widehat{Q}). Using (2.3), we find that

$$t(G) = \frac{1}{n} \lambda_2 \lambda_3 \cdots \lambda_n = \frac{\det \widehat{Q}}{n^2}. \tag{5.1}$$

We obtain from (4.4) that

$$\lambda_n = \|Q\|_2 \leq \|\widehat{Q}\|_2 \leq \|\widehat{Q}\|_1 = \max_j \sum_{k=1}^n |\widehat{Q}_{jk}| = n. \tag{5.2}$$

By the hypothesis of Theorem 2.1 we have

$$\|\widehat{Q}^{-1}\|_2 = \frac{1}{\lambda_2} \leq \frac{1}{\gamma n}. \tag{5.3}$$

Using (2.2), we get

$$n - 1 \geq d_j \geq \lambda_2 \frac{n - 1}{n} \geq \gamma(n - 1), \tag{5.4}$$

where d_j is the degree of the vertex v_j .

Consider the integral from Proposition 3.1:

$$\begin{aligned} \text{Int} &= \int_{U_n(n^{-1/2+\varepsilon})} \exp\left(i \vec{\xi}^T \vec{\beta} - \frac{1}{2} \vec{\xi}^T \widehat{Q} \vec{\xi} - \frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + \frac{R(\vec{\xi})}{2}\right) d\vec{\xi}, \\ R(\vec{\xi}) &= \text{tr}(\Lambda(\vec{\xi}) \widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1}), \quad \vec{\beta} = Q\vec{\alpha}, \end{aligned} \tag{5.5}$$

where $\Lambda(\vec{\xi})$ is the diagonal matrix whose diagonal entries are equal to the corresponding components of the vector $Q\vec{\xi}$, and where $\vec{\alpha}$ is the vector whose components are equal to the corresponding diagonal entries of \widehat{Q}^{-1} .

We define $\vec{\xi}(\vec{\theta}) = (\xi_1(\vec{\theta}), \xi_2(\vec{\theta}), \dots, \xi_n(\vec{\theta}))^T$ by putting

$$\theta_k = \sqrt{\frac{d_k + 1}{2}} \xi_k. \tag{5.6}$$

Then (5.5) can be rewritten in the notation of § 4:

$$\text{Int} = \langle e^{i\vec{p}^T \vec{\theta}} \rangle_{F, \Omega} \prod_{j=1}^n \frac{1}{\sqrt{(d_j + 1)/2}}, \tag{5.7}$$

where $\vec{p} = (p_1, p_2, \dots, p_n)$,

$$p_k = \frac{\beta_k}{\sqrt{(d_j + 1)/2}}, \tag{5.8}$$

$$F(\vec{\theta}) = -\vec{\theta}^T A \vec{\theta} + H(\vec{\theta}), \tag{5.9}$$

$$\vec{\theta}^T A \vec{\theta} = \frac{1}{2} \vec{\xi}(\vec{\theta})^T \widehat{Q} \vec{\xi}(\vec{\theta}), \quad A_{jk} = \frac{1}{\sqrt{(d_j + 1)(d_k + 1)}} \widehat{Q}_{jk}, \tag{5.10}$$

$$H(\theta) = -\frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + \frac{R(\vec{\xi}(\vec{\theta}))}{2}, \tag{5.11}$$

$$\Omega = \{\vec{\theta} \in \mathbb{R}^n : \vec{\xi}(\vec{\theta}) \in U_n(n^{-1/2+\varepsilon})\}. \tag{5.12}$$

Using Lemma 4.2, we shall express the integral Int from (5.7) in terms of

$$\langle 1 \rangle = \int_{\mathbb{R}^n} e^{-\vec{\theta}^T A \vec{\theta}} d\vec{\theta} = \frac{\pi^{n/2}}{\sqrt{\det A}} = \frac{(2\pi)^{n/2}}{\sqrt{\det \widehat{Q}}} \prod_{j=1}^n \sqrt{\frac{d_j + 1}{2}}. \tag{5.13}$$

We proceed as follows. First we must verify that all the hypotheses of Lemma 4.2 hold. Then we successively eliminate the oscillatory term $e^{i\vec{p}^T \vec{\theta}}$, the quadratic term $\frac{R(\vec{\xi}(\vec{\theta}))}{2}$ and the residual term $-\frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4$.

In the rest of this section we use the notation $f = O(g)$ meaning that $|f| \leq c|g|$ for some constant $c > 0$ depending only on γ and ε .

5.1. Verification of the hypotheses of Lemma 4.2. Combining (5.3), (5.4), (5.6), (5.10) and (5.12), we obtain the following assertion.

Assertion 5.1. *Consider the matrix A and the domain Ω defined in (5.10) and (5.12). Then A satisfies the conditions (i), (ii) in § 4 and $U_n(r_1 n^\varepsilon) \subset \Omega \subset U_n(r_2 n^\varepsilon)$ for some $a, b, r_1, r_2 > 0$ depending only on γ .*

We define $\vec{e}^{(k)} = (e_1^{(k)}, \dots, e_n^{(k)})^T \in \mathbb{R}^n$ by putting $e_j^{(k)} = \delta_{jk}$, where δ_{jk} is Kronecker's delta. Since $\Lambda(\vec{\xi})$ and $\text{tr}(\cdot)$ are linear, we find that

$$R(\vec{\xi}(\vec{\theta})) = \vec{\xi}(\vec{\theta})^T R \vec{\xi}(\vec{\theta}) = \vec{\theta}^T S \vec{\theta}, \tag{5.14}$$

where

$$\begin{aligned} R_{jk} &= \text{tr}(\Lambda(\vec{e}^{(j)}) \widehat{Q}^{-1} \Lambda(\vec{e}^{(k)}) \widehat{Q}^{-1}), \\ S_{jk} &= \frac{R_{jk}}{\sqrt{(d_j + 1)/2} \sqrt{(d_k + 1)/2}}. \end{aligned} \tag{5.15}$$

We use the following inequalities for $n \times n$ matrices X, Y :

$$\begin{aligned} |\text{tr}(XY)| &\leq \|X\|_{HS} \|Y\|_{HS}, \\ \|XY\|_{HS} &\leq \|X\|_{HS} \|Y^T\|_2, \end{aligned} \tag{5.16}$$

where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm:

$$\|X\|_{HS} = \sqrt{\sum_{j=1}^n \sum_{k=1}^n |X_{jk}|^2}. \tag{5.17}$$

Combining (5.2), (5.3) and (5.14)–(5.16), we find that

$$\begin{aligned} \vec{\xi}_1^T R \vec{\xi}_2 &\leq \|\Lambda(\vec{\xi}_1) \widehat{Q}^{-1}\|_{HS} \|\Lambda(\vec{\xi}_2) \widehat{Q}^{-1}\|_{HS} \leq \|\Lambda(\vec{\xi}_1)\|_{HS} \|\Lambda(\vec{\xi}_2)\|_{HS} \|\widehat{Q}^{-1}\|_2^2 \\ &= \|Q \vec{\xi}_1\|_2 \|Q \vec{\xi}_2\|_2 \|\widehat{Q}^{-1}\|_2^2 \leq \|Q\|_2^2 \|\widehat{Q}^{-1}\|_2^2 \|\vec{\xi}_1\|_2 \|\vec{\xi}_2\|_2 = O(1) \|\vec{\xi}_1\|_2 \|\vec{\xi}_2\|_2. \end{aligned} \tag{5.18}$$

Using (5.2), (5.10), (5.11) and (5.18), we obtain

$$\begin{aligned} H(\vec{\theta}) &\leq R(\vec{\xi}(\vec{\theta})) = \vec{\xi}(\vec{\theta})^T R \vec{\xi}(\vec{\theta}) = O(1) \|\vec{\xi}(\vec{\theta})\|_2^2 = O(1) \frac{\vec{\xi}(\vec{\theta})^T \widehat{Q} \vec{\xi}(\vec{\theta})}{n} \\ &= O(n^{-1}) \vec{\theta}^T A \vec{\theta}. \end{aligned} \tag{5.19}$$

Let $(\widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1})_{kk}$ be the (k, k) -th entry of the matrix $\widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1}$. For every $k, 1 \leq k \leq n$, we have

$$\begin{aligned} \frac{\partial R(\vec{\xi})}{\partial \xi_k} &= 2 \text{tr} \left(\frac{\partial \Lambda(\vec{\xi})}{\partial \xi_k} \widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1} \right) = 2 \text{tr}(\Lambda(\vec{e}^{(k)}) \widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1}) \\ &= 2d_k (\widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1})_{kk} + 2 \text{tr}(\widetilde{\Lambda} \widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1}), \end{aligned} \tag{5.20}$$

where $\widetilde{\Lambda}$ is the diagonal matrix with entries $\widetilde{\Lambda}_{jj} = \Lambda(\vec{e}^{(k)})_{jj}$ for $j \neq k$ and $\widetilde{\Lambda}_{kk} = 0$ on the diagonal. In particular,

$$\|\widetilde{\Lambda}\|_2 \leq 1. \tag{5.21}$$

Since $\Lambda(\vec{\xi})$ is a diagonal matrix, we get

$$|d_k(\widehat{Q}^{-1}\Lambda(\vec{\xi})\widehat{Q}^{-1})_{kk}| = |d_k\Lambda(\vec{\xi})_{kk}| \|(\widehat{Q}^{-1})_k\|_2^2 \leq d_k\|\Lambda(\vec{\xi})\|_2 \|(\widehat{Q}^{-1})_k\|_2^2, \tag{5.22}$$

where $(\widehat{Q}^{-1})_k$ is the k -th column of the matrix \widehat{Q}^{-1} . Note that

$$\|(\widehat{Q}^{-1})_k\|_2 \leq \|\widehat{Q}^{-1}\|_2 \|\widehat{Q}(\widehat{Q}^{-1})_k\|_2 = \|\widehat{Q}^{-1}\|_2. \tag{5.23}$$

We also observe that

$$\|\Lambda(\vec{\xi})\|_2 = \|Q\vec{\xi}\|_\infty \leq 2n\|\vec{\xi}\|_\infty. \tag{5.24}$$

Combining (5.3), (5.4) and (5.21)–(5.24), we get

$$\begin{aligned} |\text{tr}(\widetilde{\Lambda}\widehat{Q}^{-1}\Lambda(\vec{\xi})\widehat{Q}^{-1})| &\leq n\|\widetilde{\Lambda}\widehat{Q}^{-1}\Lambda(\vec{\xi})\widehat{Q}^{-1}\|_2 \leq n\|\widetilde{\Lambda}\|_2 \|\Lambda(\vec{\xi})\|_2 \|\widehat{Q}^{-1}\|_2^2 = O(1)\|\vec{\xi}\|_\infty, \\ d_k(\widehat{Q}^{-1}\Lambda(\vec{\xi})\widehat{Q}^{-1})_{kk} &= O(1)\|\vec{\xi}\|_\infty. \end{aligned} \tag{5.25}$$

Using (5.4), (5.6), (5.14), (5.20) and (5.25) for all k such that $1 \leq k \leq n$, we can see that

$$\begin{aligned} 2\|R\vec{\xi}\|_\infty &= \left\| \frac{\partial R(\vec{\xi})}{\partial \vec{\xi}} \right\|_\infty = O(1)\|\vec{\xi}\|_\infty, \\ 2\|S\vec{\theta}\|_\infty &= \left\| \frac{\partial R(\vec{\xi}(\vec{\theta}))}{\partial \vec{\theta}} \right\|_\infty = O(n^{-1})\|\vec{\theta}\|_\infty. \end{aligned} \tag{5.26}$$

For every k such that $1 \leq k \leq n$, we have

$$\left| \frac{\partial}{\partial \xi_k} \sum_{\{v_j, v_l\} \in EG} \Delta_{jl}^4 \right| \leq 4 \sum_{j=1}^n |\Delta_{jk}|^3 = O(n)\|\vec{\xi}\|_\infty^3. \tag{5.27}$$

Combining (5.6) and (5.27), we find that

$$\left\| \frac{\partial}{\partial \vec{\theta}} \sum_{\{v_j, v_l\} \in EG} \Delta_{jl}^4 \right\|_\infty = O(n^{-1})\|\vec{\theta}\|_\infty^3. \tag{5.28}$$

Using (4.9), the equality

$$(\widehat{Q}^{-1})_{jk} = \frac{(A^{-1})_{jk}}{\sqrt{(d_j + 1)/2}\sqrt{(d_k + 1)/2}} \tag{5.29}$$

and (5.4), we get

$$\|\vec{p}\|_\infty = \sup_{1 \leq k \leq n} \frac{\beta_k}{\sqrt{(d_j + 1)/2}} = O(n^{-\frac{1}{2}})\|Q\alpha\|_\infty = O(n^{-\frac{1}{2}}), \tag{5.30}$$

where the vector \vec{p} is defined in (5.8).

Putting together (5.10), (5.19), (5.26), (5.28) and (5.30), we can see from Assertion 5.1 that all the hypotheses of Lemma 4.2 with data (5.8)–(5.12) hold for some constants $a, b, r_1, r_2, c_1, c_2 > 0$ depending only on γ .

5.2. The oscillatory term. We define $\vec{p}^{(k)} = (p_1^{(k)}, p_2^{(k)}, \dots, p_n^{(k)})^T \in \mathbb{R}^n$ as follows:

$$p_j^{(k)} = \begin{cases} 0 & \text{for } j \leq k, \\ p_j & \text{for } j > k. \end{cases} \tag{5.31}$$

Put

$$F^{(k)}(\vec{\theta}) = F(\vec{\theta}) - \frac{1}{2} \sum_{j=1}^k p_j^2 \theta_j^2. \tag{5.32}$$

Note that

$$F^{(0)}(\vec{\theta}) = F(\vec{\theta}). \tag{5.33}$$

Using (5.30) and the equality

$$\frac{\partial}{\partial \theta_k} \sum_{j=1}^n p_j^2 \theta_j^2 = 2p_k^2 \theta_k = O(n^{-1}) \|\vec{\theta}\|_\infty, \tag{5.34}$$

we can see that all the hypotheses of Lemma 4.2 hold (if we take $F^{(k)}(\vec{\theta})$ for the function $F(\vec{\theta})$ and $\vec{p}^{(k)}$ for the vector \vec{p}) for some constants $a, b, r_1, r_2, c_1, c_2 > 0$ depending only on γ .

Note that

$$\vec{\theta} \in \Omega \implies \|\vec{\theta}\|_\infty = O(n^\epsilon). \tag{5.35}$$

Taking (5.30) into account and using the Taylor series expansion, we get

$$e^{ip_k \theta_k} = e^{-\frac{1}{2} p_k^2 \theta_k^2} + ip_k \theta_k + O(n^{-\frac{3}{2} + 3\epsilon}), \quad \vec{\theta} \in \Omega. \tag{5.36}$$

Using (5.30) and (4.19), (4.25) for $F^{(k-1)}$ and $\vec{p}^{(k-1)}$, we find that

$$\begin{aligned} ip_k \langle \theta_k e^{i\vec{\theta}^T \vec{p}^{(k)}} \rangle_{F^{(k-1)}, \Omega} \\ = -\frac{p_k}{2} \sum_{j=k+1}^n p_j (A^{-1})_{jk} \langle e^{i\vec{\theta}^T \vec{p}^{(k)}} \rangle_{F^{(k)}, \Omega} + O(n^{-\frac{3}{2} + 5\epsilon}) \langle 1 \rangle. \end{aligned} \tag{5.37}$$

Combining (4.19), (5.36) and (5.37), we get

$$\begin{aligned} \langle e^{i\vec{\theta}^T \vec{p}^{(k-1)}} \rangle_{F^{(k-1)}, \Omega} = \left(1 - \frac{p_k}{2} \sum_{j=k+1}^n p_j (A^{-1})_{jk} \right) \langle e^{i\vec{\theta}^T \vec{p}^{(k)}} \rangle_{F^{(k)}, \Omega} \\ + O(n^{-\frac{3}{2} + 5\epsilon}) \langle 1 \rangle. \end{aligned} \tag{5.38}$$

Using (4.9), (5.8), (5.10) and (5.30), we also obtain

$$\begin{aligned} p_j p_k (A^{-1})_{jk} = \beta_j \beta_k (\widehat{Q}^{-1})_{jk} = O(n^{-2}), \quad j \neq k, \\ p_k \sum_{j \neq k, j \leq n} p_j (A^{-1})_{jk} = O(n^{-1}). \end{aligned} \tag{5.39}$$

By (5.33), (5.39) and (5.38) for $k = 1, 2, \dots, n$, we have

$$\langle e^{i\vec{\theta}^T \vec{p}} \rangle_{F, \Omega} = C_1 \langle 1 \rangle_{F^{(n)}, \Omega} + O(n^{-\frac{1}{2} + 5\epsilon}) \langle 1 \rangle, \tag{5.40}$$

where

$$\begin{aligned}
 C_1 &= \exp\left(-\sum_{k=1}^{n-1} \sum_{j=k+1}^n \beta_k(\widehat{Q}^{-1})_{jk}\beta_j\right) \\
 &= \prod_{k=1}^{n-1} \left(1 - \frac{p_k}{2} \sum_{j=k+1}^n (\widehat{A}^{-1})_{jk}p_j\right) + O(n^{-1}) = O(1).
 \end{aligned}
 \tag{5.41}$$

Taking (5.30) into account and using the Taylor series expansion, we get

$$e^{-\frac{1}{2}p_k^2\theta_k^2} = 1 - \frac{1}{2}p_k^2\theta_k^2 + O(n^{-2+4\varepsilon}), \quad \vec{\theta} \in \Omega.
 \tag{5.42}$$

Combining (5.42) and (4.19), (4.20) for $F^{(k)}$, we obtain

$$\langle 1 \rangle_{F^{(k)}, \Omega} = \left(1 - \frac{1}{4}p_k^2\right) \langle 1 \rangle_{F^{(k-1)}, \Omega} + O(n^{-2+4\varepsilon}) \langle 1 \rangle.
 \tag{5.43}$$

Using (5.30), (5.33) and (5.43) for $k = 1, 2, \dots, n$, we find that

$$\langle 1 \rangle_{F^{(n)}, \Omega} = C_2 \langle 1 \rangle_{F, \Omega} + O(n^{-1+4\varepsilon}) \langle 1 \rangle,
 \tag{5.44}$$

where

$$C_2 = \exp\left(-\sum_{k=1}^n \frac{\beta_k^2}{2(d_k + 1)}\right) = \prod_{k=1}^n \left(1 - \frac{1}{4}p_k^2\right) + O(n^{-1}) = O(1).
 \tag{5.45}$$

5.3. The quadratic term. We define $\vec{\theta}^k = (\theta_1^k, \theta_2^k, \dots, \theta_n^k)^T \in \mathbb{R}^n$ as follows:

$$\theta_j^k = \begin{cases} 0 & \text{for } j \leq k, \\ \theta_j & \text{for } j > k. \end{cases}
 \tag{5.46}$$

Put

$$F^k(\vec{\theta}) = -\vec{\theta}^T A \vec{\theta} + H^k(\vec{\theta}), \quad H^k(\vec{\theta}) = -\frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + \frac{1}{2} R(\vec{\xi}(\vec{\theta}^k)).
 \tag{5.47}$$

Arguing as in §5.1, we can see that all the hypotheses of Lemma 4.2 hold (in the case when $F^k(\vec{\theta})$ is taken in place of $F(\vec{\theta})$) for some $a, b, r_1, r_2, c_1, c_2 > 0$ depending only on γ .

Note that

$$F^0(\vec{\theta}) = F(\vec{\theta}),
 \tag{5.48}$$

$$R(\vec{\xi}(\vec{\theta}^k)) = \vec{\xi}(\vec{\theta}^k)^T R \vec{\xi}(\vec{\theta}^k) = (\vec{\theta}^k)^T S \vec{\theta}^k,
 \tag{5.49}$$

where R, S are the same matrices as in (5.14), (5.15).

Combining (5.47), Lagrange’s mean value theorem and (5.26), we get

$$\begin{aligned}
 |F^{k-1}(\vec{\theta}) - F^k(\vec{\theta})| &= \frac{1}{2} |R(\vec{\xi}(\vec{\theta}^{k-1})) - R(\vec{\xi}(\vec{\theta}^k))| \\
 &= \frac{1}{2} \left| \frac{\partial R(\vec{\xi}(\vec{\theta}^*))}{\partial \theta_k} \theta_k \right| = O(n^{-1}) \|\vec{\theta}\|_\infty^2,
 \end{aligned}
 \tag{5.50}$$

where $\vec{\theta}^*$ lies in the interval between $\vec{\theta}^{k-1}$ and $\vec{\theta}^k$. Using (5.35), (5.50) and the Taylor series expansion, we find that

$$\begin{aligned}
 e^{F^{k-1}(\vec{\theta}) - F^k(\vec{\theta})} &= 1 + F^{k-1}(\vec{\theta}) - F^k(\vec{\theta}) + O(n^{-2+4\epsilon}) \\
 &= 1 + \frac{1}{2} S_{kk} \theta_k^2 + \sum_{j=k+1}^n S_{jk} \theta_j \theta_k + O(n^{-2+4\epsilon}), \quad \vec{\theta} \in \Omega.
 \end{aligned}
 \tag{5.51}$$

Using (5.4), (5.15) and (5.18), we can see that

$$S_{kk} = \frac{(\vec{e}^{(k)})^T R \vec{e}^{(k)}}{(d_k + 1)/2} = O(n^{-1}) \|\vec{e}^{(k)}\|_2^2 = O(n^{-1}).
 \tag{5.52}$$

Since (5.26) implies that $\|S\|_\infty = O(n^{-1})$, we obtain from (4.22) that

$$\begin{aligned}
 \left| \sum_{j=k+1}^n S_{jk} \langle \theta_j \theta_k \rangle_{F^{k-1}, \Omega} \right| &\leq \left| \sup_{k < j \leq n} \langle \theta_j \theta_k \rangle_{F^{k-1}, \Omega} \right| \sum_{j=k+1}^n |S_{jk}| \\
 &= O(n^{-1+5\epsilon}) \|S\|_\infty = O(n^{-2+5\epsilon}).
 \end{aligned}
 \tag{5.53}$$

Combining (4.19), (4.20), (5.51) and (5.53), we find that

$$\langle 1 \rangle_{F^{k-1}, \Omega} = \left(1 + \frac{1}{4} S_{kk} \right) \langle 1 \rangle_{F^k, \Omega} + O(n^{-2+5\epsilon}) \langle 1 \rangle.
 \tag{5.54}$$

Using (5.48), (5.52) and (5.54) for $k = 1, 2, \dots, n$, we get

$$\langle 1 \rangle_{F, \Omega} = C_3 \langle 1 \rangle_{F^n, \Omega} + O(n^{-1+5\epsilon}) \langle 1 \rangle,
 \tag{5.55}$$

where

$$C_3 = \exp \left(\sum_{k=1}^n \frac{R_{kk}}{2(d_k + 1)} \right) = \prod_{k=1}^n \left(1 + \frac{1}{4} S_{kk} \right) + O(n^{-1}) = O(1).
 \tag{5.56}$$

5.4. The residual term. For every subset Θ of the edge set EG we define

$$\begin{aligned}
 F_\Theta(\vec{\theta}) &= -\vec{\theta}^T A \vec{\theta} + H_\Theta(\vec{\theta}), \\
 H_\Theta(\vec{\theta}) &= -\frac{1}{12} \sum_{\{v_j, v_k\} \in \Theta} \Delta_{jk}^4.
 \end{aligned}
 \tag{5.57}$$

Arguing as in §5.1, we can see that all the hypotheses of Lemma 4.2 hold (in the case when $F_\Theta(\vec{\theta})$ is taken in place of $F(\vec{\theta})$) for some constants $a, b, r_1, r_2, c_1, c_2 > 0$ depending only on γ .

Note that

$$F_{EG}(\vec{\theta}) = F^n(\vec{\theta}), \tag{5.58}$$

$$\begin{aligned} \Delta_{jk}^4 = & \frac{4\theta_j^4}{(d_j + 1)^2} - 4 \frac{4\theta_j^3\theta_k}{(d_j + 1)^{3/2}(d_k + 1)^{1/2}} + 6 \frac{4\theta_j^2\theta_k^2}{(d_j + 1)(d_k + 1)} \\ & - 4 \frac{4\theta_k^3\theta_j}{(d_k + 1)^{3/2}(d_j + 1)^{1/2}} + \frac{4\theta_k^4}{(d_k + 1)^2}. \end{aligned} \tag{5.59}$$

Combining (4.21), (4.23), (4.24), (5.4) and (5.59), we get

$$\begin{aligned} \langle \Delta_{jk}^4 \rangle_{F_{\Theta}, \Omega} = & \left(\frac{3}{4} \frac{4}{(d_j + 1)^2} + \frac{6}{4} \frac{4}{(d_j + 1)(d_k + 1)} \right. \\ & \left. + \frac{3}{4} \frac{4}{(d_k + 1)^2} \right) \langle 1 \rangle_{F_{\Theta}, \Omega} + O(n^{-3+7\epsilon}) \langle 1 \rangle. \end{aligned} \tag{5.60}$$

Using (4.19) and (5.60), we find that

$$\langle e^{-\frac{1}{2}\Delta_{jk}^4} \rangle_{F_{\Theta}, \Omega} = P_{jk} \langle 1 \rangle_{F_{\Theta}, \Omega} + O(n^{-3+7\epsilon}) \langle 1 \rangle, \tag{5.61}$$

where

$$P_{jk} = 1 - \frac{1}{4(d_j + 1)^2} - \frac{1}{2(d_j + 1)(d_k + 1)} - \frac{1}{4(d_k + 1)^2}. \tag{5.62}$$

Note that

$$1 - \frac{1}{n^2} \leq P_{jk} \leq 1. \tag{5.63}$$

Using (4.19), (5.61), we gradually remove all the edges from the residual term H_{EG} and obtain

$$\langle 1 \rangle_{F_{EG}, \Omega} = \prod_{\{v_j, v_k\} \in EG} P_{jk} \langle 1 \rangle_{\Omega} + O(n^{-1+7\epsilon}) \langle 1 \rangle. \tag{5.64}$$

Combining (4.18) and (5.64), we get

$$\langle 1 \rangle_{F_{EG}, \Omega} = C_4 \langle 1 \rangle + O(n^{-1+7\epsilon}) \langle 1 \rangle, \tag{5.65}$$

where

$$C_4 = \exp \left(-\frac{1}{4} \sum_{\{v_j, v_k\} \in EG} \left(\frac{1}{d_j + 1} + \frac{1}{d_k + 1} \right)^2 \right). \tag{5.66}$$

Using (5.40), (5.44), (5.55), (5.58) and (5.65), we can see that

$$\langle e^{i\vec{p}^T \vec{\theta}} \rangle_{F, \Omega} = C_1 C_2 C_3 C_4 \langle 1 \rangle + O(n^{-\frac{1}{2}+7\epsilon}) \langle 1 \rangle. \tag{5.67}$$

Taking (4.9), (5.10), (5.41) and (5.45) into account, we obtain

$$\begin{aligned} C_1 C_2 = & \exp \left(-\frac{1}{2} \vec{\beta}^T \widehat{Q}^{-1} \vec{\beta} \right), \\ \vec{\beta}^T \widehat{Q}^{-1} \vec{\beta} = & \vec{\alpha}^T Q^T \widehat{Q}^{-1} (Q + J) \vec{\alpha} - \vec{\alpha}^T Q \widehat{Q}^{-1} J \vec{\alpha} \\ = & \sum_{\{v_j, v_k\} \in EG} \left(\frac{1}{d_j + 1} - \frac{1}{d_k + 1} \right)^2 + O(n^{-1}). \end{aligned} \tag{5.68}$$

Using (4.9) and (5.10) once again we find that

$$\begin{aligned}
 R_{kk} &= \text{tr}(\Lambda(\vec{e}^{(k)})\widehat{Q}^{-1}\Lambda(\vec{e}^{(k)})\widehat{Q}^{-1}) \\
 &= \sum_{j,m=1}^n \Lambda_{jj}(\vec{e}^{(k)})(\widehat{Q}^{-1})_{jm}\Lambda_{mm}(\vec{e}^{(k)})(\widehat{Q}^{-1})_{mj} = 1 + O(n^{-1}), \tag{5.69}
 \end{aligned}$$

$$\begin{aligned}
 C_3 &= \exp\left(\sum_{k=1}^n \frac{R_{kk}}{2(d_k + 1)}\right) = \exp\left(\sum_{k=1}^n \frac{1}{2(d_k + 1)}\right) + O(n^{-1}) \\
 &= \exp\left(\frac{1}{2} \sum_{\{v_j, v_k\} \in EG} \left(\frac{1}{(d_j + 1)^2} + \frac{1}{(d_k + 1)^2}\right)\right) + O(n^{-1}). \tag{5.70}
 \end{aligned}$$

Putting together (3.1), (3.6), (3.12), (5.1), (5.7), (5.13) and (5.65)–(5.70), we obtain (2.5) and (2.6) for $n \geq n_0(\gamma, \varepsilon) > 0$ (but with the exponent 7ε instead of ε). Estimate (2.6) holds for $n \leq n_0$ if we choose a sufficiently large constant C .

§ 6. Proof of Lemma 4.2

In this section we use the notation $f = O(g)$ meaning that $|f| \leq c|g|$ for some $c > 0$ depending only on r_1, r_2, c_1, c_2, a, b and ε .

Let

$$\vec{\varphi}(\vec{\theta}) = (\varphi_1(\vec{\theta}), \varphi_2(\vec{\theta}), \dots, \varphi_n(\vec{\theta}))^T = A\vec{\theta}. \tag{6.1}$$

By conditions (i), (ii) in § 4, we have $A = I + X, X_{jj} = 0$, whence

$$\vec{\theta}^T A\vec{\theta} = \varphi_1^2(\vec{\theta}) + g_1(\vec{\theta}) \tag{6.2}$$

for some $g_1(\vec{\theta}) = g_1(\theta_2, \dots, \theta_n)$. Using conditions (i), (ii) in § 4 and (6.2) and estimating the various terms in a Gaussian integral of type

$$\int (\max\{|x|, k_1\})^s e^{-(x-k_2)^2} dx, \tag{6.3}$$

we find that, for all $r > 0$ and $s \geq 0$,

$$\begin{aligned}
 \langle \|\vec{\theta}\|_\infty^s \rangle &= \int_{\mathbb{R}^n} \|\vec{\theta}\|_\infty^s e^{-\vec{\theta}^T A\vec{\theta}} d\vec{\theta} \\
 &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-g_1(\theta_2, \dots, \theta_n)} \left(\int_{-\infty}^{+\infty} \|\vec{\theta}\|_\infty^s e^{-\varphi_1^2(\vec{\theta})} d\theta_1 \right) d\theta_2 \dots d\theta_n \\
 &= (1 + O(\exp(-c_4 n^{2\varepsilon}))) \int_{|\varphi_1(\vec{\theta})| \leq rn^\varepsilon} \|\vec{\theta}\|_\infty^s e^{-\vec{\theta}^T A\vec{\theta}} d\vec{\theta}, \tag{6.4}
 \end{aligned}$$

where $c_4 = c_4(r, \varepsilon, s) > 0$. Combining expressions similar to (6.4) for $\varphi_1, \varphi_2, \dots, \varphi_n$, we get

$$\int_{\|\vec{\varphi}(\vec{\theta})\|_\infty \leq rn^\varepsilon} \|\vec{\theta}\|_\infty^s e^{-\vec{\theta}^T A\vec{\theta}} d\vec{\theta} = (1 + O(\exp(-c_5 n^{2\varepsilon}))) \langle \|\vec{\theta}\|_\infty^s \rangle, \tag{6.5}$$

where $c_5 = c_5(r, \varepsilon, s) > 0$. Combining (4.8), (6.1) and (6.5) for $s = 0$, we obtain (4.18).

Using (4.15), we find that

$$|\langle 1 \rangle_{F,\Omega}| \leq \int_{\Omega} |e^{F(\vec{\theta})}| d\vec{\theta} \leq \int_{\mathbb{R}^n} \exp\left(-\vec{\theta}^T A \vec{\theta} + \frac{c_1}{n} \vec{\theta}^T A \vec{\theta}\right) d\vec{\theta} = O(\langle 1 \rangle). \tag{6.6}$$

To prove (4.20)–(4.24), we use the following two lemmas. Their proofs are given in §7.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $U_n(r_1 n^\varepsilon) \subset \Omega \subset U_n(r_2 n^\varepsilon)$ for some $r_2 > r_1 > 0$. Suppose that A satisfies conditions (i), (ii) in §4, and the assumptions (4.15), (4.16) hold for some constants $c_1, c_2 > 0$. Suppose that $P = P(x) = O(|x|^s)$ for some fixed $s \geq 0$. Then for every function $T(\vec{\theta})$ such that $|T(\vec{\theta})| \leq P(\|\vec{\theta}\|_\infty)$, we have*

$$\langle T(\vec{\theta}) \rangle_{\mathbb{R}^n \setminus \Omega} = O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle \tag{6.7}$$

and for every function $T(\vec{\theta})$ such that $\tilde{T}(\vec{\theta}) = \tilde{T}(\theta_1, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n)$ and $\tilde{T}(\vec{\theta}) \leq P(\|\vec{\theta}\|_\infty)$ we have

$$\langle \varphi_k^2(\vec{\theta}) \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} = \frac{1}{2} \langle \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} + O(n^{-1+4\varepsilon}) \langle |\tilde{T}(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle, \tag{6.8}$$

$$\langle \varphi_k^4(\vec{\theta}) \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} = \frac{3}{4} \langle \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} + O(n^{-1+4\varepsilon}) \langle |\tilde{T}(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle, \tag{6.9}$$

$$\langle \varphi_k(\vec{\theta}) \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} = O(n^{-1+4\varepsilon}) \langle |\tilde{T}(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle, \tag{6.10}$$

$$\langle \varphi_k^3(\vec{\theta}) \tilde{T}(\vec{\theta}) \rangle_{F,\Omega} = O(n^{-1+6\varepsilon}) \langle |\tilde{T}(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle, \tag{6.11}$$

where the function F is defined in (4.14), the vector $\vec{\varphi}(\vec{\theta})$ is defined in (6.1), and the constant $c_6 = c_6(r_1, r_2, c_1, c_2, a, b, \varepsilon, P)$ is positive.

Lemma 6.2. *Suppose that the hypotheses of Lemma 6.1 hold. For all non-negative integers $s_1, s_2, \dots, s_n \in \mathbb{N} \cup \{0\}$ we put*

$$M(\vec{x}) = x_1^{s_1} \cdots x_n^{s_n}, \quad s = s_1 + \cdots + s_n > 0. \tag{6.12}$$

Suppose that

$$s_k = 0, \quad |\{j: s_j \neq 0\}| \leq 3. \tag{6.13}$$

Then

$$\langle \varphi_k(\vec{\theta}) M(\vec{\varphi}(\vec{\theta})) \rangle_{F,\Omega} = O(s n^{-1+(s+4)\varepsilon}) \langle 1 \rangle. \tag{6.14}$$

Using (4.9), we find that

$$\theta_k = \varphi_k + \vec{q}_k^T \vec{\varphi}, \quad \|\vec{q}_k\|_\infty = O(n^{-1}). \tag{6.15}$$

Combining (6.6), (6.15), Lemma 6.1 and Lemma 6.2, we get

$$\begin{aligned} \langle \delta_k^2(\vec{\theta}) \rangle_{F,\Omega} &= \langle (\vec{q}_k^T \vec{\varphi})^2 \rangle_{F,\Omega} = O(n^{-2}) \left(\sum_j \langle \varphi_j^2(\vec{\theta}) \rangle_{F,\Omega} + \sum_{j_1 \neq j_2} |\langle \varphi_{j_1}(\vec{\theta}) \varphi_{j_2}(\vec{\theta}) \rangle_{F,\Omega}| \right) \\ &= (O(n^{-1}) + O(n^{-1+5\varepsilon})) \langle 1 \rangle = O(n^{-1+5\varepsilon}) \langle 1 \rangle, \end{aligned} \tag{6.16}$$

$$\begin{aligned}
\langle \delta_k^4(\vec{\theta}) \rangle_{F,\Omega} &= \langle (\vec{q}_k^T \vec{\varphi})^4 \rangle_{F,\Omega} = O(n^{-4}) \sum_j \langle \varphi_j^4(\vec{\theta}) \rangle_{F,\Omega} \\
&+ O(n^{-4}) \sum_{j_1 \neq j_2} \langle \varphi_{j_1}^2(\vec{\theta}) \varphi_{j_2}^2(\vec{\theta}) \rangle_{F,\Omega} + O(n^{-4}) \sum_{j_1 \neq j_2} |\langle \varphi_{j_1}(\vec{\theta}) \varphi_{j_2}^3(\vec{\theta}) \rangle_{F,\Omega}| \\
&+ O(n^{-4}) \sum_{j_1 \neq j_2 \neq j_3} |\langle \varphi_{j_1}(\vec{\theta}) \varphi_{j_2}(\vec{\theta}) \varphi_{j_3}^2(\vec{\theta}) \rangle_{F,\Omega}| \\
&+ O(n^{-4}) \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} |\langle \varphi_{j_1}(\vec{\theta}) \varphi_{j_2}(\vec{\theta}) \varphi_{j_3}(\vec{\theta}) \varphi_{j_4}(\vec{\theta}) \rangle_{F,\Omega}| \\
&= (O(n^{-3}) + O(n^{-2+4\epsilon}) + O(n^{-3+7\epsilon}) + O(n^{-2+7\epsilon}) + O(n^{-1+7\epsilon})) \langle 1 \rangle \\
&= O(n^{-1+7\epsilon}) \langle 1 \rangle, \tag{6.17}
\end{aligned}$$

where

$$\delta_k(\vec{\theta}) = \theta_k - \varphi_k(\vec{\theta}). \tag{6.18}$$

By conditions (i), (ii) in § 4 we have

$$\delta_k(\vec{\theta}) = \delta_k(\theta_1, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n). \tag{6.19}$$

Using (6.6), (6.15)–(6.17), (6.19) and Lemma 6.1, we obtain

$$\langle \varphi_k(\vec{\theta}) \delta_k(\vec{\theta}) \rangle_{F,\Omega} = O(n^{-1+4\epsilon}) \langle |\delta_k(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\epsilon})) \langle 1 \rangle = O(n^{-1+5\epsilon}) \langle 1 \rangle, \tag{6.20}$$

$$\begin{aligned}
\langle \theta_k^2 \rangle_{F,\Omega} &= \langle (\varphi_k(\vec{\theta}) + \delta_k(\vec{\theta}))^2 \rangle_{F,\Omega} = \langle \varphi_k^2(\vec{\theta}) \rangle_{F,\Omega} + O(n^{-1+5\epsilon}) \langle 1 \rangle \\
&= \frac{1}{2} \langle 1 \rangle_{F,\Omega} + O(n^{-1+5\epsilon}) \langle 1 \rangle, \tag{6.21}
\end{aligned}$$

$$\langle \varphi_k(\vec{\theta}) \delta_k^3(\vec{\theta}) \rangle_{F,\Omega} = O(n^{-1+4\epsilon}) \langle |\delta_k^3(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\epsilon})) \langle 1 \rangle = O(n^{-1+7\epsilon}) \langle 1 \rangle, \tag{6.22}$$

$$\begin{aligned}
\langle \varphi_k^2(\vec{\theta}) \delta_k^2(\vec{\theta}) \rangle_{F,\Omega} &= \left(\frac{1}{2} + O(n^{-1+4\epsilon}) \right) \langle \delta_k^2(\vec{\theta}) \rangle_{F,\Omega} \\
&+ O(\exp(-c_6 n^{2\epsilon})) \langle 1 \rangle = O(n^{-1+5\epsilon}) \langle 1 \rangle, \tag{6.23}
\end{aligned}$$

$$\langle \varphi_k^3(\vec{\theta}) \delta_k(\vec{\theta}) \rangle_{F,\Omega} = O(n^{-1+6\epsilon}) \langle |\delta_k(\vec{\theta})| \rangle_{F,\Omega} + O(\exp(-c_6 n^{2\epsilon})) \langle 1 \rangle = O(n^{-1+7\epsilon}) \langle 1 \rangle, \tag{6.24}$$

$$\begin{aligned}
\langle \theta_k^4 \rangle_{F,\Omega} &= \langle (\varphi_k(\vec{\theta}) + \delta_k^4(\vec{\theta})) \rangle_{F,\Omega} = \langle \varphi_k^4(\vec{\theta}) \rangle_{F,\Omega} + O(n^{-1+7\epsilon}) \langle 1 \rangle \\
&= \frac{3}{4} \langle 1 \rangle_{F,\Omega} + O(n^{-1+7\epsilon}) \langle 1 \rangle. \tag{6.25}
\end{aligned}$$

Similarly to (6.16), (6.17), using (6.20)–(6.24), we find that

$$\begin{aligned}
\langle \delta_k(\vec{\theta}) \theta_l \rangle_{F,\Omega} &= O(n^{-1}) \sum_j \langle \varphi_j(\vec{\theta}) \theta_l \rangle_{F,\Omega} \\
&= (O(n^{-1+2\epsilon}) + O(n^{-1+5\epsilon})) \langle 1 \rangle = O(n^{-1+5\epsilon}) \langle 1 \rangle, \tag{6.26}
\end{aligned}$$

$$\begin{aligned} \langle \delta_k^2(\vec{\theta})\theta_l^2 \rangle_{F,\Omega} &= O(n^{-2}) \sum_j \langle \varphi_j^2(\vec{\theta})\theta_l^2 \rangle_{F,\Omega} \\ &\quad + O(n^{-2}) \sum_{j_1 \neq j_2} |\langle \varphi_{j_1}(\vec{\theta})\varphi_{j_2}(\vec{\theta})\theta_l^2 \rangle_{F,\Omega}| \\ &= (O(n^{-1+2\varepsilon}) + O(n^{-1+7\varepsilon}))\langle 1 \rangle = O(n^{-1+7\varepsilon})\langle 1 \rangle, \end{aligned} \tag{6.27}$$

$$\begin{aligned} \langle \delta_k(\vec{\theta})\theta_l^3 \rangle_{F,\Omega} &= O(n^{-1}) \sum_{j \neq l} \langle \varphi_j(\vec{\theta})\theta_l^3 \rangle_{F,\Omega} + O(n^{-1})\langle \varphi_l(\vec{\theta})\theta_l^3 \rangle_{F,\Omega} \\ &= (O(n^{-1+7\varepsilon}) + O(n^{-1+4\varepsilon}))\langle 1 \rangle = O(n^{-1+7\varepsilon})\langle 1 \rangle. \end{aligned} \tag{6.28}$$

Using (6.6), (6.19), (6.26)–(6.28) and Lemma 6.1, we obtain

$$\begin{aligned} \langle \varphi_k(\vec{\theta})\delta_k(\vec{\theta})\theta_l^2 \rangle_{F,\Omega} &= O(n^{-1+4\varepsilon})\langle |\delta_k(\vec{\theta})\theta_l^2| \rangle_{F,\Omega} \\ &\quad + O(\exp(-c_6 n^{2\varepsilon}))\langle 1 \rangle = O(n^{-1+7\varepsilon})\langle 1 \rangle, \end{aligned} \tag{6.29}$$

$$\langle \theta_k\theta_l \rangle_{F,\Omega} = \langle (\varphi_k(\vec{\theta}) + \delta_k(\vec{\theta}))\theta_l \rangle_{F,\Omega} = O(n^{-1+5\varepsilon})\langle 1 \rangle, \tag{6.30}$$

$$\langle \theta_k\theta_l^3 \rangle_{F,\Omega} = \langle (\varphi_k(\vec{\theta}) + \delta_k(\vec{\theta}))\theta_l^3 \rangle_{F,\Omega} = O(n^{-1+7\varepsilon})\langle 1 \rangle, \tag{6.31}$$

$$\begin{aligned} \langle \theta_k^2\theta_l^2 \rangle_{F,\Omega} &= \langle (\varphi_k(\vec{\theta}) + \delta_k(\vec{\theta}))^2\theta_l^2 \rangle_{F,\Omega} = \langle \varphi_k^2(\vec{\theta})\theta_l^2 \rangle_{F,\Omega} + O(n^{-1+7\varepsilon})\langle 1 \rangle \\ &= \frac{1}{2}\langle (\varphi_l(\vec{\theta}) + \delta_l(\vec{\theta}))^2 \rangle_{F,\Omega} + O(n^{-1+7\varepsilon})\langle 1 \rangle = \frac{1}{4}\langle 1 \rangle_{F,\Omega} + O(n^{-1+7\varepsilon})\langle 1 \rangle. \end{aligned} \tag{6.32}$$

Since $\|\vec{p}\|_\infty = O(n^{-1/2})$, using the expression

$$e^{\frac{1}{2}p_k^2\theta_k^2} = 1 + O(n^{-1+2\varepsilon}), \quad \vec{\theta} \in \Omega, \tag{6.33}$$

and (6.6), we find that

$$\langle \theta_k e^{i\vec{\theta}^T \vec{p} - ip_k\theta_k} \rangle_{F,\Omega} = \langle \theta_k e^{i\vec{\theta}^T \vec{p}' - ip_k\theta_k} \rangle_{F',\Omega} + O(n^{-1+3\varepsilon})\langle 1 \rangle, \tag{6.34}$$

where $F' = F - \frac{1}{2}p_k^2\theta_k^2$. Clearly, F' satisfies all the hypotheses of Lemma 6.1. For every $\vec{p}' = (p'_1, p'_2, \dots, p'_n) \in \mathbb{R}^n$ with $\|\vec{p}'\|_\infty = O(n^{-1/2})$ we have

$$e^{ip'_l\theta_l} = 1 + ip'_l\theta_l + O(n^{-1+2\varepsilon}), \quad e^{-ip'_l\theta_l} = 1 + O(n^{-\frac{1}{2}+\varepsilon}), \quad \vec{\theta} \in \Omega. \tag{6.35}$$

Using (6.6), (6.8) and (6.10) with F' instead of F , as well as (6.35), we get

$$\begin{aligned} \langle \varphi_l(\vec{\theta})e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} &= O(n^{-1+4\varepsilon})\langle 1 \rangle, \\ \langle \varphi_l(\vec{\theta})\delta_l(\vec{\theta})e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} &= O(n^{-1+5\varepsilon})\langle 1 \rangle, \\ \langle \varphi_l(\vec{\theta})\theta_l e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} &= \langle \varphi_l^2(\vec{\theta})e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} + \langle \varphi_l(\vec{\theta})\delta_l(\vec{\theta})e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} \\ &= \frac{1}{2}\langle e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} + O(n^{-1+5\varepsilon})\langle 1 \rangle = \frac{1}{2}\langle e^{i\vec{\theta}^T \vec{p}'} \rangle_{F',\Omega} + O(n^{-\frac{1}{2}+5\varepsilon})\langle 1 \rangle. \end{aligned} \tag{6.36}$$

Combining (6.35) and (6.36), we find that

$$\begin{aligned} \langle \varphi_l(\vec{\theta})e^{i\vec{\theta}^T \vec{p}'} \rangle_{F',\Omega} &= \langle \varphi_l(\vec{\theta})e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} \\ &\quad + ip'_l\langle \varphi_l(\vec{\theta})\theta_l e^{i\vec{\theta}^T \vec{p}' - ip'_l\theta_l} \rangle_{F',\Omega} + O(n^{-1+3\varepsilon})\langle 1 \rangle \\ &= \frac{i}{2}p'_l\langle e^{i\vec{\theta}^T \vec{p}'} \rangle_{F',\Omega} + O(n^{-1+5\varepsilon})\langle 1 \rangle. \end{aligned} \tag{6.37}$$

Using (4.8), (6.1) and (6.37), we get

$$\langle \theta_k e^{i\vec{\theta}^T \vec{p}'} \rangle_{F', \Omega} = \frac{i}{2} \sum_{l=1}^n p'_l (A^{-1})_{lk} \langle e^{i\vec{\theta}^T \vec{p}'} \rangle_{F', \Omega} + O(n^{-1+5\varepsilon}) \langle 1 \rangle. \tag{6.38}$$

Combining (6.34) and (6.38) for $\vec{p}' = \vec{p} - p_k \vec{e}^{(k)}$, we obtain (4.25).

§ 7. Proofs of Lemma 6.1 and Lemma 6.2

In this section we continue to use the notation $f = O(g)$ meaning that $|f| \leq c|g|$ for some $c > 0$ depending only on the constants r_1, r_2, c_1, c_2, a, b and ε .

Let

$$\vec{\theta}^{(k)} = (\theta_1, \dots, \theta_{k-1}, 0, \theta_{k+1}, \dots, \theta_n)^T. \tag{7.1}$$

Proof of Lemma 6.1. Using (4.8), (6.1) and (6.5), we find that

$$|\langle T \rangle_{\mathbb{R}^n \setminus \Omega}| \leq \int_{\mathbb{R}^n \setminus \Omega} P(\|\vec{\theta}\|_\infty^s) e^{-\vec{\theta}^T A \vec{\theta}} d\vec{\theta} = O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle. \tag{7.2}$$

For the sake of simplicity, let $k = 1$. Using (7.2), we get

$$\begin{aligned} \langle \varphi_1^p(\vec{\theta}) \tilde{T}(\vec{\theta}) \rangle_{F, \Omega} &= \int_{\Omega} \varphi_1^p(\vec{\theta}) \tilde{T}(\theta_2, \dots, \theta_n) e^{-\vec{\theta}^T A \vec{\theta} + H(\vec{\theta})} d\vec{\theta} \\ &= \int_{U_n(r_2 n^\varepsilon)} \varphi_1^p(\vec{\theta}) \tilde{T}(\theta_2, \dots, \theta_n) e^{-\vec{\theta}^T A \vec{\theta} + H(\vec{\theta})} d\vec{\theta} \\ &\quad + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle, \quad p = 1, \dots, 4, \end{aligned} \tag{7.3}$$

where $H(\vec{\theta}) \equiv 0$ for $\vec{\theta} \in \mathbb{R}^n \setminus \Omega$. Combining (4.16) and Lagrange’s mean value theorem, we find that

$$H(\vec{\theta}) - H(\vec{\theta}^{(1)}) = O(n^{-1+4\varepsilon}), \quad \vec{\theta} \in U_n(r_2 n^\varepsilon). \tag{7.4}$$

Using (6.2), we get

$$\begin{aligned} &\int_{U_n(r_2 n^\varepsilon)} \varphi_1^p(\vec{\theta}) \tilde{T}(\theta_2, \dots, \theta_n) e^{-\vec{\theta}^T A \vec{\theta} + H(\vec{\theta})} d\vec{\theta} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle \\ &= \int_{-r_2 n^\varepsilon}^{r_2 n^\varepsilon} \dots \int_{-r_2 n^\varepsilon}^{r_2 n^\varepsilon} \tilde{T} e^{-g_1(\theta_2, \dots, \theta_n) + H(\vec{\theta}^{(1)})} \\ &\quad \times \left(\int_{-r_2 n^\varepsilon}^{r_2 n^\varepsilon} \varphi_1^p e^{-\varphi_1^2(\vec{\theta}) + H(\vec{\theta}) - H(\vec{\theta}^{(1)})} d\theta_1 \right) d\theta_2 \dots d\theta_n, \quad p = 0, \dots, 4. \end{aligned} \tag{7.5}$$

Combining (4.8) and (7.4), we obtain, for $\vec{\theta}^{(1)} \in U_n(r_2 n^\varepsilon)$,

$$\begin{aligned} &\int_{-r_2 n^\varepsilon}^{r_2 n^\varepsilon} \varphi_1^p e^{-\varphi_1^2(\vec{\theta}) + H(\vec{\theta}) - H(\vec{\theta}^{(1)})} d\theta_1 \\ &= \int_{-\infty}^{+\infty} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} d\theta_1 + O(\exp(-c_7 n^{2\varepsilon})) \int_{-\infty}^{+\infty} e^{-\varphi_1^2(\vec{\theta})} d\theta_1 \\ &\quad + \int_{|\varphi_1(\vec{\theta})| \leq r_3 n^\varepsilon} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} (e^{H(\vec{\theta}) - H(\vec{\theta}^{(1)})} - 1) d\theta_1, \quad p = 0, \dots, 4, \end{aligned} \tag{7.6}$$

where $c_7 = c_7(r_2, c_1, c_2, a, b, \varepsilon) > 0$, $r_3 = r_3(r_2, c_1, c_2, a, b, \varepsilon) > 0$.

For $p = 2, 4$ we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \varphi_1^2 e^{-\varphi_1^2(\vec{\theta})} d\theta_1 &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\varphi_1^2(\vec{\theta})} d\theta_1, \\ \int_{-\infty}^{+\infty} \varphi_1^4 e^{-\varphi_1^2(\vec{\theta})} d\theta_1 &= \frac{3}{4} \int_{-\infty}^{+\infty} e^{-\varphi_1^2(\vec{\theta})} d\theta_1, \end{aligned} \tag{7.7}$$

$$\begin{aligned} &\int_{|\varphi_1(\vec{\theta})| < r_3 n^\varepsilon} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} (e^{H(\vec{\theta}) - H(\vec{\theta}^{(1)})} - 1) d\theta_1 \\ &= O(n^{-1+4\varepsilon}) \int_{-\infty}^{+\infty} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} d\theta_1 = O(n^{-1+4\varepsilon}) \int_{-\infty}^{+\infty} e^{-\varphi_1^2(\vec{\theta})} d\theta_1 \end{aligned} \tag{7.8}$$

for $\vec{\theta}^{(1)} \in U_n(r_2 n^\varepsilon)$, $p = 0, 2, 4$.

Combining (7.2)–(7.8), we obtain (6.8) and (6.9).

For $p = 1, 3$ we have

$$\int_{-\infty}^{+\infty} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} d\theta_1 = 0, \tag{7.9}$$

$$\begin{aligned} &\int_{|\varphi_1(\vec{\theta})| \leq r_3 n^\varepsilon} \varphi_1^p e^{-\varphi_1^2(\vec{\theta})} (e^{H(\vec{\theta}) - H(\vec{\theta}^{(1)})} - 1) d\theta_1 \\ &= \int_{0 \leq \varphi_1(\vec{\theta}) \leq r_3 n^\varepsilon} |\varphi_1|^p e^{-\varphi_1^2(\vec{\theta})} (e^{H(\vec{\theta}) - H(\vec{\theta}^{(1)})} - 1) d\theta_1 \\ &\quad - \int_{r_3 n^\varepsilon \leq \varphi_1(\vec{\theta}) \leq 0} |\varphi_1|^p e^{-\varphi_1^2(\vec{\theta})} (e^{H(\vec{\theta}) - H(\vec{\theta}^{(1)})} - 1) d\theta_1 \\ &= O(n^{-1+4\varepsilon}) \int_{\varphi_1 \geq 0} |\varphi_1|^p e^{-\varphi_1^2(\vec{\theta})} d\theta_1 = O(n^{-1+4\varepsilon}) \int_{-\infty}^{+\infty} e^{-\varphi_1^2(\vec{\theta})} d\theta_1 \end{aligned} \tag{7.10}$$

for $\vec{\theta}^{(1)} \in U_n(r_2 n^\varepsilon)$, $p = 1, 3$.

Combining (7.2)–(7.6), (7.9) and (7.10), we obtain (6.10) and (6.11). \square

Proof of Lemma 6.2. Suppose that $T(\vec{\theta})$ satisfies

$$\begin{aligned} |T(\vec{\theta})| &= O(\|\vec{\theta}\|_\infty^\varepsilon), \\ \frac{\partial T(\vec{\theta})}{\partial \theta_k} &= O(sn^{-1-\varepsilon}) \sup_{\vec{\theta} \in \Omega} |T(\vec{\theta})|, \quad \vec{\theta} \in \Omega. \end{aligned} \tag{7.11}$$

Combining (6.10) and (7.11), we get

$$\begin{aligned} \langle \varphi_k(\vec{\theta}) T(\vec{\theta}) \rangle_{F, \Omega} &= \langle \varphi_k(\vec{\theta}) T(\vec{\theta}^{(k)}) \rangle_{F, \Omega} + \langle \varphi_k(\vec{\theta}) (T(\vec{\theta}) - T(\vec{\theta}^{(k)})) \rangle_{F, \Omega} \\ &= \langle \varphi_k(\vec{\theta}) T(\vec{\theta}^{(k)}) \rangle_{F, \Omega} + O(sn^{-1-\varepsilon}) \left\langle \sup_{\vec{\theta} \in \Omega} |\varphi_k \theta_k T(\vec{\theta})| \right\rangle_{F, \Omega} \\ &= O(sn^{-1+4\varepsilon}) \left\langle \sup_{\vec{\theta} \in \Omega} |T(\vec{\theta})| \right\rangle_{F, \Omega} + O(\exp(-c_6 n^{2\varepsilon})) \langle 1 \rangle. \end{aligned} \tag{7.12}$$

Using conditions (i), (ii) in § 4, we find, for $\vec{\theta} \in \Omega$,

$$\frac{\partial M(\vec{\varphi}(\vec{\theta}))}{\partial \theta_k} = O(sn^{(s-1)\varepsilon}) \sum_{j: s_j \neq 0} \frac{\partial \varphi_j(\vec{\theta})}{\partial \theta_k} = O(sn^{-1-\varepsilon}) \sup_{\vec{\theta} \in \Omega} |M(\vec{\varphi}(\vec{\theta}))|. \tag{7.13}$$

Combining (6.6) and (7.12) for $T = M$, we obtain (6.14). \square

§ 8. Proof of Proposition 3.1

In this section we use the notation $f = O(g)$ meaning that $|f| \leq c|g|$ for some $c > 0$ depending only on γ and ε .

The following lemma will be used to estimate the determinant of a matrix which is close to the identity matrix I .

Lemma 8.1. *Let $\|\cdot\|$ be any matrix norm, and let X be an $n \times n$ matrix with $\|X\| < 1$. Then, for every fixed $m \geq 2$,*

$$\det(I + X) = \exp\left(\sum_{r=1}^{m-1} \frac{(-1)^{r+1}}{r} \operatorname{tr}(X^r) + E_m(X)\right), \tag{8.1}$$

where $\operatorname{tr}(\cdot)$ is the trace of a matrix and

$$|E_m(X)| \leq \frac{n}{m} \frac{\|X\|^m}{1 - \|X\|}. \tag{8.2}$$

The proof of Lemma 8.1 is based on estimating the trace of the matrix $\ln(I + X)$ by using a convergent series representation. Lemma 8.1 was also stated and proved in detail in [7].

We have

$$S_0 = \frac{1}{n} \sum_{r=1}^n \int_{V_0} \prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \tan \Delta_{jk}) d\vec{\xi}, \tag{8.3}$$

where $\Delta_{jk} = \xi_j - \xi_k$ and

$$V_0 = \left\{ \vec{\xi} \in U_n(\pi/2) : |\xi_j - \xi_k|_\pi \leq n^{-\frac{1}{2} + \varepsilon}, 1 \leq j, k \leq n \right\}, \tag{8.4}$$

$$|\xi_j - \xi_k|_\pi = \min_{l \in \mathbb{Z}} |\xi_j - \xi_k + \pi l|.$$

Since the integrand in (8.3) and the domain V_0 are invariant under uniform translations of all $\xi_j \pmod{\pi}$, we can calculate an integral of lower dimension for $\{\vec{\xi} \in V_0 \cap U_n(n^{-\frac{1}{2} + \varepsilon}) : \sum_{k=1}^n \xi_k = 0\}$ and multiply it by the ratio of the integration interval π to the length $n^{-1/2}$ of the vector $\frac{1}{n}(1, 1, \dots, 1)^T$. Thus we get

$$S_0 = \pi n^{\frac{1}{2}} \int_{L \cap V_0 \cap U_n(n^{-1/2 + \varepsilon})} K(\vec{\xi}) dL, \tag{8.5}$$

$$K(\vec{\xi}) = \frac{1}{n} \sum_{r=1}^n \prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \operatorname{tg} \Delta_{jk}),$$

where L is the orthogonal complement to the vector $(1, 1, \dots, 1)^T$.

We define an $n \times n$ matrix B by

$$B_{jk} = \begin{cases} -\tan \Delta_{jk} & \text{if } \{v_j, v_k\} \in EG, \\ \sum_{l: \{v_j, v_l\} \in EG} \tan \Delta_{jl} & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases} \tag{8.6}$$

Using Theorem 3.1 for the matrix $Q + iB$, we get

$$\sum_{r=1}^n \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \tan \Delta_{jk}) = \sum_{r=1}^n M_r, \tag{8.7}$$

where M_r is the principal minor of $Q + iB$ obtained by removing the r -th row and r -th column. Since $(1, 1, \dots, 1)^T$ is a common eigenvector of Q and B with eigenvalue 0, we have

$$\sum_{r=1}^n M_r = \frac{\det(\widehat{Q} + iB)}{n}, \tag{8.8}$$

where $\widehat{Q} = Q + J$ and J is the matrix all of whose entries are equal to 1. Note that

$$|\Delta_{jk}| \leq n^{-\frac{1}{2} + \varepsilon}, \quad \vec{\xi} \in V_0 \cap U_n(n^{-\frac{1}{2} + \varepsilon}), \tag{8.9}$$

$$\|B\|_1 = \max_j \sum_{k=1}^n |B_{jk}| = O(n^{\frac{1}{2} + \varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2} + \varepsilon}). \tag{8.10}$$

Put $\Phi = B\widehat{Q}^{-1}$. Using (4.8), (5.4), (5.29) and (8.10), we get

$$\|\Phi\|_1 \leq \|B\|_1 \|\widehat{Q}^{-1}\|_1 = O(n^{-\frac{1}{2} + \varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2} + \varepsilon}). \tag{8.11}$$

Applying Lemma 8.1 for the matrix $i\Phi$, we find that

$$\det(I + i\Phi) = \exp\left(i \operatorname{tr} \Phi + \frac{\operatorname{tr} \Phi^2}{2} + O(n^{-\frac{1}{2} + 3\varepsilon})\right), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2} + \varepsilon}). \tag{8.12}$$

Write

$$B = B_{\text{skew}} + B_{\text{diag}}, \tag{8.13}$$

where B_{skew} is a skew-symmetric matrix and B_{diag} is a diagonal matrix. Since the matrix \widehat{Q} is symmetric, we have

$$\operatorname{tr}(B_{\text{skew}}\widehat{Q}^{-1}) = 0. \tag{8.14}$$

Using (8.9), we find that

$$\|B_{\text{diag}} - \Lambda\|_2 = O(n^{-\frac{1}{2} + 3\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2} + \varepsilon}), \tag{8.15}$$

where Λ is the diagonal matrix whose diagonal entries are equal to the corresponding components of the vector $Q\vec{\xi}$. Combining (5.3) and (8.15), we get

$$|\operatorname{tr}((B_{\text{diag}} - \Lambda)\widehat{Q}^{-1})| \leq n\|B_{\text{diag}} - \Lambda\|_2 \|\widehat{Q}^{-1}\|_2 = O(n^{-\frac{1}{2} + 3\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2} + \varepsilon}). \tag{8.16}$$

Using (8.14) and (8.16), we obtain

$$\begin{aligned} \operatorname{tr} \Phi &= \operatorname{tr}(B_{\text{diag}} \widehat{Q}^{-1}) = \operatorname{tr}(\Lambda \widehat{Q}^{-1}) + O(n^{-\frac{1}{2}+3\varepsilon}) \\ &= \vec{\xi}^T Q \vec{\alpha} + O(n^{-\frac{1}{2}+3\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}), \end{aligned} \quad (8.17)$$

where $\vec{\alpha}$ is the vector whose components are equal to the corresponding diagonal entries of \widehat{Q}^{-1} .

Using the property

$$\operatorname{tr}(XY) = \operatorname{tr}(YX) \quad (8.18)$$

of the trace, we have

$$\operatorname{tr} \Phi^2 = \operatorname{tr}(B_{\text{skew}} \widehat{Q}^{-1})^2 + \operatorname{tr}(B_{\text{diag}} \widehat{Q}^{-1})^2 + 2 \operatorname{tr}(B_{\text{skew}} \widehat{Q}^{-1} B_{\text{diag}} \widehat{Q}^{-1}). \quad (8.19)$$

Since B_{skew} is skew-symmetric and $\widehat{Q}^{-1} B_{\text{diag}} \widehat{Q}^{-1}$ is symmetric, we find that

$$\operatorname{tr}(B_{\text{skew}} \widehat{Q}^{-1} B_{\text{diag}} \widehat{Q}^{-1}) = 0. \quad (8.20)$$

By (5.16) we have

$$\operatorname{tr} X^2 \leq \|X\|_{HS}^2, \quad \|XY\|_{HS} \leq \|X\|_{HS} \|Y^T\|_2. \quad (8.21)$$

Therefore we obtain

$$|\operatorname{tr}(B_{\text{skew}} \widehat{Q}^{-1})^2| \leq \|B_{\text{skew}} \widehat{Q}^{-1}\|_{HS}^2. \quad (8.22)$$

Combining (5.3) and (8.9), we find that

$$\|B_{\text{skew}} \widehat{Q}^{-1}\|_{HS} \leq \|\widehat{Q}^{-1}\|_2 \|B_{\text{skew}}\|_{HS} = O(n^{-\frac{1}{2}+\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}). \quad (8.23)$$

Using (5.3), (8.9) and (8.15), we get

$$\begin{aligned} &|\operatorname{tr}((B_{\text{diag}} - \Lambda) \widehat{Q}^{-1} B_{\text{diag}} \widehat{Q}^{-1})| \\ &\leq n \|\widehat{Q}^{-1}\|_2^2 \|B_{\text{diag}} - \Lambda\|_2 \|B_{\text{diag}}\|_2 = O(n^{-1+4\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}), \end{aligned} \quad (8.24)$$

$$\begin{aligned} &|\operatorname{tr}((B_{\text{diag}} - \Lambda) \widehat{Q}^{-1} (B_{\text{diag}} - \Lambda) \widehat{Q}^{-1})| \\ &\leq n \|\widehat{Q}^{-1}\|_2^2 \|B_{\text{diag}} - \Lambda\|_2^2 = O(n^{-2+6\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}). \end{aligned} \quad (8.25)$$

Combining (8.18), (8.24) and (8.25), we deduce that

$$\operatorname{tr}(B_{\text{diag}} \widehat{Q}^{-1})^2 = \operatorname{tr}(\Lambda \widehat{Q}^{-1})^2 + O(n^{-1+4\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}). \quad (8.26)$$

Combining (8.19), (8.20), (8.22), (8.23) and (8.26), we now obtain

$$\operatorname{tr} \Phi^2 = \operatorname{tr}(\Lambda \widehat{Q}^{-1})^2 + O(n^{-1+4\varepsilon}), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}). \quad (8.27)$$

Substituting (8.17) and (8.27) into (8.12), we can see that

$$\det(I + i\Phi) = \exp\left(i\vec{\xi}^T Q \vec{\alpha} + \frac{\operatorname{tr}(\Lambda \widehat{Q}^{-1})^2}{2} + O(n^{-\frac{1}{2}+4\varepsilon})\right), \quad \vec{\xi} \in U_n(n^{-\frac{1}{2}+\varepsilon}). \quad (8.28)$$

Using the Taylor series expansion, we get, for $\vec{\xi} \in U_n(n^{-\frac{1}{2}+\epsilon})$,

$$\prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} = \exp\left(-\frac{1}{2} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^2 - \frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + O(n^{-1+6\epsilon})\right). \tag{8.29}$$

Note that

$$\sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^2 = \vec{\xi}^T Q \vec{\xi}. \tag{8.30}$$

Putting together (8.5) for $r = 1, 2, \dots, n$, (8.7), (8.8) and (8.28)–(8.30), we arrive at

$$S_0 = \pi n^{-\frac{3}{2}} \det \widehat{Q} (\text{Int}' + O(n^{-\frac{1}{2}+6\epsilon}) \text{Int}''), \tag{8.31}$$

where

$$\begin{aligned} \text{Int}' &= \int_{L \cap V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} \exp(i\vec{\xi}^T Q \vec{\alpha} + F(\vec{\xi})) dL, & \text{Int}'' &= \int_{L \cap V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} e^{F(\vec{\xi})} dL, \\ F(\vec{\xi}) &= -\frac{1}{2} \vec{\xi}^T \widehat{Q} \vec{\xi} - \frac{1}{12} \sum_{\{v_j, v_k\} \in EG} \Delta_{jk}^4 + \frac{1}{2} R(\vec{\xi}) \end{aligned} \tag{8.32}$$

and $R(\vec{\xi}) = \text{tr}(\Lambda(\vec{\xi}) \widehat{Q}^{-1} \Lambda(\vec{\xi}) \widehat{Q}^{-1})$.

Let $\text{Pr}(\vec{\xi})$ be the orthogonal projection of the vector $\vec{\xi}$ onto the space L , where L is the orthogonal complement to $(1, 1, \dots, 1)^T$. Note that

$$\text{Pr}(\vec{\xi}) = \vec{\xi} - \bar{\xi}(1, 1, \dots, 1)^T, \tag{8.33}$$

where

$$\bar{\xi} = \frac{\xi_1 + \xi_2 + \dots + \xi_n}{n}. \tag{8.34}$$

Therefore,

$$U_n\left(\frac{1}{2}n^{-\frac{1}{2}+\epsilon}\right) \subset \{\vec{\xi}: \text{Pr}(\vec{\xi}) \in V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})\}. \tag{8.35}$$

We also note that

$$Q \vec{\xi} = Q \text{Pr}(\vec{\xi}). \tag{8.36}$$

Therefore the integrand in (8.32) does not change if we replace $\vec{\xi}$ by the vector $\text{Pr}(\vec{\xi})$, and

$$\begin{aligned} \text{Int}' &= \int_{\text{Pr}(\vec{\xi}) \in L \cap V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} e^{i\vec{\xi}^T Q \vec{\alpha} + F(\vec{\xi})} d\vec{\xi} \Big/ \int_{-\infty}^{+\infty} e^{-\frac{1}{2}nx^2} dx \\ &= \frac{n^{1/2}}{\sqrt{2\pi}} \int_{\text{Pr}(\vec{\xi}) \in V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} e^{i\vec{\xi}^T Q \vec{\alpha} + F(\vec{\xi})} d\vec{\xi}, \end{aligned} \tag{8.37}$$

where

$$\begin{aligned} \text{Int}'' &= \int_{\text{Pr}(\vec{\xi}) \in L \cap V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} e^{F(\vec{\xi})} d\vec{\xi} \Big/ \int_{-\infty}^{+\infty} e^{-\frac{1}{2}nx^2} dx \\ &= \frac{n^{1/2}}{\sqrt{2\pi}} \int_{\text{Pr}(\vec{\xi}) \in V_0 \cap U_n(n^{-\frac{1}{2}+\epsilon})} e^{F(\vec{\xi})} d\vec{\xi}. \end{aligned} \tag{8.38}$$

Using the notation (5.6)–(5.11) and formulae (4.19), (8.35), (5.67), we get

$$\text{Int}' = (1 + O(\exp(-c_7 n^{2\varepsilon}))) \frac{n^{1/2}}{\sqrt{2\pi}} \text{Int}, \quad \text{Int}'' = O(1) \frac{n^{1/2}}{\sqrt{2\pi}} \text{Int}, \quad (8.39)$$

where $c_7 > 0$ depends only on γ and ε .

Combining (8.31) and (8.39), we obtain (3.12).

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