

# Majority Colourings of Digraphs

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## Abstract

We prove that every digraph has a vertex 4-colouring such that for each vertex  $v$ , at most half the out-neighbours of  $v$  receive the same colour as  $v$ . We then obtain several results related to the conjecture obtained by replacing 4 by 3.

## 1 Introduction

A *majority colouring* of a digraph is a function that assigns each vertex  $v$  a colour, such that at most half the out-neighbours of  $v$  receive the same colour as  $v$ . In other words, at least half the out-neighbours of  $v$  receive a colour different from  $v$  (hence the name ‘majority’). Whether every digraph has a majority colouring with a bounded number of colours was posed as an open problem on mathoverflow [10]. In response, Ilya Bogdanov proved that a bounded number of colours suffice for tournaments. The following is our main result.

**Theorem 1.** *Every digraph has a majority 4-colouring.*

*Proof.* Fix a vertex ordering. First, 2-colour the vertices left-to-right so that for each vertex  $v$ , at most half the out-neighbours of  $v$  to the left of  $v$  in the ordering receive the same colour as  $v$ . Second, 2-colour the vertices right-to-left so that for each vertex  $v$ , at most half the out-neighbours of  $v$  to the right of  $v$  in the ordering receive the same colour as  $v$ . The product colouring is a majority 4-colouring.  $\square$

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Note that this proof implicitly uses two facts: (1) every digraph has an edge-partition into two acyclic subgraphs, and (2) every acyclic digraph has a majority 2-colouring.

The following conjecture naturally arises:

**Conjecture 2.** *Every digraph has a majority 3-colouring.*

This conjecture would be best possible. For example, a majority colouring of an odd directed cycle is proper (since each vertex has out-degree 1), and therefore three colours are necessary. There are examples with large minimum out-degree as well. For odd  $k \geq 1$  and prime  $n \gg k$ , let  $G$  be the directed graph with  $V(G) = \{v_0, \dots, v_{n-1}\}$  where  $N_G^+(v_i) = \{v_{i+1}, \dots, v_{i+k}\}$  and vertex indices are taken modulo  $n$ . Suppose that  $G$  has a majority 2-colouring. If some sequence  $v_i, v_{i+1}, \dots, v_{i+k}$  contains more than  $\frac{k+1}{2}$  vertices of one colour, say red, and  $v_i$  is the leftmost red vertex in this sequence, then more than  $\frac{k-1}{2}$  out-neighbours of  $v_i$  are red, which is not allowed. Thus each sequence  $v_i, v_{i+1}, \dots, v_{i+k}$  contains exactly  $\frac{k+1}{2}$  vertices of each colour. This implies that  $v_i$  and  $v_{i+k+1}$  receive the same colour, as otherwise the sequence  $v_{i+1}, \dots, v_{i+k+1}$  would contain more than  $\frac{k+1}{2}$  vertices of the colour assigned to  $v_{i+k+1}$ . For all vertices  $v_i$  and  $v_j$ , if  $\ell = \frac{j-i}{k+1}$  in the finite field  $\mathbb{Z}_n$ , then  $j = i + \ell(k+1)$  and  $v_i, v_{i+(k+1)}, v_{i+2(k+1)}, \dots, v_{i+\ell(k+1)} = v_j$  all receive the same colour. Thus all the vertices receive the same colour, which is a contradiction. Hence the claimed 2-colouring does not exist.

Note that being majority  $c$ -colourable is not closed under taking induced subgraphs. For example, let  $G$  be the digraph with  $V(G) = \{a, b, c, d\}$  and  $E(G) = \{ab, bc, ca, cd\}$ . Then  $G$  has a majority 2-colouring: colour  $a$  and  $c$  by 1 and colour  $b$  and  $d$  by 2. But the subdigraph induced by  $\{a, b, c\}$  is a directed 3-cycle, which has no majority 2-colouring.

The remainder of the paper takes a probabilistic approach to Theorem 2, proving several results that provide evidence for Theorem 2. A probabilistic approach is reasonable, since in a random 3-colouring, one would expect that a third of the out-neighbours of each vertex  $v$  receive the same colour as  $v$ . So one might hope that there is enough slack to prove that for *every* vertex  $v$ , at most half the out-neighbours of  $v$  receive the same colour as  $v$ . Section 2 proves Theorem 2 for digraphs with very large minimum out-degree (at least logarithmic in the number of vertices), and then for digraphs with large minimum out-degree (at least a constant) and not extremely large maximum in-degree. Section 3 shows that large minimum out-degree (at least a constant) is sufficient to prove the existence of one of the colour classes in Theorem 2. Section 4 discusses multi-colour generalisations of Theorem 2.

Before proceeding, we mention some related topics in the literature:

- For undirected graphs, the situation is much simpler. Lovász [7] proved that for every undirected graph  $G$  and integer  $k \geq 1$ , there is a  $k$ -colouring of  $G$  such that every vertex  $v$  has at most  $\frac{1}{k} \deg(v)$  neighbours receiving the same colour as  $v$ . The proof is simple. Consider a  $k$ -colouring of  $G$  that minimises the number of monochromatic edges. Suppose that some vertex  $v$  coloured  $i$  has greater than  $\frac{1}{k} \deg(v)$  neighbours coloured  $i$ . Thus less than  $\frac{k-1}{k} \deg(v)$  neighbours of  $v$  are not coloured  $i$ , and less than  $\frac{1}{k} \deg(v)$  neighbours of  $v$  receive some colour  $j \neq i$ . Thus,

if  $v$  is recoloured  $j$ , then the number of monochromatic edges decreases. Hence no vertex  $v$  has greater than  $\frac{1}{k} \deg(v)$  neighbours with the same colour as  $v$ .

- Seymour [9] considered digraph colourings such that every non-sink vertex receives a colour different from some outneighbour, and proved that a strongly-connected digraph  $G$  admits a 2-colouring with this property if and only if  $G$  has an even directed cycle. The proof shows that every digraph has such a 3-colouring, which we repeat here: We may assume that  $G$  is strongly connected. In particular, there are no sink vertices. Choose a maximal set  $X$  of vertices such that  $G[X]$  admits a 3-colouring where every vertex has a colour different from some outneighbour. Since any directed cycle admits such a colouring,  $X \neq \emptyset$ . If  $X \neq V(G)$ , then choose an edge  $uv$  entering  $X$  and colour  $u$  different from the colour of  $v$ , contradicting the maximality of  $X$ . So  $X = V(G)$ . (The same proof shows two colours suffice if you start with an even cycle.)
- Alon [1, 2] posed the following problem: Is there a constant  $c$  such that every digraph with minimum out-degree at least  $c$  can be vertex-partitioned into two induced digraphs, one with minimum out-degree at least 2, and the other with minimum out-degree at least 1?
- Wood [11] proved the following edge-colouring variant of majority colourings: For every digraph  $G$  and integer  $k \geq 2$ , there is a partition of  $E(G)$  into  $k$  acyclic subgraphs such that each vertex  $v$  of  $G$  has out-degree at most  $\lceil \frac{\deg^+(v)}{k-1} \rceil$  in each subgraph. The bound  $\lceil \frac{\deg^+(v)}{k-1} \rceil$  is best possible, since in each acyclic subgraph at least one vertex has out-degree 0.

## 2 Large Outdegree

We now show that minimum out-degree at least logarithmic in the number of vertices is sufficient to guarantee a majority 3-colouring. All logarithms are natural.

**Theorem 3.** *Every  $n$ -vertex graph  $G$  with minimum out-degree  $\delta > 72 \log(3n)$  has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.*

*Proof.* Randomly and independently colour each vertex of  $G$  with one of three colours  $\{1, 2, 3\}$ . Consider a vertex  $v$  with out-degree  $d_v$ . Let  $X(v, c)$  be the random variable that counts the number of out-neighbours of  $v$  coloured  $c$ . Of course,  $\mathbf{E}(X(v, c)) = d_v/3$ . Let  $A(v, c)$  be the event that  $X(v, c) > d_v/2$ . Note that  $X(v, c)$  is determined by  $d_v$  independent trials and changing the outcome of any one trial changes  $X(v, c)$  by at most 1.

By the simple concentration bound<sup>1</sup>,

$$\mathbf{P}(A(v, c)) \leq \exp(-(d_v/6)^2/2d_v) = \exp(-d_v/72) \leq \exp(-\delta/72).$$

The expected number of events  $A(v, c)$  that hold is

$$\sum_{v \in V(G)} \sum_{c \in \{1,2,3\}} \mathbf{P}(A(v, c)) \leq 3n \exp(-\delta/72) < 1,$$

where the last inequality holds since  $\delta > 72 \log(3n)$ . Thus there exist colour choices such that no event  $A(v, c)$  holds. That is, a majority 3-colouring exists.  $\square$

The following result shows that large out-degree (at least a constant) and not extremely large in-degree is sufficient to guarantee a majority 3-colouring.

**Theorem 4.** *Every digraph  $G$  with minimum out-degree  $\delta \geq 1200$  and maximum in-degree at most  $\exp(\delta/72)/12\delta$  has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.*

*Proof.* We assume  $\delta \geq 1200$ , as otherwise the minimum out-degree  $\delta$  is greater than the maximum in-degree  $\exp(\delta/72)/12\delta$ , which does not make sense.

We use the following weighted version of the Local Lemma [4, 8]: Let  $\mathcal{A} := \{A_1, \dots, A_n\}$  be a set of ‘bad’ events, such that each  $A_i$  is mutually independent of  $\mathcal{A} \setminus (D_i \cup \{A_i\})$ , for some subset  $D_i \subseteq A$ . Assume there are numbers  $t_1, \dots, t_n \geq 1$  and a real number  $p \in [0, \frac{1}{4}]$  such that for  $1 \leq i \leq n$ ,

$$(a) \mathbf{P}(A_i) \leq p^{t_i} \quad \text{and} \quad (b) \sum_{A_j \in \mathcal{D}_i} (2p)^{t_j} \leq t_i/2.$$

Then with positive probability no event  $A_i$  occurs.

Define  $p := \exp(-\delta/72)$ . Since  $\delta \geq 1200$  we have  $p \in [0, \frac{1}{4}]$ . Randomly and independently colour each vertex of  $G$  with one of three colours  $\{1, 2, 3\}$ . Consider a vertex  $v$  with out-degree  $d_v$ . Let  $X(v, c)$  be the random variable that counts the number of out-neighbours of  $v$  coloured  $c$ . Of course,  $\mathbf{E}(X(v, c)) = d_v/3$ . Let  $A(v, c)$  be the event that  $X(v, c) > d_v/2$ . Let  $\mathcal{A} := \{A(v, c) : v \in V(G), c \in \{1, 2, 3\}\}$  be our set of events. Let  $t(v, c) := t_v := d_v/\delta$  be the associated weight. Then  $t_v \geq 1$ . It suffices to prove that conditions (a) and (b) hold.

Note that  $X(v, c)$  is determined by  $d_v$  independent trials and changing the outcome of any one trial changes  $X(v, c)$  by at most 1. By the simple concentration bound,

$$\mathbf{P}(A(v, c)) \leq \exp(-(d_v/6)^2/2d_v) = \exp(-d_v/72) = \exp(-\delta t_v/72) = p^{t_v}.$$

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<sup>1</sup>The simple concentration bound says that if  $X$  is a random variable determined by  $d$  independent trials, such that changing the outcome of any one trial can affect  $X$  by at most  $c$ , then  $\mathbf{P}(X > \mathbf{E}(X) + t) \leq \exp(-t^2/2c^2d)$ ; see [8, Chapter 10]. With  $\mathbf{E}(X_v) = d_v/3$  and  $t = d_v/6$  and  $c = 1$  we obtain the desired upper bound on  $\mathbf{P}(X_v > d_v/2)$ .

Thus condition (a) is satisfied. For each event  $A(v, c)$  let  $D(v, c)$  be the set of all events  $A(w, c') \in \mathcal{A}$  such that  $v$  and  $w$  have a common out-neighbour. Then  $A(v, c)$  is mutually independent of  $\mathcal{A} \setminus (D(v, c) \cup \{A(v, c)\})$ . Since  $t_w \geq 1$ ,

$$\sum_{A(w, c') \in D(v, c)} (2p)^{t_w} \leq \sum_{A(w, c') \in D(v, c)} (2p)^1 = 2p|D(v, c)|.$$

Since each out-neighbour of  $v$  has in-degree at most  $\exp(\delta/72)/12\delta$ , we have  $|D(v, c)| \leq d_v \exp(\delta/72)/4\delta$  and

$$\sum_{A(w, c') \in D(v, c)} (2p)^{t_w} \leq p d_v \exp(\delta/72)/2\delta = \exp(-\delta/72)t_v \exp(\delta/72)/2 = t_v/2.$$

Thus condition (b) is satisfied. By the local lemma, with positive probability, no event  $A(v, c)$  occurs. That is, a majority 3-colouring exists.  $\square$

Note that the conclusion in Theorems 3 and 4 is stronger than in Conjecture 2. We now show that such a conclusion is impossible (without some extra degree assumption).

**Lemma 5.** *For all integers  $k$  and  $\delta$ , there are infinitely many digraphs  $G$  with minimum out-degree  $\delta$ , such that for every vertex  $k$ -colouring of  $G$ , there is a vertex  $v$  such that all the out-neighbours of  $v$  receive the same colour.*

*Proof.* Start with a digraph  $G_0$  with at least  $k\delta$  vertices and minimum out-degree  $\delta$ . For each set  $S$  of  $\delta$  vertices in  $G_0$ , add a new vertex with out-neighbourhood  $S$ . Let  $G$  be the digraph obtained. In every  $k$ -colouring of  $G$ , at least  $\delta$  vertices in  $G_0$  receive the same colour, which implies that for some vertex  $v \in V(G) \setminus V(G_0)$ , all the out-neighbours of  $v$  receive the same colour.  $\square$

### 3 Stable Sets

A set  $T$  of vertices in a digraph  $G$  is a *stable set* if for each vertex  $v \in T$ , at most half the out-neighbours of  $v$  are also in  $T$ . A majority colouring is a partition into stable sets. Of course, if a digraph has a majority 3-colouring, then it contains a stable set with at least one third of the vertices. The next lemma provides a sufficient condition for the existence of such a set.

**Theorem 6.** *Every  $n$ -vertex digraph  $G$  with minimum out-degree at least 22 has a stable set with at least  $\frac{n}{3}$  vertices.*

Theorem 6 is proved via the following more general lemma.

**Lemma 7.** *For  $0 < \alpha < p < \beta < 1$ , every digraph  $G$  with minimum out-degree at least*

$$\delta := \left\lceil \frac{(\beta + p) \log\left(\frac{p}{p-\alpha}\right)}{(\beta - p)^2} \right\rceil$$

*contains a set  $T$  of at least  $\alpha n$  vertices, such that  $|N_G^+(v) \cap T| \leq \beta |N_G^+(v)|$  for every vertex  $v \in T$ .*

*Proof.* Let  $d_v := |N_G^+(v)|$  be the out-degree of each vertex  $v$  of  $G$ . Initialise  $S := \emptyset$ . For each vertex  $v$  of  $G$ , add  $v$  to  $S$  independently and randomly with probability  $p$ . Let  $X_v := |N_G^+(v) \cap S|$ . Note that  $X_v \sim \text{Bin}(d_v, p)$  and

$$\mathbf{P}(X_v > \beta d_v) = \sum_{k \geq \lfloor \beta d_v \rfloor + 1}^{d_v} \binom{d_v}{k} p^k (1-p)^{d_v-k}. \quad (1)$$

By the Chernoff bound<sup>2</sup>,

$$\mathbf{P}(X_v > \beta d_v) \leq \exp\left(-\frac{(\beta-p)^2}{\beta+p} d_v\right) \leq \exp\left(-\frac{(\beta-p)^2}{\beta+p} \delta\right) \leq \frac{p-\alpha}{p}, \quad (2)$$

where the last inequality follows from the definition of  $\delta$ . Let  $B := \{v \in S : X_v > \beta d_v\}$ . Then

$$\mathbf{E}(|B|) = \sum_{v \in V(G)} \mathbf{P}(v \in S \text{ and } X_v > \beta d_v).$$

Since the events  $v \in S$  and  $X_v > \beta d_v$  are independent,

$$\mathbf{E}(|B|) = \sum_{v \in V(G)} \mathbf{P}(v \in S) \mathbf{P}(X_v > \beta d_v) = p \sum_{v \in V(G)} \mathbf{P}(X_v > \beta d_v) \leq (p-\alpha)n.$$

Let  $T := S \setminus B$ . Thus  $|N_G^+(v) \cap T| \leq \beta d_v$  for each vertex  $v \in T$ , as desired. By the linearity of expectation,

$$\mathbf{E}(|T|) = \mathbf{E}(|S|) - \mathbf{E}(|B|) = pn - \mathbf{E}(|B|) \geq \alpha n.$$

Thus the desired set  $T$  exists. □

*Proof of Theorem 6.* The proof follows that of Theorem 7 with one change. Let  $\alpha := \frac{1}{3}$  and  $\beta := \frac{1}{2}$  and  $p := 0.38$ . Then  $\delta = 129$ . If  $22 \leq d_v \leq 128$  then direct calculation of the formula in (1) verifies that  $\mathbf{P}(X_v > \beta d_v) \leq \frac{p-\alpha}{p}$ , as in (2). For  $d_v \geq 129$  the Chernoff bound proves (2). The rest of the proof is the same as in Theorem 7. □

Note the following corollary of Theorem 7 obtained with  $\alpha = \frac{1}{2} - \epsilon$  and  $p = \frac{1}{2} - \frac{\epsilon}{2}$ . This says that graphs with large minimum out-degree have a stable set with close to half the vertices.

**Proposition 8.** *For  $0 < \epsilon < \frac{1}{2}$ , every  $n$ -vertex digraph  $G$  with minimum out-degree at least  $2\epsilon^{-2}(2-\epsilon) \log(\frac{1-\epsilon}{\epsilon})$  contains a stable set of at least  $(\frac{1}{2} - \epsilon)n$  vertices.*

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<sup>2</sup>The Chernoff bound implies that if  $X \sim \text{Bin}(d, p)$  then  $\mathbf{P}(X \geq (1+\epsilon)pd) \leq \exp(-\frac{\epsilon^2}{2+\epsilon} pd)$  for  $\epsilon \geq 0$ . With  $\epsilon = \frac{\beta}{p} - 1$  we have  $\mathbf{P}(X > \beta d) \leq \exp(-\frac{(\beta-p)^2}{p+\beta} d)$ .

## 4 Multi-Colour Generalisation

In the initial version of this paper we suggested the following natural generalisation of Theorem 2:

**Conjecture 9.** *For  $k \geq 2$ , every digraph has a vertex  $(k+1)$ -colouring such that for each vertex  $v$ , at most  $\frac{1}{k} \deg^+(v)$  out-neighbours of  $v$  receive the same colour as  $v$ .*

Gregory Gauthier [personal communication, Banff workshop “New Trends in Graph Coloring”, October 2016] and independently Girão et al. [5] observed that this conjecture is false, as shown by the cyclic orientation of  $K_{2c-1}$ , which has out-degree  $c-1$  at every vertex, and thus all vertices receive distinct colors. Because of this example, we wonder if Theorem 9 holds if every vertex has out-degree at least  $c$ , which is a natural assumption in this context. Without this extra assumption, the following conjecture would be best possible by the above example.

**Conjecture 10.** *For  $k \geq 2$ , every digraph has a vertex  $(2k-1)$ -colouring such that for each vertex  $v$ , at most  $\frac{1}{k} \deg^+(v)$  out-neighbours of  $v$  receive the same colour as  $v$ .*

This conjecture would be best possible by the above example. The proof of Theorem 1 generalises to give an upper bound of  $k^2$  on the number of colours in Theorems 9 and 10. It is open whether the number of colours is  $O(k)$ .

Theorem 7 with  $\alpha = \frac{1}{k} - \epsilon$  and  $\beta = \frac{1}{k}$  and  $p = \frac{1}{k} - \frac{\epsilon}{2}$  implies the following ‘stable set’ version of Theorem 9 for digraphs with large minimum out-degree.

**Proposition 11.** *For  $k \geq 2$  and  $\epsilon \in (0, \frac{1}{k})$ , every  $n$ -vertex digraph  $G$  with minimum out-degree at least  $2\epsilon^{-2}(\frac{4}{k} - \epsilon) \log(\frac{2}{\epsilon k} - 1)$  contains a set  $T$  of at least  $(\frac{1}{k} - \epsilon)n$  vertices, such that for every vertex  $v \in T$ , at most  $\frac{1}{k} \deg^+(v)$  out-neighbours of  $v$  are also in  $T$ .*

## 5 Open Problems

In addition to resolving Theorem 2, the following open problems arise from this paper:

1. Is there a constant  $\beta < 1$  for which every digraph has a 3-colouring, such that for every vertex  $v$ , at most  $\beta \deg^+(v)$  out-neighbours receive the same colour as  $v$ ?
2. Does every tournament have a majority 3-colouring?
3. Does every Eulerian digraph have a majority 3-colouring? Note that for an Eulerian digraph  $G$ , if each vertex  $v$  has in-degree and out-degree  $\deg(v)$ , then by the result for undirected graphs mentioned in Section 1, the underlying undirected graph of  $G$  has a 4-colouring such that each vertex  $v$  has at most  $\frac{1}{2} \deg(v)$  in- or- out-neighbours with the same colour as  $v$ . In particular,  $G$  has a majority 4-colouring. By an analogous argument every Eulerian digraph has a 3-colouring such that each vertex  $v$  has at most  $\frac{2}{3} \deg(v)$  in- or- out-neighbours with the same colour as  $v$ , thus proving a special case of the first question above.

4. Does every digraph in which every vertex has in-degree and out-degree  $k$  have a majority 3-colouring? A variant of Theorem 4 proves this result for  $k \geq 144$ .
5. Is there a characterisation of digraphs that have a majority 2-colouring (or a polynomial time algorithm to recognise such digraphs)?
6. Does every digraph have a  $O(k)$ -colouring such that for each vertex  $v$ , at most  $\frac{1}{k} \deg^+(v)$  out-neighbours receive the same colour as  $v$  (for all  $k \geq 2$ )?
7. A digraph  $G$  is *majority  $c$ -choosable* if for every function  $L : V(G) \rightarrow \mathbb{Z}$  with  $|L(v)| \geq c$  for each vertex  $v \in V(G)$ , there is a majority colouring of  $G$  with each vertex  $v$  coloured from  $L(v)$ . Is every digraph majority  $c$ -choosable for some constant  $c$ ? The proof of Theorem 1 shows that acyclic digraphs are majority 2-choosable, and obviously Theorem 3 and Theorem 4 extend to the setting of choosability.
8. Consider the following fractional setting. Let  $S(G)$  be the set of all stable sets of a digraph  $G$ . Let  $S(G, v)$  be the set of all stable sets containing  $v$ . A *fractional majority colouring* is a function that assigns each stable set  $T \in S(G)$  a weight  $x_T \geq 0$  such that  $\sum_{T \in S(G, v)} x_T \geq 1$  for each vertex  $v$  of  $G$ . What is the minimum number  $k$  such that every digraph  $G$  has a fractional majority colouring with total weight  $\sum_{T \in S(G)} x_T \leq k$ ?

## 6 Update

Since this paper was submitted there has been significant progress on several of the above open problems.

Question 1 was answered by Anholcer et al. [3] with  $\beta = \frac{2}{3}$ . More generally, Girão et al. [5] and Knox and Šámal [6] proved that for  $k \geq 2$  every digraph has a  $k$ -colouring such that for each vertex  $v$ , at most  $\frac{2}{k} \deg^+(v)$  out-neighbours of  $v$  receive the same colour as  $v$ . This answers Question 6 and answers Question 1 with  $\beta = \frac{2}{3}$ .

Question 2 was partially answered by Girão et al. [5]. They proved that every tournament with minimum out-degree at least 50 is majority 3-colourable, and that every tournament can be 3-coloured so that all but at most 7 vertices receive the same colour as at most half of their out-neighbours.

Question 7 was answered by Anholcer et al. [3], Girão et al. [5] and Knox and Šámal [6] who all proved that every digraph is majority 4-choosable. Moreover, the above results concerning Questions 1 and 6 were established in the choosability setting.

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