Maximal 2-local subgroups of $E_7(q)$

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Abstract. Let $G = E_7(q)$ be the finite exceptional group of Lie type (universal version). We classify, up to conjugacy, all maximal-proper 2-local subgroups of $G$, that is, all 2-local $M < G$ which are maximal with respect to inclusion among all proper subgroups of $G$ which are 2-local. For this purpose, we first determine, up to conjugacy, all elementary-abelian 2-subgroups of $G$ containing the center $Z(G)$. These classifications are an important first step towards a classification of the 2-radical subgroups of $G$.

1. Introduction

Many of the central conjectures in representation theory, like the Dade, McKay, or Alperin Weight Conjecture, have inductive versions, that is, reductions to finite simple groups. Navarro & Tiep [21], for example, proved that the Alperin Weight Conjecture is true for every finite group, if every finite simple group satisfies a stronger condition, namely, AWC goodness. Verifying that the latter holds, requires a detailed study of the finite simple groups and their covering groups. An important role in all these inductive conjectures is played by $p$-radical subgroups, that is, nontrivial subgroups $R \leq G$ with $R = O_p(N_G(R))$, the largest normal $p$-subgroup of the normaliser $N_G(R)$. Such subgroups are omnipresent in modular representation theory: for example, a defect group of a block is radical, the subgroup $R$ of a weight $(R, \varphi)$ is radical, and the first nontrivial subgroup in any radical chain is radical. Moreover, if the radical subgroups of $G$ are known, then the essential rank of the Frobenius category $\mathcal{F}_G(D)$, $D \leq G$ a Sylow subgroup, can be determined, cf. [1]. Another application of radical subgroups is the study of so-called $p$-local geometries, cf. [18]: Finite group theorists in the 1980’s tried to generalise Tits’ classification of spherical buildings to a suitable subclass of diagram geometries which would incorporate the class of sporadic groups. More generally, large parts of the classification of the finite simple groups are based on the analysis of $p$-local subgroups; one reason for this is that the Alperin-(Goldschmidt) Fusion Theorem implies that the fusion of $p$-elements is controlled by the normaliser of $p$-groups.

The classification of radical subgroups therefore is an important (and open) research problem; classifications are known for the the symmetric, classical, sporadic, and the exceptional groups $^2G_2(q)$, $G_2(q)$, $^3D_4(q)$, $^2F_4(q)$, $^2B_2(q)$, $E_6(q)$, $^2E_7(q)$, see [3] and the references given there. This paper is part of a current research project ([2–5]) which considers the remaining exceptional groups of Lie type, especially when $p$ is small and a bad prime. Here we investigate $E_7(q)$, the universal version of the finite group of Lie type $E_7$, and the prime $p = 2$, which is bad for $E_7(q)$, see [8, Table 1].

A subgroup $M \leq G$ is $p$-local if it is the normaliser of a nontrivial $p$-subgroup of $G$. It is maximal $p$-local if $M$ is maximal with respect to inclusion among all $p$-local subgroups of $G$. It is local maximal if it is $p$-local for some prime $p$ and maximal among all subgroups of $G$. If $G$ has a normal $p$-subgroup, then the only maximal $p$-local subgroup is $G$ itself. We say $M < G$ maximal-proper $p$-local if $M$ is $p$-local and maximal with respect to inclusion among all proper subgroups of $G$ which are $p$-local. Clearly, if $O_p(G) = 1$, then the maximal-proper $p$-local subgroups are exactly the maximal $p$-local subgroups. If $R \leq G$ is $p$-radical, then $N_G(R)$ is $p$-local and $N_G(R) \leq N_G(C)$ for every characteristic subgroup $C \leq R$. In particular, $N_G(R)$ is contained in some maximal-proper $p$-local $M \leq G$, so that $N_G(R) = N_M(R)$ and $R$ is $p$-radical in $M$. Hence, every radical $p$-subgroup of $G$ is radical in some maximal-proper $p$-local subgroup of $G$. A strategy for classifying radical $p$-subgroups is as follows:

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Classify all maximal-proper $p$-local $M \leq G$ up to conjugacy.

Classify the radical $p$-subgroups $R$ of each such $M$.

Decide which of these $R$ are $p$-radical in $G$, and fuse these in $G$.

Cohen et al. [11] classified local maximal subgroups of exceptional groups of Lie type. However, not every maximal-proper $p$-local subgroup is local maximal, and the details obtained in the classification of maximal-proper $p$-local subgroups have been proved to be very useful for the classification of the radical subgroups, cf. [2,3,5].

In this paper, we consider the first step of this classification, namely, the classification of maximal-proper $2$-local subgroups of $G = E_7(q)$. If $q$ is even, then $O_2(G) = 1$ and the maximal $2$-local subgroups (and $2$-radical subgroups) of $G$ are known by the Borel-Tits Theorem [16, Corollary 1.3.5]; these are the maximal parabolic subgroups, of which, up to conjugacy, there are six. Thus, throughout the paper, we suppose that $q$ is odd.

Then $G$ has a nontrivial center $Z = Z(G)$ of order $2$, hence $G = N_G(Z)$ is maximal $2$-local itself; we classify all maximal-proper $2$-local of $G$, up to conjugacy.

1.1. Strategy of the proof. Let $G = E_7(q)$ with $q$ odd, and write $Z = Z(G)$. To classify the maximal-proper $2$-local subgroups of $G$, we first show (Lemma 3.3) that every such $M \leq G$ has the form $M = N_G(E)$, where $E$ is an abelian elementary $2$-subgroup (with $Z < E$ if $|E| \neq 2$), cyclic of order $4$ (with $Z < E$), or extraspecial (with $Z(E) = Z$). The main part of the paper then deals with the classification of the elementary abelian $2$-subgroups of $G$, and their local structure. For this, we use the obvious inductive strategy: If the elementary abelian groups of order $2^i$ are known up to conjugacy, say $K_1, \ldots, K_i$, then the groups of order $2^{i+1}$ can be constructed as $E = (K_i, k)$ with $E \leq C_G(K_i)$ and $k \in C_G(K_i) \setminus K_i$. The elementary abelian groups of order $2^2$ containing $Z$ can readily be determined since there are only two $G$-classes of non-central involutions in $G$. Clearly, if $E = (K_i, k)$, then $C_G(E) = C_{C_G(K_i)}(k)$, which helps to determine the structure of $C_G(E)$; determining the normaliser structure is usually the harder part.

An elementary abelian $2$-subgroup $E \leq G$ either is toral, that is, lies in a maximal split torus $(q \pm 1)^7$ with Weyl group $W$, or non-toral. We show (Section 4) that if $E$ is toral, then the $G$-class of $E$ is uniquely determined by $C_W(E)$; in particular, we can compute $C_W(E)$ and $N_W(E)$ explicitly in $2^7W \leq (q \pm 1)^7W$, using the computer algebra system Magma [7]. We will see that $N_G(E)/C_G(E) \cong N_W(E)/C_W(E)$, hence $N_G(E)$ can easily be determined for toral $E \leq G$.

Comment: This strategy for classifying all elementary abelian $2$-subgroups of $G$ is straightforward, but technical; it involves a number of case distinctions which makes the exposition lengthy and sometimes tedious. However, for the sake of completeness (and for actually proving our claims), we have decided to give all details.

There is the danger that some proofs (in particular, the technical ones in Sections 6.1 and 6.2) detract from the general, straightforward, progression of the paper. To increase comprehension and to gain an understanding of the general proceeding, we recommend that these proofs are skipped at first reading.

1.2. Structure of the paper. In Section 2, we state our main results, that is, the classification of the elementary abelian $2$-subgroups containing $Z$, and the classification of the maximal-proper $2$-local subgroups. In Section 3, we provide preliminary results on maximal-proper $p$-local subgroup. The elementary abelian subgroups of $G$ lying in maximal split tori are discussed in Section 4. In Section 5, we classify the maximal proper $2$-local subgroups $N_G(E)$ with $E$ cyclic. One of these subgroups, $M_1$, will play an important role in the classification of the maximal proper $2$-local $N_G(E)$ with $E$ elementary abelian; we investigate this group in Section 6 and provide some preliminary results. The proofs in Sections 6.1 and 6.2 are particularly technical and tedious, and can be skipped for the understanding of the remaining results. The maximal-proper $2$-local $N_G(E)$ with $E$ extraspecial or elementary abelian of order at most $2^2$ are classified in Sections 7 and 8. Section 9 and 10 complete the classification of the elementary abelian subgroup $E \leq G$ with $Z < E$; this also completes the classification of the maximal-proper $2$-local subgroups of $G$.

1.3. Notation. Our notation for simple groups, group extensions, and conjugacy classes are as in [2,5], thus taken from [12,16]. If not indicated by brackets, then we read group extensions $A.B.C$ from the left, that
is, $A.B.C = (A.B).C$. If $n,m$ are positive integers, then $n^m$ denotes the direct product of $m$ copies of cyclic groups of order $n$. We denote by $n_p$ largest $p$-power dividing $n$; if $q$ is fixed and $\varepsilon \in \{\pm 1\}$, then
\[
n_\varepsilon = \gcd(n, q - \varepsilon).
\]

Let $\mathcal{R}_p(H)$ be the set of all $p$-radical subgroups of a finite group $H$; if $K \leq H$, then write $\text{Out}_H(K) = N_H(K)/KC_H(K)$. If $H_1, H_2 \leq H$ and $Z \leq Z(H_1) \cap Z(H_2)$, then $H_1 \circ H_2$ is the central product of $H_1$ and $H_2$ over $Z$. We write $H_1 \cong H_2$ whenever $H_1^x \leq H_2$ for some $x \in H$. Analogously, $H_1 = H_1 H_1$ and $y \in H_1$ with $y \in H$ are defined.

Recall the notation $\text{SL}_n^i(q)$ and $\text{GL}_n^i(q)$; if $\varepsilon = 1$, then these are the special linear and general linear groups of degree $n$ over the field $\text{GF}(q)$; if $\varepsilon = -1$, then these are the corresponding special unitary and unitary group, respectively, defined over $\text{GF}(q^2)$. Recall that $\text{SL}_2(q) = \text{SL}_2^{-1}(q) = \text{Sp}_2(q)$. We use the definition of “graph” and “field” automorphisms used in [16], see [16, Warning 2.5.2]; for example, the order 2 field automorphism of $\text{SL}_n^{-1}(q)$ is called a graph automorphism. For a suitable integer $z$, we denote by $\text{SL}_n^z(q)/\zeta$ the quotient of $\text{SL}_n^i(q)$ by the unique central subgroup of order $z$.

We denote by $S_n$, $D_n$, and $Q_n$ the symmetric group of degree $n$, the dihedral group of order $n$, and the quaternion group of order $n$, respectively.

The following notation will be retained throughout the paper.

**Notation 1.1.** Let $G = E_2(q)$ (universal) with $q$ odd; write $Z = Z(G) = \langle z \rangle = 2$. Let $a \geq 2$ be the largest integer with $2^{a+1} \mid q - 1$; let $\varepsilon \in \{\pm 1\}$ with $2^a \mid q - \varepsilon$. It follows from [16, Table 2.2] that $|G|_2 = 2^{7a+10}$. We use the notation of [16], and denote by $2Z = \langle z \rangle$ the $G$-class of $z$. Let $r$ be the characteristic of the underlying field $\text{GF}(q)$. Let $T = (q - \varepsilon)^2$ and $T^* = (q + \varepsilon)^2$ be maximal tori of $G$, with Weyl group $W = W(E_7)$.

### 2. Main results

We use the previous notation; in particular, $G = E_2(q)$ (universal version) with $q$ odd, and $Z = Z(G)$. The next theorem classifies the elementary abelian 2-subgroups $E < G$ with $Z \leq E$ up to conjugacy. We denote by $E^i$ such a group of order $2^i$; if it is not unique up to conjugacy, then we label non-conjugate groups by $E^i_1, E^i_2, \ldots$. An elementary abelian 2-subgroup $E$ is $E^i_1$-pure if every subgroup $2^j \leq E$ is conjugate in $G$ to $E^i_1$.

**Theorem 2.1.** If $E \leq G$ is an elementary abelian 2-group with $Z = Z(G) \leq E$, then the following hold:

a) If $E = 2$, then $E = Z = \langle z \rangle$ and $C_G(E) = N_G(E) = G$.

b) If $E = 2^2$, then $E = \langle E^2 \rangle$ with $C_G(E^2) = N_G(E^2) = (L_1 \circ L_2).2$ where $L_1 = \text{SL}_2(q)$ and $L_2 = \text{Spin}^3_2(q)$.

c) If $E = 2^3$, then $E \in \{E^1_1, \ldots, E^1_4\}$ with
\[
C_G(E^1_1) = (\text{SL}_2(q) \circ \text{Spin}^3_2(q) \circ \text{Spin}^3_2(q)) \cdot 2^2,
\]
\[
C_G(E^1_2) = ((q - \varepsilon)^2 \circ 3, \text{SL}_6(q)) \cdot 2^3,
\]
\[
C_G(E^1_3) = ((q + \varepsilon)^2 \circ 3, \text{SL}_6(q)) \cdot 2^3.
\]

We can assume $E^i_1 = \langle E^i \rangle$ with $v \in L_2$, $v \in (L_1 \circ L_2) \setminus L_2$, and $v \in N_G(E^2) \setminus (L_1 \circ L_2)$, respectively.

d) If $E = 2^4$, then $E \in \{E^1_1, \ldots, E^1_5\}$ with
\[
C_G(E^1_1) = (\text{SL}_2(q) \circ \text{Spin}^3_2(q) \circ \text{Spin}^3_2(q) \circ \text{Spin}^3_2(q)) \cdot 2^5,
\]
\[
C_G(E^1_2) = (\varepsilon \circ 5, \text{SL}_4(q) \times \text{SL}_4(q)) \cdot 2^3,
\]
\[
C_G(E^1_3) = (\varepsilon \circ 5, \text{SL}_4(q) \times \text{SL}_4(q)) \cdot 2^3.
\]

We can assume $E^1_2 = \langle E^1_2, E^1_3 \rangle$, $E^1_4 = \langle E^1_1, E^1_3 \rangle$, and $E^1_5 = \langle E^1_2, E^1_3 \rangle$. The group $E^1_1$ is $E^1_1$-pure; $E^1_5$ is $E^1_1$-pure. We have $\text{Out}_G(E^1_4) \in \{S_4, D_8\}$ and $\text{Out}_G(E^1_4) = S_4$ for $i \in \{2, 3\}$. 


In particular, all groups are maximal-proper 2-local subgroups of $G$.

We can suppose
\[ E_2^5 = \langle E_1^4, E_2^5 \rangle, \quad E_3^5 = \langle E_2^4, E_3^5 \rangle. \]

If $i \in \{2, 3\}$, then $\text{Out}_G(E_i^5) = 2^4 \cdot (S_3 \times S_3)$.

We can suppose $E_1^5 \leq E_1^6$ and $E_2^5 \leq E_3^5$. If $i \in \{1, 4\}$, then $\text{Out}_G(E_i^5) = 2^4 \cdot S_6$.

If $E = 2^7$, then $E \in G \{ E_1^1, \ldots, E_5^1 \}$ with $T = (q - \varepsilon)^7$ and $T^* = (q + \varepsilon)^7$, and
\[ C_G(E_1^1) = T.2 = T.Z(W), \quad C_G(E_2^1) = T^*.2 = T^*.Z(W), \]
\[ C_G(E_3^1) = T.Z(W), \quad C_G(E_4^1) = T.W, \quad C_G(E_5^1) = T.W. \]

In particular, all groups $E_i^1$ exist.

Theorem 2.1 is proved in Sections 5, 7, 9, and 10.

**Theorem 2.2.** Up to conjugacy, the maximal-proper 2-local subgroups of $G$ are $M_1, \ldots, M_{14}$ as in Table I.

Theorem 2.2 is proved by using Theorem 2.1 (and the additional information given in Sections 7, 9, and 10), and Propositions 5.2 and 8.1. Our results confirm the constructions we did for $q \in \{3, 5\}$ using the computer algebra system Magma [7].

### 3. On maximal-proper $p$-local subgroups

This preliminary section provides some results on maximal-proper $p$-local subgroups. In the following, let $H$ be a finite group and $p$ a prime. If $E \leq H$ is a $p$-group and $N_H(E)$ is maximal $p$-local, then
\[ N_H(E) = N_H(\Omega_1(Z(O_p(N_H(E))))), \]
where $\Omega_i(H) = \langle h \in H \mid |h| = p^i \rangle$. Since $\Omega_1(Z(O_p(N_H(E))))$ is elementary abelian, this shows that every maximal $p$-local $M$ has the form $M = N_H(E)$ with $E$ in
\[ \mathcal{E}R_p(H) = \{ E \leq H \mid 1 \neq E = \Omega_1(Z(O_p(N_H(E)))) \}. \]

**Lemma 3.1.** ([8, Lemma 2.3]) Let $E \in \mathcal{E}R_p(H)$ and $R = O_p(N_H(E))$. Then $N_H(E)$ is maximal $p$-local if and only if $N_H(E) = N_H(Y)$ for every nontrivial elementary abelian $p$-subgroup $Y$ of $\Omega_1(R)$ which is normal in $N_H(E)$; in particular, if $R$ is abelian, then $Y \leq E$.

In the following two lemmas let $H = Z.K$ be a central extension of $Z = p$ by a finite group $K \neq 1$ with $O_p(K) = 1$. For a subgroup $M \leq H$ let \[ \overline{M} = MZ/Z \]
be the image under the projection $H \to K$. Note that if $M < H$ is $p$-local, say $M = N_H(E)$ for a $p$-subgroup $E$, then $Z < O_p(M)$: clearly, $Z \leq O_p(M)$; if $Z = O_p(M)$, then $E = Z$, a contradiction to $M \neq H$. If $Z \leq E \leq H$, then $N_H(E) \to N_K(E)$, $g \to gZ$, is surjective with kernel $Z = Z \cap N_H(E)$: if $hZ \in N_K(E)$, then $E h Z / Z = E / Z$, and $Z \leq E$ proves that $E h = E$; we have shown that $N_H(E) = N_K(E)$.

\begin{lemma}
Let $H = Z.K$ as above.
\begin{enumerate}
\item The group $M < H$ is maximal-proper $p$-local if and only if $Z \leq M$ and $\overline{M} \leq K$ is maximal-proper $p$-local.
\end{enumerate}
\end{lemma}

\begin{proof}
a) Suppose $M$ is maximal-proper $p$-local, say $M = N_H(E)$ for some $p$-subgroup $1 < E < H$: as shown above, $Z < M$. Suppose, for a contradiction, that $\overline{M} \leq \overline{N} < K$ for some $p$-local $\overline{N} = N_K(\overline{E})$ with $Z < \overline{F} < H$, a $p$-subgroup. Now $N = N_H(F)$ is $p$-local with $\overline{N} = \overline{N}$, hence $M < N < H$. This is a contradiction to the maximality of $M$, thus $\overline{M}$ must be maximal-proper $p$-local. For the converse, suppose that $Z \leq M$ and $\overline{M} < K$ is maximal-proper $p$-local. If $M < N < H$ for some $N = N_H(F)$ with $Z < \overline{F} < H$, a $p$-subgroup, then $\overline{N} = N_K(\overline{F})$, and $Z \leq M$ implies that $\overline{M} \leq \overline{N} < K$. This contradicts the maximality of $\overline{M}$, hence $M$ is maximal-proper $p$-local. To prove the last assertion, suppose that $M$ is maximal-proper $p$-local; as shown above, $Z < O_p(M)$. If $M \neq N_H(O_p(M))$, then $N_H(O_p(M)) = H$ by the maximality of $M$, thus $1 \neq O_p(M) \leq K$, a contradiction to $O_p(K) = 1$. This proves $M = N_H(O_p(M))$, from which it follows that $\overline{M} = N_K(O_p(\overline{M}))$.

b) Since $O_p(N_K(\overline{E}))$ is abelian and $\overline{E} \in E R_p(K)$, it follows from a) and Lemma 3.1 that $M = N_H(E)$ is maximal-proper $p$-local if and only if $\overline{M} = N_K(\overline{E})$ is maximal $p$-local, if and only if $N_K(\overline{E}) \leq N_K(\overline{F})$ for every $\overline{F} \in E R_p(K)$ with $\overline{F} < \overline{E}$, if and only if $N_H(E) \leq N_H(F)$ for every $Z < F < E$ with $\overline{F} \in E R_p(K)$.
\end{proof}

The next lemma shows that every maximal-proper $p$-local $M < H$ has the form $M = N_H(E)$ for some $Z < E < H$ with $E$ elementary-abelian, cyclic of order $p^2$, or extraspecial (of exponent $p$ if $p$ is odd).
Lemma 3.3. Let $H = ZK$ be as before.

a) Let $M < H$ be maximal-proper $p$-local. If $Z < E < H$ is defined by $E = \Omega_1(Z(O_p(M)))$, then $M = N_H(E)$ and $E \in \mathcal{ER}_p(K)$. Also, $M = N_H(Y)$ for some $Z < Y \leq E$ such that one of the following holds:

1) $Y = \Omega_1(Z(E))$ elementary abelian; if $O_p(M)$ is abelian, then $Y = \Omega_1(O_p(M)) \in \mathcal{ER}_p(H)$,

2) $Z(\Omega_1(E))$ elementary abelian, $p$ odd, and $E$ extraspecial with $Z = Z(E)$ and exponent $p^2$,

3) $Y = Z(E)$ cyclic of order $p^2$ with $Z = \Omega_1(Y)$,

4) $Y = E$ extraspecial with $Z = Z(Y)$; if $p$ is odd, then $Y$ has exponent $p$.

b) If $E \in \mathcal{R}_p(H)$ with $Z < E$ is extraspecial and $N_K(E) \not\leq N_K(X)$ for every $Z < X < E$ with $X \in \mathcal{ER}_p(K)$, then $N_H(E)$ is maximal-proper $p$-local.

c) If $Z < E \leq H$ is cyclic of order $p^2$ and $O_p(N_H(E))$ is cyclic, then $N_H(E)$ is maximal-proper $p$-local.

Proof. a) Write $Q = O_p(M)$, so that $M = N_H(Q)$ and $M = N_K(\overline{Q})$ by Lemma 3.2. Let $Z < E \leq Q$ be defined by $E = \Omega_1(Z(\overline{Q}))$. The maximality of $M$ and $O_p(K) = 1$ imply that $N_K(E) = M$, so $E \in \mathcal{ER}_p(K)$.

In particular, $N_H(E)/Z = N_K(E) = M$, which proves that $M = N_H(Q) = N_H(E)$.

Let $X = \Omega_1(Z(E))$, so that $Z \leq X$ and $N_H(E) \leq N_H(X)$. If $Z < X$, then an argument as above shows that $M = N_H(E) = N_H(X)$, so we can take $Y = X$ and the first assertion of (1) holds. If $Q$ is abelian, then so are $E$ and $\overline{Q}$, thus $X = \Omega_1(E) \leq \Omega_1(Q)$. By definition, $E = \Omega_1(Z(\overline{Q})) = \Omega_1(\overline{Q})$ and $\Omega_1(Q)/Z = \Omega_1(Q) = \overline{E}$, so $\Omega_1(Q) \leq E$ and $\Omega_1(Q) \leq \Omega_1(E) = X$. In conclusion, $X = \Omega_1(E) = \Omega_1(Q) \in \mathcal{ER}_p(H)$ and (1) holds.

Now suppose that $Z = X$, so that $Z(E)$ is cyclic. Recall that $E$ is elementary abelian; hence, if $Z < Z(E)$, then $Z(E) = p^2$ and, by an argument as above, $N_H(E) = N_H(Z(E))$. It follows that (3) holds with $Y = Z(E)$.

Finally, suppose $Z = Z(E)$, so $Z = X$ and $E$ is nonabelian as $E \neq 1$. Since $Z$ is cyclic and $E/Z = \overline{E}$ is elementary abelian, $E$ is extraspecial. Now suppose that $p$ is odd. If $E$ has exponent $p$, then (4) holds with $Y = E$. If $E$ has exponent $p^2$, then $U = Z(\Omega_1(E))$ is elementary abelian with $Z < U < E$, hence $N_H(E) = N_H(U)$, and (2) holds with $Y = U$: note that $E \leq O_p(M)$ is not abelian.

b) By assumption, $E = O_p(M)$ and $E = O_p(M)$ is elementary abelian, hence $\overline{E} = \Omega_1(Z(O_p(M)))$ with $M = N_K(E)$, and $E \in \mathcal{ER}_p(K)$. By assumption, $N_K(E) \not\leq N_K(F)$ for every $F \in \mathcal{ER}_p(K)$ with $Z < F < E$, and $N_K(E)$ is maximal $p$-local by Lemma 3.1. Now $M$ is maximal-proper $p$-local by Lemma 3.2.

c) Since $O_p(M)$ is cyclic, so is $O_p(M) = O_p(N_K(E))$, hence $E = p = \Omega_1(O_p(N_K(E)))$ and $E \in \mathcal{ER}_p(K)$. It follows from Lemma 3.1 that $M$ is maximal $p$-local, and $M$ is maximal-proper $p$-local by Lemma 3.2. □

4. Weyl group action

Recall Notation 1.1. Let $\overline{G}$ be a simply connected algebraic group with $\overline{G}^F = G$, where $F$ a suitable Frobenius morphism. Let $\overline{T}$ be a maximal $F$-stable torus. It is well-known that the Weyl group $N_{\overline{T}}(\overline{T})/\overline{T}$ has the form

$$W = W(E_\gamma) = 2 \times \text{Sp}_6(2),$$

and the maximal tori of $G$ are parametrised by the conjugacy classes of $W$: note that all maximal tori in $G$ fuse into an algebraic torus of $\overline{G}$, see [6], and the theory of tori described in [10] implies that $F$-classes of Weyl group elements parametrised maximal tori; these $F$-classes of tori for $G$ are determined in [13]. Let $1_W \in W$ be the identity and let $w_0 \in W$ be the element of maximal length; recall that $w_0$ is unique and of order 2, hence $Z(W) = \langle w_0 \rangle$ by the simplicity of $\text{Sp}_6(2)$. It follows from [13, Table III and p. 902] that the two maximal tori of $G$ corresponding to the classes of $1_W$ and $w_0$ are $T = (q - \varepsilon)^7$ and $T^* = (q + \varepsilon)^7$.

Corollary 4.1. The element $w_0 \in W$ of maximal length acts by inversion on $\overline{T}$.

Proof. We use the notation of [10, Section 3.3]. Let $g\overline{T} = g\overline{T}g^{-1}$ with $g \in \overline{G}$ be the maximal $F$-stable torus corresponding to the conjugacy class of $w_0$, that is, $g^{-1}F(\overline{T}) = w_0 \in N_G(\overline{T})/\overline{T}$. Now $T^* = (g\overline{T})^F$ consists of those $gtg^{-1} \in g\overline{T}$ with $gtg^{-1} = F(g)F(t)(g^{-1}) = gF(t)w_0g^{-1}$, that is, of those $gtg^{-1}$ with $t \in \overline{T}$ and
$t^{w_0} = F(t)$. It follows from $T^* = (q + 1)^T$ that $t^{w_0} = F(t)$ exactly for those $t \in \mathcal{T}$ with $t^{q+1} = 1$. We can assume that $F(t) = t^q$ for all $t \in \mathcal{T}$, cf. [13, p. 894]; this implies $t^{w_0} = t^{-1}$.

Note that $C_G(T) = T.w_0$ if $q = 3$; thus, in this case, $T$ is a degenerate maximal torus.

Since $\mathcal{T}$ is a maximally split torus with $T = \mathcal{T}^\mathcal{F}$ and $T^* = (q\mathcal{T})^T$ with $g^{-1}F(g)\mathcal{T} = w_0$, it follows from [10, Prop. 3.3.6] that $N_G(T)/T \cong W^F \cong N_G(T^*)/T^*$, where, as usual, $W^F \leq N_G(\mathcal{T})/\mathcal{T}$ denotes the set of elements fixed by $F$. It is well-known that $W^F = W$; in fact, we have $N_G(T)/T \leq W$ and our results in Section 5 also imply that $N_G(T)/T = W$, hence $W = W^F$. Thus, $N_G(T) = T.W$ and, by abuse of notation, we also write $N_G(T^*) = T^*.W$.

For later reference, we note the following corollary.

**Corollary 4.2.** If $y \in G$ has order coprime $q$, then $y^q = y^{-1}$ for some $g \in G$.

**Proof.** We use the previous notation. By assumption, $y$ is semisimple, and therefore lies in some maximal torus $S \leq G$. Let $S$ be a maximal $F$-stable torus of $G$ with $S = \mathcal{S}^F$, say $S = h\mathcal{T}$ for some $h \in G$. Let $\mathcal{N} = N_G(S) = h(N_G(\mathcal{T}))$ and $w = h.w_0\mathcal{S} \in \mathcal{N}/\mathcal{S}$. Note that $w$ acts on $\mathcal{S}$ via inversion, and $w$ is the unique element of maximal length in $\mathcal{H}W$, generating $Z(hW) = 2$; thus it follows that $F(w) = w$. Since $S$ is normal in $\mathcal{N}^F$, we have $w \in \mathcal{N}^F \leq N_G(S)$. Now $g \in G$ with $gS = wS$ satisfies $y^g = y^w = y^{-1}$.

Let $E = \Omega_1(O_2(T)) = 2^7$ note that $Z \subset E$. If we identify $E = V = 2^7$, then $W$ acts on $V$ as a subgroup of $GL_7(2)$, stabilising the 1-dimensional subspace of $V$ corresponding to $Z$. Recall that $W' = Sp_6(2)$ is simple and $Z(W) = 2$ acts trivially on $V$, which implies that $W$ acts on $V$ as a subgroup of $GL_7(2)$ of type $W' = Sp_6(2)$.

A direct computation, for example using MAGMA [7], shows that $GL_7(2)$ has, up to conjugacy, three subgroups $W_1, W_2, W_3 \leq GL_7(2)$ with $W_i \cong Sp_6(2) = W'$ for $i \in \{1, 2, 3\}$. Only two of these subgroups, say $W_1$ and $W_2$, stabilise a 1-dimensional subspace of $V$ which shows that $W$ acts on $V$ as $W_1$ or $W_2$. A direct computation shows that, up to conjugacy, we can assume that $W_1$ and $W_2$ stabilise the same 1-dimensional subspace $Z$, and, moreover, that $W_1$ and $W_2$ have the same orbits on 1-dimensional subspaces of $V$ containing $Z$, for all $i \in \{1, \ldots, 7\}$. Also, if $L \leq V$ is such a subspace containing $Z$, then $C_{W_1}(L) \cong C_{W_2}(L)$ and $N_{W_1}(L) \cong N_{W_2}(L)$.

This shows that $W$ and both $W_1, W_2 \leq GL_7(2)$ have the same action on the set of subspaces of $V$ containing $Z$. If $L \leq V$ is such a subspace, then $C_W(L) \cong 2 \times C_{W_1}(L)$ and $N_W(L) \cong 2 \times N_{W_1}(L)$ where $i \in \{1, 2\}$ is arbitrary.

The results of this computation is summarised in Table II. Clearly, the above argument also holds if we identify $V = E^* = \Omega_1(O_2(T^*))$. This proves that Table II describes the action of $W$ on $E$ and $E^*$, respectively, when acting on sets of subgroups containing $Z$.

The next lemma is useful as it implies that $\text{Out}_G(E) = \text{Out}_W(E)$ for elementary abelian $E \leq T$ or $E \leq T^*$.

**Lemma 4.3.** Let $A, B \leq S$ with $S \in \{T, T^*\}$. If $A^g = B$ for some $g \in G$, then $g = wc$ for some $c \in C_G(B)$ and $w \in N_G(S) = S.W$; in particular, $\text{Out}_G(A) = \text{Out}_W(A)$ and $N_G(A) = C_G(A).\text{Out}_W(A)$.

**Proof.** Let $h \in G$ with $h^{-1}F(h)\mathcal{T} = w_0$, such that $S = \mathcal{T}$ or $S = h\mathcal{T}$, depending on whether $S = T$ or $S = T^*$. By [10, Theorems 3.5.2 & 3.5.4], if $b \in B$, then $C_G(b)^g$ is a connected reductive group containing $S$; since $B$ is finite, induction proves that $C_G(B)^g$ is connected, reductive, and contains $S$ and $S^g$. Any two maximal tori in a connected reductive group are conjugate, see [10, §1.7], thus $S = S^{gc}$ for some $c \in C_G(B)$, and $gc \in N_G(S)$.

First, we consider $S = (q - 1)^T$. We now use the notation of [22, Notation 2.1] and write $N_G(\mathcal{T}) = (\mathcal{T}, V)$, where $V$ is the extended Weyl subgroup of $G$, which is generated by $x_\alpha(1)$ with $\alpha$ running over an irreducible root system $R$ for $G$. Thus, $V = \langle x_\alpha(1) \mid \alpha \in R \rangle$, and $x_\alpha(1) = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ as shown in [9, Section 12.1]. By [9, p. 61], we have $x_\alpha(\zeta) = \exp(\text{ad}(e_\alpha))$, where $e_\alpha$ comes from a Chevalley basis with integral structure constants. Hence, $x_\alpha(1)^F = x_\alpha(F(1))$, which implies that $x_\alpha(1)^F = x_\alpha(F(1)) = x_\alpha(1)$. This proves that $V^F = V$, hence $V \leq G$, and $N_G(S) = \langle S, V \rangle$. 


Now suppose $S = (q + 1)^7$, and recall that $N_\mathcal{T}(T) = \langle T, V \rangle$ and $V/(V \cap T) \cong W$. It is mentioned in [22, Notation 2.1] that $V \cap T = \langle h_{\alpha}(-1) \mid \alpha \in R \rangle \cong (-1)^7$ for any odd $q$, so $V$ is independent of the underlying field GF(q). A direct computation for $q = 3$ shows that $Z(V) = (Z(G), x) \cong 2^2$ with $xT = w_0$.

(Alternatively, note that $V \cap T = \Omega_1(O_2(S))$, and the action of $W$ on $\Omega_1(S)$ is given above; again, a direct computation shows that $Z(\Omega_1(O_2(S), W) \cong 2^2 = (Z(G), w_0)$, so $xT = w_0$ for some $x \in Z(V)$.) By the Lang-Steinberg Theorem [20, Theorem 21.7], there is $y \in G$ with $y^{-1}F(y) = x$; recall that $xT = w_0 = h^{-1}F(h)T$, so may assume that $h = y$, and so $F(h) = hx$ with $x \in Z(V)$ as defined above. Now, if $h^t \in hV$, then $F(h^t) = F(h)F(t) = hx^t = h^t$, and so $hV \leq G$. Recall that $\langle \overline{S}, hV \rangle$ and hence $N_{\mathcal{T}}(\overline{S}) = hN_{\mathcal{T}}(T) = \langle \overline{S}, hV \rangle$.

To complete the proof, write $U = V$ or $hV$ depending on whether $S = (q - 1)^7$ or $S = (q + 1)^7$, respectively, so that, as shown above, $N_G(S) = \langle S, U \rangle$ and $N_{\mathcal{T}}(\overline{S}) = \langle \overline{S}, U \rangle$. Recall from the first paragraph that $gc \in N_G(S)$.

Now we can deduce that $gc = vt$ for some $t \in \overline{S}$ and $v \in U$, hence $gct^{-1} = v$ with $ct^{-1} \in C_G(B)$. We may suppose $gc = v$, and therefore $c = g^{-1}v \in G \cap C_G(B) = C_G(B)$ and $g = vc^{-1}$ with $v \in N_G(S)$ and $c^{-1} \in C_G(B)$.

This proves the first claim of the lemma.

Finally, if $B = A$, then every $g \in N_G(A)$ can be written as $g = vc$ with $c \in C_G(A)$ and $v \in N_G(A) \leq N_{N_G(S)}(A)$, so $N_G(A) = N_{N_G(S)}(A)C_G(A)$; recall that $C_G(A) \leq N_G(A)$. This proves that

\[
\text{Out}_G(A) \cong N_G(A)/C_G(A) \\
\cong \frac{N_{N_G(S)}(A)/(N_{N_G(S)}(A) \cap C_G(A))}{N_{N_G(S)}(A)/C_{N_G(S)}(A)} \\
\cong \frac{N_{N_G(S)}(A)/S}{(C_{N_G(S)}(A)/S)} \\
\cong N_W(A)/C_W(A) \\
\cong \text{Out}_W(A).
\]

This completes the proof.

5. Maximal-proper 2-local subgroups: the cyclic case

Recall Notation 1.1. By Lemma 3.3, if $M < G$ is maximal-proper 2-local, then $M = N_G(E)$ for some $E < G$ with $E$ elementary abelian (with $Z < E$ if $E \neq 2$), cyclic of order 4 with $Z < E$, or extraspecial with $Z(E) = Z$.

The aim of this section is to discuss the case that $E$ is cyclic; this will yield the maximal-proper 2-local groups $M_1$, $M_2$, and $M_3$ in Theorem 2.2.
First, we introduce some more notation. Let $pX$ be the conjugacy class of an element of order $p$ in a finite group $H$. An elementary abelian $p$-subgroup $E \leq H$ of order $p^n$ is called $pX$-pure if all its nontrivial elements lie in $pX$; in this case we say that $E$ has type $pX^n$ and write $E = pX^n$. Analogously, we write $E = pX_{m_1,Y_{m_2}}$ if $E$ has exactly $m_1$ and $m_2$ elements lying in $pX$ and $pY$, respectively, and $m_1 + m_2 = p^n - 1$; similarly, $E = pX_{m_1,Y_{m_2}}Z_{m_3}$ is defined. Note that if $E = pX$, then $E$ is uniquely defined up to conjugacy; we pose this condition for the other types as well, that is, if we write $E = pX_{m_1,Y_{m_2}}Z_{m_3}$ (with possibly $m_2 = 0$ or $m_3 = 0$), then $E$ is uniquely defined up to conjugacy in $H$. If for some type $pX_{m_1,Y_{m_2}}Z_{m_3}$ there are non-conjugate groups, then we label these groups and write $E = (pX_{m_1,Y_{m_2}}Z_{m_3}), i = 1,2,\ldots$. More generally, we say that a $p$-group has type $pX_{m_1,Y_{m_2}}Z_{m_3}$ if it contains exactly $m_1$ elements of $pX_{m_1}$, and so on.

Recall that $r$ is the prime dividing $q$; for a group $N$ we denote by $O^r(N)$ the largest normal subgroup of $N$ which yields a quotient group of $r'$-order.

**Lemma 5.1.** a) There are two $G$-classes of non-central involutions in $G$, called $2\text{A}$ and $2\text{A}'$, with representatives $z_A$ and $zz_A$, and $N_G((z_A)) = N_G((zz_A)) = C_G(z_A) = C_G(zz_A) = \langle SL_2(q) \circ Spin_1^+(q), x_A \rangle$, where $x_A = s_1 \circ s_2$ with $s_1$ and $s_2$ acting on $SL_2(q)$ and $Spin_1^+(q)$, respectively, as outer-diagonal (non-classical) automorphisms.

b) There are two $G$-classes of elements $y \in G$ with $y^2 = z$, with representatives $z_B$ and $z_C$, and

$$C_G(z_B) = \begin{cases} 4 \circ_2 (SL_2(q)/2) & \text{if } q \geq 5 \\ (SL_2(q)/2).2 & \text{if } q = 2,5 \end{cases}$$

$$N_G((z_B)) = C_G(z_B).2.$$
Proof. If $X \in \{B, C\}$ and $E = \langle z_X \rangle$, then $E = 4$ cyclic, $Z < E$, and $O_2(N_G(E)) = O_2(C_G(E))$ is cyclic. Now Lemma 3.3c shows that $M_2$ and $M_3$ are maximal-proper 2-local. If $E = \langle z_A, z \rangle$, then $E = 2^2$ is elementary abelian and $Z < E$. As before, for a subgroup $L \leq G$ denote $\mathcal{E} = L/Z \leq G/Z = K$. Note that $N_K(\mathcal{E}) = N_G(E) = N_G(E)/Z$, which implies that $\mathcal{E} \in \mathcal{ER}_4(K)$. Since $\mathcal{E} = 2$ and $O_2(N_K(\mathcal{E}))$ is cyclic, the group $N_G(E) = N_G(\langle z_A \rangle) = M_1$ is maximal-proper 2-local by Lemma 3.2.

If $E = 2$, then $E = G(\langle z_A \rangle)$ or $E = G(\langle z_A z \rangle)$; in both cases, $N_G(E) = G(M_1)$. If $E = \langle e \rangle = 4$ with $e^2 = z$, then $e = G z_X$ for some $X \in \{B, C\}$, hence $N_G(E) \in \langle M_2, M_3 \rangle$.

We conclude this section with two useful lemmas.

**Lemma 5.3.** If $E \leq G$ is elementary abelian with $Z \leq E$, then $C_{G/Z}(E/Z) = C_G(E)/Z$ and $N_{G/Z}(E/Z) = N_G(E)/Z$, hence $Out_{G/Z}(E/Z) = Out_G(E)$.

**Proof.** It is straightforward to prove $N_{G/Z}(E/Z) = N_G(E)/Z$. Now we consider $\psi: C_G(E) \rightarrow C_{G/Z}(E/Z)$, $g \mapsto gZ$. Suppose, for a contradiction, that there is $hZ \in C_{G/Z}(E/Z)$ with $h \not\in C_G(E)$, that is, $e^h = ze$ for some $e \in E \setminus Z$. Up to conjugacy, $e \in G(\langle z_A, z z_A \rangle)$, thus it follows from $e^b = ze$ that $z_A = G z z_A$, a contradiction. Hence, $\psi$ is surjective with kernel $Z$, thus $C_G(E)/Z \cong C_{G/Z}(E/Z)$.

**Lemma 5.4.** Write $M_1 = (L_1 \circ_2 L_2).2$ with $L_1 = SL_2(q)$ and $L_2 = Spin^+_2(q)$.

a) If $X \in \{B, C\}$, then $L_2 \cap 2X \neq \emptyset$. If $z_X \in L_2$, then $O^r(C_{M_1}(z_X)) = SL_2(q) \circ_2 SL_6(q)$.

b) If $X \in \{A, B, C\}$, then $T \cap 2X \neq \emptyset$ where $T = (q - \varepsilon)^7$.

**Proof.** a) Note that $M_1$ contains a Sylow 2-subgroup of $G$, hence we can assume that $z_X \in M_1$. Let $K = O^r(C_G(z_X))$, so that $K = E^z_6(q)$ or $SL^z_6(q)/2$. By [16, Table 4.5.2], there exists an involution $u \in K$ with $O^r(C_K(u)) = SL_2(q) \circ_2 SL_6(q)$.

By Lemma 5.1, we can assume $u \in \{z_A, z z_A\}$, hence $C_K(u) \leq C_G(z_A) = M_1$ and $O^r(C_K(u)) \leq M_1 \cap C_G(z_X) = C_{M_1}(z_X)$. Note that if $z_X \not\in L_1 \circ_2 L_2$, then we can assume $M_1 = (L_1 \circ_2 L_2).z_X$; moreover, $C_{L_i}(z_X)$ is abelian and $z_X$ induces an outer involutionary automorphism on $L_2$. In particular, $O^r(C_K(u)) \leq C_{M_1}(z_X)$ implies that $O^r(C_K(u)) \leq C_{L_2}(z_X)$. However, by [16, Table 4.5.2], there is no subgroup of $C_{L_2}(z_X)$ of type $O^r(C_K(u)) = SL_2(q) \circ_2 SL_6(q)$. This forces $z_X \in L_1 \circ_2 L_2$, and we can write $z_X = y_1 y_2$ with $y_1 \in L_1$. If $y_1 \not\in Z(L_1)$, then $C_{L_i}(z_X)$ is abelian, implying that $O^r(C_K(u)) \leq C_{L_2}(z_X)$, which is not possible as shown above. Thus we must have $z_X \in L_2$, and $z_X$ induces an inner involutionary automorphism on $L_2$. Now it follows from [16, Table 4.5.2] that $C_{L_2}(z_X) = ((2 \times (q - \varepsilon)) \circ_2 SL^z_6(q)).3$; note that $6 \varepsilon = 2 \times 3 \varepsilon$. Together with $C_{L_1}(z_X) = L_1$, this proves the assertion.

b) By part a), if $X \in \{B, C\}$, then $O^r(C_{M_1}(z_X))$ has a maximal torus $S = (q - \varepsilon) \circ (q - \varepsilon)^6$. Since the outer 2 in $M_1 = (L_1 \circ_2 L_2).2$ acts as an outer-diagonal automorphism on $L_1$ and $L_2$, respectively, a conjugate of 2 centralises $S$, and we can assume that $T = S \leq C_{M_1}(z_X)$ is a maximal torus isomorphic to $T$. Since $z_A, z_X \in T$ and by uniqueness of $T$ (up to conjugacy), see [13], we may suppose $z_A, z_B, z_C \in T \cap L_2$.

6. Some preliminary results on $M_1$

Recall that, up to conjugacy, $G$ has two non-central involutions, $z_A$ and $z z_A$, and $M_1 = N_G(\langle z_A \rangle) = N_G(\langle z z_A \rangle)$. Thus, if $M = N_G(E)$ is maximal-proper 2-local for some elementary abelian $Z < E$, then we can assume that $z_A \in E$, in particular, $E \leq M_1$. This shows that we can classify the elementary abelian subgroups of $G$ as subgroups of $C_G(z_A)$. For this purpose, we need to introduce more notation: Notation 6.1 is retained throughout paper. The two subsequent subsections are of technical nature. We define, as subgroups of $M_1$, certain quaternion subgroups $Q_1, \ldots, Q_4 \cong Q_8$, and two dihedral subgroups $D_1, D_2 \cong D_8$; these group will be important in the classification of the maximal-proper 2-local $N_G(E)$ with $E$ elementary abelian or extraspecial.
Notation 6.1. This notation extends Notation 1.1. Let
\[ M_1 = C_G(z_A) = (L_1 \circ \langle z_A \rangle L_2).x_A \]
with \( L_1 = \text{SL}_2(q) \), \( L_2 = \text{Spin}^+_1(q) \), and \( x_A = s_1 : s_2 \), where each \( s_i \) acts as an outer-diagonal automorphism on \( L_i \) (and \( s_2 \) is non-classical, cf. [16, p. 177]). Note that \( Z(L_2) = Z(M_1) = \{ 1, z, z_A, zz_A \} \) and \( Z(L_1) = \{ 1, z_A \} \).

Suppose \( L_2 \) is given in natural representation with underlying space \( V \), that is, \( L_2 = \text{Spin}(V) \). Let
\[ \pi : L_2 = \text{Spin}(V) \rightarrow \Omega(V) = \text{Spin}^+_1(q) \]
be the natural projection, so that \( Y = \ker(\pi) = 2 \leq Z(L_2) \), see [16, Theorem 2.2.6 & p. 71]. Note that
\[ M_1 = (L_1 \circ \langle z_A \rangle L_2).2 \]
and it follows from [16, Table 4.5.2 & p. 70] that \( z_A \notin \ker \pi \), thus \( \pi(z_A) = -1.Y \). This implies that \( Y = (z) \) or \( Y = (zz_A) \). If \( Y = (z) \) and \( X \in \{ B, C \} \), then \( \pi(z_X) \) is a diagonizable involution in \( \Omega(V) \), hence \( O^\epsilon(C_{L_2}(z_X)) \neq \text{SL}_2(q) \), a contradiction to the proof of Lemma 5.4. Thus, we must have
\[ Y = \ker(\pi) = (zz_A). \]
Choose an orthogonal decomposition \( V = V_1 \perp V_2 \perp V_3 \) with
\[ S_i = \text{Spin}(V_i) = \text{Spin}^+_1(q) \quad \text{and} \quad O_i = \Omega(V_i) = \Omega^+_1(q) \]
for \( i \in \{ 1, 2, 3 \} \). Let \( H = S_1S_2S_3 \) and note that \( \pi|_H : H \rightarrow O_1 \times O_2 \times O_3 \), hence \( H = S_1 \circ Y \circ S_2 \circ Y \circ S_3 \). Let \( \pi_1 = \pi|_{S_1} : S_1 \rightarrow O_1 \), so that \( \ker(\pi_1) = Y \). Due to [19, Proposition 2.9.1] and [16, Proposition 6.2.1], in the following we identify
\[ S_1 = \text{Spin}^+_1(q) = \text{SL}_2(q) \times \text{SL}_2(q) \quad \text{and} \quad O_i = \Omega(V_i) = \Omega^+_1(q) = \text{SL}_2(q) \otimes \text{SL}_2(q); \]
in particular, \( O_1 = \text{SL}_2(q) \circ \text{SL}_2(q) \) and, if \( x = (a, b) \in S_1 \) with \( a, b \in \text{SL}_2(q) \), then \( \pi_1(x) = a \circ b \) is the usual Kronecker product. Note that \( Z(S_1) = \langle z_A, z_A \rangle \) with \( z_A = (-1, 12) \) and \( z = (-1, 2, 12) \), where \( 1_m \) denotes an \( m \times m \) identity matrix. We assume \( z_A = z_A \otimes \imath_3 \) and \( z = z \otimes \imath_3 \); identifying \( z_A = z_A \) and \( z = z \), we have \( Y = \ker(\pi) = (-1, 12) \), and \( z_A \) and \( z \) act on \( S_1 = \text{SL}_2(q) \times \text{SL}_2(q) \) as \( (-1, 12) \) and \( (1, 2, 12) \), respectively.

6.1. Some quaternion subgroups. As mentioned above, we construct four subgroups \( Q_1, \ldots, Q_4 \) of \( M_1 \) isomorphic to the quaternion group \( Q_8 \). The proof is highly technical and makes use of the following lemma.

Lemma 6.2. Let \( K = \text{SL}_m^+(q) \) with \( m \) even. Let \( \gamma \) be the inverse-transpose map on \( K \) and let \( A \in K \) be the matrix with whose only nonzero entries are a string of alternating 1’s and -1’s running diagonally from upper right to lower left. Let \( B = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) with \( a \) a non-square. If \( \rho = \gamma A, \rho_+ = \gamma, \) and \( \rho_- = \gamma B \), then \( C_K(\rho) = \text{Sp}_m(q), C_K(\rho_+) = \Omega^+_m(q) \), and \( C_K(\rho_-) = \Omega^-_m(q) \). If \( \gamma' \) is a graph involution in Aut(K), then, up to Inndiag(K)-conjugacy, \( \gamma' \in \{ \rho, \rho_+, \rho_- \} \).

Proof. Recall that \( \rho \) is the graph automorphism of \( K \), see [16, p. 68]. The first assertion follows from a direct computation; for \( \rho_- \) see also [19, Proposition 2.5.13]. The last assertion is [16, Table 4.5.2].

Recall that for subgroups \( A, B \leq H \) we write \( A =_H B \) if and only if \( A \) and \( B \) are conjugate in \( H \), see Section 1.3.

Lemma 6.3. We use Notation 6.1.

a) There exist \( Q_1, \ldots, Q_3 \leq L_2 \) such that \( Q_i = \langle x_i, y_i \rangle \cong Q_8 \) with \( Z(Q_i) = \{ 1, z_A \}, Z(Q_2) = Z(Q_3) = \{ 1, z \} \); moreover, \( O^\epsilon(C_{L_2}(x_i)) = \text{SL}_m^+(q) \) and \( C_{L_2}(Q_i) = 2 \times \text{Sp}_d(q) \) for \( i \in \{ 1, 2 \} \); both groups \( Q_1 \) and \( Q_2 \) commute.

b) If \( b \in 2B \cap L_2 \) and \( c \in 2C \cap L_2 \), then \( b = l_2 z_A c \) and \( c = l_2 z_A b \).

c) We can suppose \( Q_2 \) has type \( 2Z_1C_6 \) with \( C_G(Q_2) = 2 \times F_4(q) \) and \( N_G(Q_2) = (Q_8 \times F_4(q)).S_3 \), where \( S_3 \) acts irreducibly on \( Q_2/Q \). 2.

d) We can suppose \( Q_3 \cap 2X \neq 0 \) for \( X \in \{ B, C \} \), and \( C_G(Q_3) = 2 \times \text{PSp}_6(q) \).

e) If \( Q \cong Q_8 \) with \( z \in Q \) is not of type \( 2Z_1B_6 \), then \( Q =_C Q_i \) for some \( i \in \{ 2, 3 \} \).
368 Proof. a) Recall that $L_2 = \text{Spin}(V)$ and $S_i = \text{Spin}(V_i)$ where $V = V_1 \oplus V_2 \oplus V_3$. Let $R_1 = Q_8$ and $R_2 = Q_8$ be subgroups of the first and the second factor of $S_i = \text{SL}_2(q) \times \text{SL}_2(q)$, respectively, such that $R_i = (u_i, w_i)$ with $u_i^4 = w_i^4 = 1$, $u_i^2 = w_i^2 = z_i \in \Z(R_1)$ and $u_i^2 = w_i^2 = z \in \Z(R_2)$; note that $\pi(z_i^3) = \pi(z_i^2)$ is a generator of $\Z(O_1)$. We may suppose that $u_i$ and $w_i$ are conjugate in $\text{GL}_2(q^2)$ to 
\[ \text{diag}(\omega, \omega^{-1}), \] where $|\omega| = 4$. Let $Q_i = R_i \otimes 1 \times H$ be the diagonal embedding of $R_i$ into $H$; since 
\[ z z_i \neq Q_i, \] we have $\pi(Q_i) = R_i = Q_8$. Let $x_i = u_i \otimes 1$ and $y_i = w_i \otimes 1$, so that $Q_i = (x_i, y_i) = Q_8$ with 
\[ x_i^2 = y_i^2 = z_i \in \Z(Q_i) \] and $x_i^2 = y_i^2 = z \in \Z(Q_2)$. It follows from [15, (1.13)] and the construction of $x_i$ that 
\[ C_{\text{SO}(V)}(\pi(x_i)) = \text{GL}_6(q); \] now [15, (2E)] shows that $O^{\prime}(C_{L_2}(x_i)) = \text{SL}_6(q)$; and [16, Table 4.5.2] yields 
\[ C_{L_2}(x_i) = ((2 \times (q - \varepsilon)) \sigma_3, \text{SL}_6(q))_3, \] with $x_i \in q - \varepsilon$. Note that each $x_i$ induces an inner involutionary automorphism on $L_2 = \text{Spin}_6(q_2)$, and 
\[ x_1 \neq \text{SL}_6(q) \] for some $g \in G$, then $x_2, x_2 \in \Z(C_{L_2}(x_1))$. Since $x_2 = \text{SL}_6(z \in X \in (B, C)$, the structure of $\Z(C_{L_2}(x_2))$ implies that $z = x_2^2 = (x_1^2)^2 = z_2$, a contradiction. Thus, $C_{L_2}(x_1) \neq \text{SL}_6(q_2)$. Now it follows from [14, p. 129] that, up to conjugacy, $C_{L_2}(x_1)$ and $C_{L_2}(x_2)$ are the only centralisers of semisimple type $A_1 + A_5$ of semisimple elements; recall that $C_{L_1}(x_1) = \text{SL}_2(q)$.

It follows from [16, Table 4.5.1] that 
\[ C_{L_2}(x_1, Z(L_2)) = ((q - \varepsilon) \sigma_3, \text{SL}_6(q))_2 \times \gamma, \] and we can suppose that $y_1 Z(L_2) \in C_{L_2}(x_1, Z(L_2))$ acts as $\gamma$ on $(q - \varepsilon) \sigma_3, \text{SL}_6(q))_2$, where $\gamma$ is a graph automorphism and 
\[ \varepsilon \] is inversion. In particular, $y_1$ acts as a graph automorphism on $\text{SL}_6(q)$. It follows that 
\[ C_{L_2}(Q_1) = C_{L_2}(x_1)(y_1) = 2 \times \text{SL}_6(q)_3, \] and $C_{\text{SL}_6(q)}(y_1)$ is given in [16, Table 4.5.2]. In particular, if $C_{\text{SL}_6(q)}(y_1) \neq \text{SL}_6(q)$, then $O^{\prime}(C_{\text{SL}_6(q)}(y_1)) = \text{SO}_6^\varepsilon(q)$ for some $\eta \in \{ \pm 1 \}$. Note that, by construction, 
\[ \text{SL}_6(q) \leq C_{S_i}(R_i) \] with $z \in \text{SL}_6(q)$ for $j = 1, 2$, thus $\text{SL}_6(q) \rightarrow \text{SL}_6(q) \rightarrow \text{SL}_6(q)$ is not possible since $\text{PO}_6^\varepsilon(q) = \text{PSL}_6^\varepsilon(q)$, see [19, Proposition 2.9.1]. This proves that $C_{L_2}(Q_1) = 2 \times \text{SL}_6(q)(y_1) = 2 \times \text{Sp}_6(q)$. The argument for $Q_2$ is the same, and we define $Q_3 = (x_3, y_3)$ where $x_3 = x_2$ and $y_3 = z_3 y_2$.

b) We continue with the notation of a) and, first, show that there are conjugates of $z_3, z_3, \zeta_3$ in $L_2$ with 
\[ z_3 = z_3 \zeta_3. \] As seen in the proof of Lemma 5.4, if $X \in (B, C)$ and $X \in L_2$, then $O^{\prime}(C_{L_2}(x_3)) = \text{SL}_6(q)$. As seen in part a), the elements $x_1$ and $x_2$ are the only two non $L_2$-conjugate projective involutions with 
\[ O^{\prime}(C_{L_2}(x_i)) = \text{SL}_6(q), \] see also [16, Table 4.5.2]. Thus, $C_{L_2}(x_i) = L_2 C_{L_2}(x_i)$ for some $i \in \{ 1, 2 \}$. If, up to conjugacy, 
\[ C_{L_2}(x_2) = C_{L_2}(x_1), \] then $x_2 \in \Z(C_{L_2}(x_1))$, and it follows that $x_2^2 = x_1^2 = z_3, a contradiction.

Thus, we can assume that $C_{L_2}(z_3) = C_{L_2}(z_3) = C_{L_2}(z_3)$ and $z_3, z_3, z_3 \in \Z(C_{L_2}(x_2))$. In particular, 
\[ z_3 = u_3 z_3 \] and $z_3 = w_3 z_3$ with $u_3 = z_3 \in \Z(C_{L_2}(x_2))$. Note that $z_3 = z_3 = z_3$ since $z_3^2 = (w_3)^2 = w_3 z_3^2 = z_3 z_3^2 = z_3^2$, thus $z_3 = z_3$ with $u_3 = z_3 \zeta_3$. Since $z_3 = z_3 = z_3$, we can assume that $z_3 = z_3 z_3$, hence $z_3 = z_3 z_3$.

Now let $b \in 2B \cap L_2$ and $c \in 2C \cap L_2$ as in the lemma. If not $z_3 z_3 b \in 2C$, then we can assume that $z_3 b = z_3$.

As shown above, we can also assume that $z_3 = z_3 z_3$, together with $z_3 b = z_3$, this implies that $b = z_3$, a contradiction. Hence, $z_3 b \in 2C$ and, similarly, $z_3 c \in 2B$.

c) Let $E = Q_2 = (x_2, y_2)$ as in part a), and write $x = x_2$ and $y = y_2$. Recall that 
\[ x^2 = y^2 \] and $y^x = y^{-1}$. Thus, $x \in 2X$ with $X \in (B, C)$. If $x = z_3 b$, then $z_3 x = 2C$ by $b$, and, since $E \leq C_{(z_3 b)}$, we have $(z_3 b)^2 = z_3, (z_3 b)^2 = (z_3 b)^{-1}$, and $y^x = y^{-1}$. Hence, by replacing $x$ by $z_3 b$, we can assume that $x \in 2C$. Recall that $z_3 \in \Z(L_2)$, hence we still have $O^{\prime}(C_{L_2}(x)) = \text{SL}_6(q)$ and $O^{\prime}(C_{L_2}(x)) = \text{SL}_6(q)$. In particular, $y \in N_{C_2}(x) = C_{M_3}$, and $y$ acts as a graph automorphism on $K = O^{\prime}(C_{L_2}(x)) = \text{SL}_6(q)$, see the proof of Lemma 5.1. Thus, $N_{C_2}(x) = C_{M_3}(y)$ and $O^{\prime}(C_{L_2}(x)) = O^{\prime}(C_{M_3}(y))$. By [16, Proposition 4.9.1], we have $C_{K}(y) = O^{\prime}(C_{K}(y)) \in \{ F_4(q), \text{PSp}_6(q) \}$, thus $C_{K}(E) = 2 \times F_4(q)$ or $C_{K}(E) = 2 \times \text{PSp}_6(q)$; note that $N_{C_2}(x) = ((q - \varepsilon) \sigma_3, \text{E}_6(q))(3z)$, $y$ where $3z$ and $y$ act as an outer-diagonal and a graph automorphism, respectively, on $\text{E}_6(q)$.\]
If $C_G(E) = 2 \times F_3(q)$, then $y \in 2C$ since $F_3(q) \not\leq C_G(zB)$, hence $C_G(E) \not\leq C_G(zB)$, by an order argument; in this case, $E$ has type $2Z_1C_6$ as required. Now let $C_G(E) = 2 \times PSP_8(q)$. Suppose, for a contradiction, that $E$ has type $2Z_1C_6$. Then $y \in 2C$, hence $z_Ay \in 2B$ by b), and $\bar{E} = (x, z_Ay)$ has not type $2Z_1C_6$.

Note that $y \in 2C$ and $z_Ay \in 2B$ are not $G$-conjugate, thus, $\langle O'(C_G(z_Ay)) = F_3(q) \rangle$, which implies that $C_G(E) \not\leq 2 \times F_4(q) \leq C_G(z_Ay) = C_G(zB)$, a contradiction. Thus, $E$ is of type different to $2Z_1C_6$, and we must have $y \in 2B$. Now $z_Ay \in 2C$, and $\bar{E} = (x, z_Ay)$ is of type $2Z_1C_6$ with $C_G(E) = 2 \times F_4(q)$.

Finally, we prove that $N_G(E) = (E \times F_4(q)).S_3$. It follows from $N_G(E)/EC_G(E) \leq \Out(E) = S_3$ that $EC_G(E) \leq N_G(E) \leq EC_G(E).S_3$. Recall that $H = S_1 \rtimes S_2 \rtimes S_3$, and $E = Q_8 \rtimes 1 \leq H$ with $Q_8 \leq S_1$; also, $x_A = s_1$ and $x_2$ induces a non-classical outer-diagonal involutory automorphism of $L'$, hence its action is induced by an element of the conformal group $CSO(V) \setminus SO(V)$, see [16, p. 71]. Thus, we may suppose $s_2 = u_1 w_2 u_3$, where each $w_i$ induces non-classical outer-diagonal involutory automorphism of $S_i$. In particular, each $w_i$ induces outer-diagonal involutory automorphism on each factor $S_{L_2}(q)$ of $S_i = SL_2(q) \times SL_2(q)$. Recall that $2^a$ is the largest 2-power dividing $q - \varepsilon$. If $a = 2$, then $Q_8$ is a Sylow 2-subgroup of $S_{L_2}(q)$ and $Out_{S_{L_2}(q), q}(Q_8) = S_3$, hence $S_3 \rtimes 1 \leq N_G(E)/EC_G(E)$. The same result follows for $a \geq 3$ since, in this case, $Out_{S_{L_2}(q)}(Q_8) = S_3$.

d) This follows from b) and c); since $Q_2 \not\in Q_2$, the proof of part c) implies that $C_G(Q_3) = 2 \times PSP_8(q)$.

e) We can suppose $Q = (x, y)$ with $x = zC$, thus $Q \leq M_3 = N_G((zC) = (q - \varepsilon) \rtimes z_C, E'_q(q)); y$, it follows from [16, Table 4.5.1] that $y$ acts as $\gamma$ on $(q - \varepsilon) \rtimes z_C, E'_q(q)$ for some graph automorphism $\gamma$ of $E'_q(q)$. By the proof of c), up to conjugacy, it follows that $i \in \{2, 3\}$ such that $y$ and $z_C$ induce the same action on $E'_q(q)$, thus $y^{-1}z_C = u \in Z(C_G(x)) = (q - \varepsilon)$. Since $z = y^2 = (y, u) = u^2 z$, it follows that $y = y_1$ or $y = y_2$, thus $Q = Q_i$.

f) We first construct $Q_4$, and then prove its uniqueness. It follows from [11, Table 1] that, up to conjugacy, $K = Indag(G/Z) = (G/Z).2$ has a unique subgroup $E = 2^2 \leq G/Z$ such that $N_K(E) = (E \times PO_{4}(q)^2)\cdot S_3$ is 2-local maximal in $K$; it satisfies $C_K(E) = E \rtimes PO_{4}(q)^2$. In particular, $C_G(E) = E \rtimes PO_{4}(q)^2$ and $N_G/E(Z) = C_G(E)^3$. Let $Q_4 \leq G$ be of order $2^3$ such that $Q_4/Z = E$. The proof of [11, Lemma 2.15] implies that $Q_4 = Q_8$. Since $z \in Q_4$, it follows from a) and c) that $Q_4$ has type $2Z_1B_6$. Clearly, we can suppose that $Q_4 = (x, y, q, z)$, where $x_A = x_B \rtimes y_A = x_B$. Let $C, N \leq G$ be a preimage of $G_{2Z_1B_6}/E$ and $N_G/E(Z)$, respectively, under the projection $G \rightarrow G/Z$.

We know $C_{G/Z}(E) = E \times PO_{4}(q)^2$ and $M_2 = N_G((x_4)) = C_G(x_4).y_4$; in particular, $C_{S_{L_2}(q)}(y_4) = \Omega_4^+(q)$ by [16, Table 4.5.2]. Since $y_4$ centralises the unique order 2 element in $Z(S_{L_2}(q))$, it follows that $PO_8^+(q) \leq C_{S_{L_2}(q)}(y_4)$, so $Q_4 \rtimes PO_8^+(q) = Q_4 \times PO_8^+(q) \leq C_G(Q_4) \leq C_C(Q_4)$. Note that $Out _G(J_4) \geq 3$, hence $Out_G(Q_4) \geq 3$. If $Out_G(Q_4) \neq 1$, then $Out_G(Q_4) = 2$ as $Out_G(Q_4) = S_3$; in this case $Out_G(Q_4) \geq Out_G(Q_4).3 = 6$, which is impossible. This forces $Out_G(Q_4) = 1$, hence $C = N_G(Q_4) = C_G(Q_4).Q_4$ and $C = Q_4 \rtimes PO_8^+(q)^2$. In conclusion, by an order argument, it follows that $C_G(Q_4) = 2 \times PO_8^+(q)^2$. Finally, it is easy to see that $N(G(Q_4))$, which implies that $N_G(Q_4) = C_G(Q_4).3$.

Now suppose $Q = (x, y, q, z)$ is of type $2Z_1B_6$. We can suppose $x = zB$, so $Q \not\leq M_2 = C_G(x).y$.

Recall that $Q_3 = (x, y, z)$ and we can suppose $x \in zC$ and $y_3 = zC$; in particular, suppose $x = y_3$. Note that $y$ induces a graph automorphism of $O'(C_G(x)) = SL_5^+(q)$. If $C_{S_{L_2}(q)}/y_4$ is $PS_{P_8}(q)$, then, up to conjugacy, $y$ and $x_3$ induce the same action on $C_G(x)$, hence $y^{-1}x_3 = u \in Z(C_G(x))$. Now $z = y^2 = 2^2 u^2 = z^2$, yields $y = x_3$ or $y = z_3$. Note that $x_3 = zC = zC = zC$ (since $zC$ and $zzC$ have the same local structure), hence $y_3 \in zC$, a contradiction. Now [16, Table 4.5.2] proves that $C_{S_{L_2}(q)}/y_4) = PO_8^+(q)$ for some $\eta \in \{\pm 1\}$. If $\eta = 1$, then $Q_4 = C_G(Q_4)$ by the definition of $Q_4$ and the uniqueness of such a graph automorphism $y$, see [16, Table 4.5.2].

It remains to show that $\eta = -1$ is not possible, so suppose $\eta = -1$. Recall $M_2 = C_G(x).y = C_G(x).x_3$ and $C_{S_{L_2}(q)}(x_3) = SP_8(q)$. We can suppose $x = mx_3$ for some $m \in SL_5^+(q)/2$ or $m \in (SL_5^+(q)/2)$. Depending on whether $a = 2$ or $a \geq 3$, where $2^a$ is the largest 2-power dividing $q - \varepsilon$. (If $a = 2$ and $m$ lies in $(4 \times 2$ $SL_5^+(q/2)) \setminus SL_5^+(q)/2$, then replace $y$ by $xy$.) If $m \in (SL_5^+(q)/2)$, then $a \geq 3$ and we can assume that $m$ is induced by some matrix $M \in GL_2^+(q)$ with square determinant. Clearly, the same holds if $a = 2$ and $m \in SL_5^+(q)/2$. Thus, in both cases, $y$ acts on $x_3$ on $SL_5^+(q)$ with $C_{SL_5^+(q)}(y) = \Omega_4^+(q)$. By Lemma 6.2, the elements $x_3$ and $Mx_3$ are conjugate in $I = Indag(SL_5^+(q))$ to inverse-transpose $\rho$ and $B\rho$, respectively, where $B = \text{diag}(1, \ldots, 1, \lambda)$ with $\lambda$ a non-square; we can suppose that $|B| = \lambda = 2^a$, so that $B$ induces a generator of $Outag(SL_5^+(q)/2) = \gcd(8, q - \varepsilon) \in \{4, 8\}$. Let $\alpha \in I$ such that $x_3^\alpha = \rho$. Write $N = M^\alpha$ so that $(Mx_3)^\alpha = N\rho$. 


Let $\beta \in I$ with $(N\rho)^{\beta} = B\rho$; thus, $N^\beta (\beta^{-1}(\beta^{-1})^t B^{-1}) \in Z(SL_n^r(q))$; this is impossible since the discriminant of $N^\beta (\beta^{-1}(\beta^{-1})^t B^{-1})$ is a non-square – a contradiction. Thus $\eta = -1$ is not possible. 

6.2. Some dihedral subgroups. We construct two subgroups $D_1, D_2$ of $G$ of type $D_8$. Recall that $M_i = C_G(z_i) = (L_1 \circ L_2, x_i$ with $L_1 = SL_2(q)$, $L_2 = Spin_1^3(q)$, and $x_i = s_i \circ s_2$, where each $s_i$ acts as an outer-diagonal automorphism on $L_i$ (and $s_2$ is non-classical, cf. [16, p. 177]). We use this notation and define

$$J_i = (L_i, s_i) = L_i, 2.$$ 

In Proposition 7.1c), we will classify the elementary abelian subgroups $E \leq G$ of order 8 with $Z \leq E$. As we will see, one can assume that $E = \langle z, z_A, v \rangle$ for some $v \in M_1$; parts (i), (ii), and (iii) of Proposition 7.1c) discuss the case $v \in L_2, v \in (L_1 \circ L_2) \setminus L_2$, and $v \in M_1 \setminus (L_1 \circ L_2)$, respectively. The proof of part (iii) will require the following Lemma 6.4. Unfortunately, the proof of Lemma 6.4 itself requires Proposition 7.1c)(ii); we stress that no circular reasoning is involved since Lemma 6.4 is not used to establish Proposition 7.1c)(ii).

**Lemma 6.4.** For $i \in \{1, 2\}$, there exists $D_i \leq J_i$ such that $D_i = \langle a_i, b_i \rangle \cong D_8$ with $a_i \in L_i$ and $a_i^2 = z_A, b_i \in J_i \setminus L_i$ of order 2, and

$$C_{L_i}(a_i) = \langle q - \varepsilon \rangle, C_{L_i}(b_i) = \langle q + \varepsilon \rangle, C_{L_i}(2) = ((2 \times \langle q + \varepsilon \rangle) \circ_{6,-} SL_n^\varepsilon(q), 3,-\varepsilon, $$

and $O^r(C_{L_i}(D_i)) = Sp_6(q)$. In addition, we may suppose $a_2 = x_1$ with $x_1$ as in Lemma 6.3a). Finally, $a_2$ acts as a graph automorphism on $O^r(C_{L_i}(D_i)) = SL_n^\varepsilon(q)$ with $O^r(C_{L_i}(D_i)(a_2)) = Sp_6(q)$.

**Proof.** Recall that $L_1 = SL_2(q) = SL_2^1(q)$; since $L_1$ has a unique outer-diagonal automorphism, we can consider $SL_2^\varepsilon(q) = L_1 \leq G_1 \cong GL_2^\varepsilon(q)$ such that $G_1 = Z(G_1) \circ (L_1, 2)$. Let $a_1 \in L_1, L_1 \setminus L_1$, so that $a_1^2 = 1, a_1 = z_A$, and $a_1^2 = 1$, that is, $\langle a_1, b_1 \rangle = D_8 \leq G_1$; for example, if $\varepsilon = -1$, then $b_1 = (q^1 1)$ and $a_1 = (1 -1)$. In both cases, it follows from [26, Table 4.5.2] that $C_{L_i}(a_1) = \langle q - \varepsilon \rangle$. Since $b_1$ is conjugate to diag$(1, -1)$, it follows that $C_{G_i}(b_1) = \langle q + \varepsilon \rangle$, hence $C_{L_i}(b_1) = \langle q + \varepsilon \rangle$.

Now consider $J_2 = \langle L_2, s_2 \rangle = L_2, s$ using Notation 6.1; we write $L_2 = Spin(V)$ and decompose $V = V_1 \perp V_2 \perp V_3$ with $S_i = Spin(V_i) = Spin_1(q)$. The element $s_2$ induces a non-classical outer-diagonal involutory automorphism of $L_2$, hence its action is induced by an element of the conformal group $SO(V) \setminus SO(V)$, see [16, p. 71]. Thus, we may suppose $s_2 = w_1 w_2 w_3$, where each $w_i$ induces non-classical outer-diagonal involutory automorphism of $S_i$. In particular, each $w_i$ induces outer-diagonal involutory automorphism on each factor $SL_2(q)$ of $S_i = SL_2(q) \times SL_2(q)$. Applying the argument for $L_1$ to each of these factors, we know that there exists an involution $g_1 \in S_1, w_1 = (SL_2(q) \times SL_2(q), 2$ and an element $h_1 \in S_1$ of order 4 such that $C_{S_1}(g_1) = (q + \varepsilon)^2, h_1^2 = z_A, h_1^4 = h_1^2$, and thus $\langle b_1, g_1 \rangle = D_8$. As in the proof of Lemma 6.3a), let $H = S_1 \circ S_2 \circ S_3$ where $Y = (zz_A)$ is the kernel of the projection Spin$(V) \rightarrow \Omega(V)$. Now $a_2 = b_1 \otimes 1_3$ lies in $H$, and $b_2 = g_1 \otimes 1_3 \in J_2$: in particular, $D_2 = \langle a_2, b_2 \rangle$ satisfies $D_2 = D_8, a_2^2 = z_a, a_2^2 = a_3^2, and$ $(q + \varepsilon)^6 \leq C_{L_2}(b_2)$. Thus $C_{L_2}(b_2)$ induces a non-classical outer-diagonal involution on $L_2$, it follows from [16, Table 4.5.2] that

$$C_{L_2}(b_2) = ((2 \times (q + \varepsilon)) \circ_{6,-} SL_n^\varepsilon(q), 3,-\varepsilon, $$

By construction, we may suppose that $a_2 = x_1$ with $x_1$ as in Lemma 6.3; thus, by the proof of that lemma, $C_{L_2}(a_2) = ((2 \times (q - \varepsilon)) \circ_{6,-} SL_n^\varepsilon(q), 3,-\varepsilon, Clearly, $b_2 Z(L_2)$ is an involution in $L_2/Z(L_2)$, centralising $a_2 Z(L_2)$ since $a_2^2 = a_2^{-1}$ and $a_2 Z$. Comparing $C_{L_2}(a_2)$ and $C_{L_2/Z(L_2)}(a_2 Z(L_2))$ in [16, Tables 4.5.1 and 4.5.2] shows that $b_2$ induces a graph automorphism $\gamma$ on $O^r(C_{L_2}(a_2)) = SL_6(q)$. Thus, we have $O^r(C_{L_2}(D_2)) = O^r(C_{L_2}(a_2)^r) \gamma = O^r(C_{SL_6^\varepsilon}(\gamma)) \gamma$, and $O^r(C_{L_2}(D_2)) \in \{Sp_6(q), SO_6^\varepsilon(q}\}$. It remains to prove that $O = Sp_6(q)$. Let $v = a_1 a_2 \in (L_1 \circ_2 L_2) \setminus L_2$ and $b = b_1 b_2 \in M_1 \setminus (L_1 \circ_2 L_2)$; then $b^2 = a_2^2 = 1$ and $b$ centralises $U = \langle z, z_A, v \rangle = 2^3$. By Proposition 7.1c)(ii), whose proof does not make use of this lemma, we can suppose $U = E_3^3 = 2^3$ with

$$C_{G}(E_3^3) = (((q - \varepsilon)^2 \circ_3 SL_6(q))) \circ_\gamma (t : t : \gamma),$$
and $b \in C_G(\mathbf{E}_3^2) \setminus ((q - \varepsilon)^2 \circ_3 \mathbf{SL}_6(q)).3$. In particular, $b$ induces $\iota: \iota: \gamma$ on $(q - \varepsilon)^2 \circ_3 \mathbf{SL}_6(q)$, where $\iota$ is inversion and $\gamma$ a graph automorphism. We show that if $O^\varepsilon(C_{\mathbf{SL}_6(q)}(\gamma)) \neq \mathbf{Sp}_6(q)$, then $\gamma$ has order greater that 2; then, since $b^2 = 1$, it follows that $O = \mathbf{Sp}_6(q)$. For this purpose, we consider the graph automorphisms of $\mathbf{SL}_6(q)$ in $C_G(\mathbf{E}_3^2)$, cf. Lemma 6.2.

First, by Proposition 7.1c(ii), the element $\gamma_1 = (\iota: \iota: \gamma) \in C_G(\mathbf{E}_3^2)$ has order 2 and $C_{\mathbf{SL}_6(q)}(\gamma_1) = C_{\mathbf{SL}_6(q)}(\gamma) = \mathbf{Sp}_6(q)$; by Lemma 6.2, we can assume it acts on $X \in \mathbf{SL}_6(q)$ as $X = A(X^{-1})^T A^{-1}$, where $A$ with $A^2 = -16$ is the matrix given on [16, p. 68]. Note that $\rho = \gamma_1 A \in C_G(\mathbf{E}_3^2)$ acts as inverse-transpose on $\mathbf{SL}_6(q)$, and $C_{\mathbf{SL}_6(q)}(\rho) = \mathbf{SO}_6^+(q)$; in particular, $\rho \in C_G(\mathbf{E}_3^2)$ is a projective involution since $\rho^2 = A^2 = -16 \in \mathbf{SL}_6(q)$.

For $\eta \in \{\pm 1\}$ let $B_\eta \in \mathbf{SL}_6(q).3, \leq \mathbf{GL}_6(q)$ be a symmetric matrix such that $\{X \in \mathbf{SL}_6(q) \mid XB_\eta X^T = B_\eta\} = \mathbf{SO}_6^+(q)$, and set $\rho_\eta = B_\eta \rho$, so that $C_{\mathbf{SL}_6(q)}(\rho_\eta) = \mathbf{SO}_6^+(q)$. We note that if no $B_\eta \in \mathbf{SL}_6(q).3, \epsilon = -1$ exists, then $C_G(\mathbf{E}_3^2)$ contains no projective involution $w$ with $C_{\mathbf{SL}_6(q)}(w) = \mathbf{SO}_6^+(q)$.

Now if $w \in C_G(\mathbf{E}_3^2) \setminus ((q - \varepsilon)^2 \circ_3 \mathbf{SL}_6(q))3$, is a (projective) involution with $C_{\mathbf{SL}_6(q)}(w) \neq \mathbf{Sp}_6(q)$, then, up to conjugacy, $w$ and $\rho_\eta$ induce the same action on $\mathbf{SL}_6(q)$ for some $\eta$. Modifying $w$ by elements of $(q - \varepsilon)^2 \leq C_G(\mathbf{E}_3^2)$, if necessary, we can assume that $w \rho_\eta^{-1} = t \in \mathbf{Z}(\mathbf{SL}_6(q))$, and hence $w^t = (t \rho_\eta)^2 = t^* \rho_\eta^2 = \rho^2 = -16 \in \mathbf{SL}_6(q)$ has order 4. Recall that $[b] = 2$; and, as explained above, $C_{\mathbf{SL}_6(q)}(b) = \mathbf{Sp}_6(q)$ follows.

Comparing $C_{L_2}(b_2)$ and $C_{L_2/G(L_2)}(b_2Z(L_2))$ in [16, Tables 4.5.1 and 4.5.2] shows that $a_2$ induces a graph automorphism on $O^\varepsilon(C_{L_2}(b_2)) = \mathbf{SL}_6^+(q)$. Thus, $\mathbf{Sp}_6(q) = O = O^\varepsilon(C_{L_2}(a_2)) = O^\varepsilon(C_{\mathbf{SL}_6^+(q)}(a_2))$. □

7. Elementary abelian subgroups I

We now classify, up to conjugacy, the elementary abelian $E \leq G$ with $Z < E < |E| \leq 2^i$. More precisely, we define the groups $\mathbf{E}_2^2$ and $\mathbf{E}_3^3, \ldots, \mathbf{E}_3^3$ as in Theorem 2.1, and we prove that each $N_G(\mathbf{E}_3^3)$ is maximal-proper 2-local. We continue with Notation 6.1; recall $M_1 = C_G(z_A) = (L_1 \circ (z_A) L_2) \cdot x_A$ with $L_1 = \mathbf{SL}_2(q), L_2 = \mathbf{Spin}_3^+(q)$, and $x_A = s_1 : s_2$.

**Proposition 7.1.** Let $E \leq G$ with $Z \leq E = 2^i$.

a) If $i = 1$, then $E = Z$ with $C_G(E) = N_G(E) = G$.

b) If $i = 2$, then $E = G \cdot E_2^2 = \langle z, z_A \rangle$ with $C_G(E) = C_G(z_A) = M_1$.

c) If $i = 3$, then we can assume $E = \langle \mathbf{E}_2^2, v \rangle \leq M_1$ and the following holds:

(i) If $v \in L_2$, then $E = G \cdot \mathbf{E}_2^2$ with $C_G(E) = (\mathbf{SL}_2(q) \circ (z_A) \mathbf{Spin}_3^+(q) \circ (z_A) \mathbf{Spin}_3^+(q)).2^2$.

(ii) If $v \in (L_1 \circ L_2) \setminus L_2$, then $E = G \cdot \mathbf{E}_2^2$ and $C_G(E) = (((q - \varepsilon)^2 \circ_3 \mathbf{SL}_6(q)).3).Z \cdot \iota$ where $\iota, \iota, \gamma \in L_1 \circ L_2$ has order 2 and $\gamma$ is a graph automorphism of $\mathbf{SL}_6(q)$ with $C_{\mathbf{SL}_6(q)}(\gamma) = \mathbf{Sp}_6(q)$.

(iii) If $v \in M_1 \setminus (L_1 \circ L_2)$, then $E = G \cdot \mathbf{E}_2^2$ and $C_G(E) = (((q - \varepsilon)^2 \circ_3 \mathbf{SL}_6(q)).(3.3)).Z \cdot \iota$ where $\iota, \iota, \gamma \in L_1 \circ L_2$ has order 2 and $\gamma$ is a graph automorphism of $\mathbf{SL}_6(q)$ with $C_{\mathbf{SL}_6(q)}(\gamma) = \mathbf{Sp}_6(q)$.

Each group $\mathbf{E}_3^3$ exists and $N_G(\mathbf{E}_3^3)$ is maximal-proper 2-local in $G$.

**Proof.** Parts a) and b) are obvious; it remains to prove c). If $E = 2^2$ with $Z < E$, then we may suppose $z_A \in E$, hence $E = \langle \mathbf{E}_2^2, v \rangle \leq M_1 = (L_1 \circ L_2).2$. We make a case distinction on $v$.

(i) Suppose $v \in L_2$, so that $v$ induces an involution in $\text{Inn}(L_2)$. It follows from [16, Table 4.5.2] that $L_2$ has exactly one conjugacy class (called $2a$) of non-central involutions, with representative $z_A$; for note, that $Z(\mathbf{Spin}_3^+(q))$ and $Z(\mathbf{Spin}_3^-(q))$ are cyclic by [16, Table 2.2]. In particular, [16, Table 4.5.2] yields

\[ C_{L_2}(z_A) = (\mathbf{SL}_2(q) \times \mathbf{SL}_2(q)) \circ_2 \mathbf{Spin}_3^+(q)2. \]

recall that $\mathbf{Spin}_3^+(q) = \mathbf{SL}_2(q) \times \mathbf{SL}_2(q)$; it follows from [15] that the central product is over $\langle z_A \rangle$. Thus, we can assume that $E = \mathbf{E}_3^3 = \langle \mathbf{E}_2^2, z_A \rangle$; in particular, $v = z_A$ and

\[ C_G(E_3^3) = (\mathbf{SL}_2(q) \circ_2 C_{L_2}(v)).2, \]
as stated in the lemma; note that $v$ is centralised by some conjugate of $x_A$.

(ii) Suppose $v \in (L_1 \circ L_2) \setminus L_2$. We use Notation 6.1. For example, let $\pi: L_2 \to \Omega_{12}^+(q)$ be the natural projection, so that $\ker \pi = \langle z, z_A \rangle$ and $\pi(z) = \pi(z_A) = -1_{12} \in \Omega_{12}^+(q)$. Since $v \in (L_1 \circ L_2) \setminus L_2$, we can write $v = v_1 \circ v_2$ with $v_1 \in L_1$ and $v_1 \not\in Z(L_1)$. Since $v$ has order 2 and $Z(L_1) = \{1, z_A\}$, it follows that $v_1^2 = v_2^2 = z_A$, and each $v_i$ is a projective involution of $L_i$; in particular, $\pi(v_i^2) = -1_{12}$. Note that $v_2$ induces an inner involution in $\text{Inn}(L_2)$; by [16, Table 4.5.2], the group $L_2$ has four (five if $\varepsilon = -1$) $\text{Inn}(L_2)$-classes of such, with representatives $u_1, \ldots, u_5$ (and $u_6$), such that

$$C_{L_2}(u_1) = ((2 \times (q - \varepsilon)) \circ_2 \text{Spin}^{-\varepsilon}_6(q)), 2$$
$$C_{L_2}(u_3) = ((2 \times (q - \varepsilon)) \circ_6 \text{SL}^{-\varepsilon}_6(q)), 3$$
$$C_{L_2}(u_5) = (\text{Spin}_{\varepsilon}^\varepsilon(q) \circ_2 \text{Spin}_{\varepsilon}^\varepsilon(q)), (2 : 2)$$

Thus, we may suppose $v_1 = v_1 \circ u_3$ and $v_2 = v_2$. Note that if $i \in \{3, 4\}$, then $\Omega_1(O_2(Z(C_{L_2}(u_i)))) = 2^3$, hence $\Omega_1(O_2(Z(C_{L_2}(u_i)))) = Z(L_2) = \langle z, z_A \rangle$. In all other cases, $Z(L_2) = \Omega_1(O_2(Z(C_{L_2}(u_i)))) = 2^3$. Note that $C_{L_2/\ker \pi}(\pi(u_i)) \leq \text{SO}_2^\varepsilon(q) \times \text{SO}_1^\varepsilon(q)$, hence we can assume that either $\pi(u_1) = \text{diag}(-1, 1_2) \in \Omega_2^+(q)$ or $\pi(u_i) = 1_2 \in \Omega_2^+(q)$; in particular, $u_1 \neq v_2$ since $\pi(v_2) = -1_{12}$. Similarly, we deduce $v_2 \not\in L_2 \{u_2, u_5\}$. On the other hand, $\pi(u_2) = \pi(u_4) = -1_{12}$, and it follows that $v_2 \in L_2 \{u_3, u_4\}$. Lemma 6.3 shows that $L_2$ contains two subgroups $Q = Q_8 = \langle x_i, y_i \rangle$ with $Z(Q_1) = \langle 1, z_A \rangle$ and $Z(Q_2) = \langle 1, z \rangle$, and $\text{O}^+(1_2) = \text{SL}_2^+(q)$ for $i = 1, 2$. In particular, each $x_i$ is a projective involution in $L_2$, and $x_i \neq v_2$. Thus we may suppose $x_1 = x_1 = u_3$ and $x_2 = x_2 = u_4$; in particular, $L_2$ contains exactly one class of projective involutions $x_1$ with $x_1^2 = z_A$. Hence, we can suppose $v_2 = v_1 = u_5$. Let $Q = \langle c, d \rangle = Q_8$ be a subgroup of $L_1 = \text{SL}_2^+(q)$, so that $c^2 = d^2 = z_A$ and $d$ acts as inversion on $C_{L_1}(c) = (q - \varepsilon)$; up to conjugacy, such a $c$ is uniquely determined in $L_1$, see [16, Table 4.5.2]. Thus, in conclusion, we may suppose

$$v_1 = v \quad \text{and} \quad v_2 = v_1,$$

hence $v = v_1 x_1$. By the proof of Lemma 6.3a), the element $g = d y_1 \in (L_1 \circ L_2) \setminus L_2$ is an involution, acting by inversion on $C_{L_1}(v_1)$ and $2 \times (q - \varepsilon) \leq C_{L_2}(v_2) = C_{L_2}(x_1)$, and inducing a graph automorphism $\gamma$ on $\text{SL}_2^+(q) = \text{O}^+(C_{L_2}(v_2))$ with $C_{\text{SL}_2^+(q)}(g) = \text{Sp}_6(q)$; clearly, $g \in C_G(v)$. In conclusion, we have proved that, up to conjugacy, there is a unique $E_8^3 = E_8 = (E_8^2, v)$ with $v \in (L_1 \circ L_2) \setminus L_2$.

Recall that $x_A = s_1 : s_2$. By Lemma 6.4, there exist $D_i \leq L_i, s_i$ with $D_i = (a_i, b_i) \in D_8$, such that $b_i \in L_i, s_i \setminus L_i$ and $a_i \in L_i$. Then $v^i = a_i : a_2 \in (L_1 \circ L_2) \setminus L_2$ and $y = b_i b_2 \in M_1 \setminus (L_1 \circ L_2)$; clearly, $v^i v = v^i$ follows from $b_i b_2 = b_i a_i$. As shown above, we can assume that $E_8^3 = (E_8^2, v) = (E_8^2, v')$; in particular, suppose $v = v'$, so that $y \in C_G(E_8^2) \setminus (L_1 \circ L_2)$. Note that if $\hat{y} \in C_G(E_8^2) \setminus (L_1 \circ L_2)$, then $\hat{y} = y \hat{c}$ for some $\hat{c} \in C_{L_1 \circ L_2}(E_8^2)$. In conclusion, we have shown that

$$C_G(E_8^2) = C_M(v) = ((C_{L_1}(v_1) \circ_2 C_{L_2}(v_2)), 2). g = (((q - \varepsilon) \circ_3 \text{SL}_6^+(q)), 3). c.$$

and $g$ induces $\nu \circ \gamma$ on $(q - \varepsilon)^2 \circ_3 \text{SL}_6^+(q)$; note that $b_i = 2 \times 3$. In particular, $C_{C_G(E_8^2)}(g) = 2^3 \times \text{Sp}_6(q)$. Since $C_{L_2}(D_2) = 2 \times \text{Sp}_6(q)$, the element $y$ acts as $\nu \circ \gamma$ on $(q - \varepsilon) \circ_3 \text{SL}_6^+(q)$ with $C_{C_G(E_8^2)}(\gamma) = \text{Sp}_6(q)$.

(iii) Suppose $v \in M_1 \setminus (L_1 \circ L_2)$. As in the proof of the previous part, by Lemma 6.4, there exist $D_i \leq L_i, s_i$ with $D_i = (a_i, b_i) \in D_8$, such that $b_i \in L_i, s_i \setminus L_i, a_i \in L_i, a_i^2 = z_A, |b_i| = 2, C_{L_1}(b_1) = (q + \varepsilon), \text{O}^+(C_{L_2}(b_2)) = \text{SL}_6^+(q)$, and $C_{L_2}(D_2) = 2 \times \text{Sp}_6(q)$. Define $v = b_2 b_1 \in M_1 \setminus (L_1 \circ L_2)$ and $a = a_1 : a_2 \in L_1 \circ L_2$; it follows from $b_2 b_1 = b_1 b_2 = b_1 z_A$ that $v^2 = v$. Thus, $E_8^3 = (E_8^2, v)^2$ satisfies

$$C_M(E_8^3) = ((q + \varepsilon)^2 \circ_3 \text{SL}_6^-(q)), 3. c. a.$$

and Lemma 6.4 shows that $a$ acts as on $\text{SL}_6^-(q)$ as a graph automorphism $\gamma$ with $C_{\text{SL}_6^-(q)}(a) = \text{Sp}_6(q)$; thus, it follows from [16, Table 4.5.1] (for $C_{L_1}(v_1)$ and $C_{L_2}(v_2)$) that $a$ acts as $\nu \circ \gamma$ on $(q + \varepsilon)^2 \circ_3 \text{SL}_6^-(q)$.

We now show that $E = (E_8^2, v)^2$ with $v \in M_1 \setminus (L_1 \circ L_2)$ is unique up to conjugacy. Note that $v = v x_A$ for some $u \in L_1 \circ L_2$, and, by abuse of notation, $v = v_1 \circ v_2$ with $v_1 \in L_1, s_i \setminus L_i$. Recall that $L_1$ has a unique outer-diagonal involution, so, up to conjugacy, $v_1^{-1} b_1 \in Z(L_1) \leq E$; in particular, we can assume that $v_1 = b_1$ and $v_1$ has order 2. Since $v_2$ induces a non-classical involution in $\text{Out}(L_2)$, it follows from [16, Table 4.5.1] that $C_{L_2}(v_2) = ((2 \times (q + \varepsilon)) \circ_3 \text{SL}_6^-(q)), 3. c$. In particular, $C_G(E) = ((q + \varepsilon)^2 \circ_2 C_{L_2}(v_2)) \circ$ contains a maximal torus $T^* = (q + \varepsilon)^7$ with $E \leq T^*$. As in part b), we have $C_W(E) \geq S_6$ and Table II proves that
Proposition 8.1. If $E < G$ is maximal-proper 2-local and $M = N_G(E)$ with $Z < E$ extraspecial, then $M \in_G \{M_4, M_5\}$ where $M_4 = N_G(Q_2) = (Q_2 \times F_4(q)).S_3$ and $M_5 = N_G(Q_3) = (Q_3 \times P\Omega^+_5(q).2^2).3$ with $Q_2, Q_4 \cong Q_8$ as in Lemma 6.3.

Proof. As a preliminary result, we first prove that, up to conjugacy, $Q_2, Q_4 \leq G$ are the only subgroups of type $Q_8$ containing $Z$ with maximal-proper 2-local normaliser. Let $E = Q_2$ be as in Lemma 6.3, so that $E = Q_8$ is of type $2Z_1C_6$ with $Z(E) = Z$, and $C_G(E)$ and $N_G(E)$ are as in the lemma. Note that $N_G(E)/Z = N_G(Z/E(Z)) = (2^2 \times F_4(q)).S_3$, and $G/Z = E_8(q)$ is adjoint. Since $S_3$ acts irreducibly on $E/Z = 2^2$, it follows that $N_G/Z(E/Z) \leq N_G(Z)/Z = C_G/Z(u)$ for every involution $1 \neq u \in E/Z$; thus, $N_G/E(Z) \leq N_G/Z((u))$. Since $E \in R_2(G)$, Lemma 3.3 shows that $M_4 = N_G(E)$ is maximal-proper 2-local. By construction, the group $N_G/Z(Q_4/Z)$ is maximal-2-local in $G/Z$, hence, by Lemma 3.2, the group $M_5$ is maximal-proper 2-local.

We now show that these groups are unique. For this purpose, suppose there is $F = Q_8 \leq G$ with $Z(F) = Z$ and $N_G(F) \neq G \{M_1, M_2, M_3\}$ maximal-proper 2-local. Write $F = \langle x, y \rangle$ with $x^2 = z = y^2, x^y = x^{-1}$, and $y^x = y^{-1}$. Up to conjugacy in $F$, there are three elements $e$ with $e^2 = z$. Thus, if $F \cap 2B \neq \emptyset \cap 2C$, then, up to $F$-conjugacy, there is a unique $e \in F$ with $e = G 2X$ for some $X \in \{B, C\}$. Now $N_G(F) < N_G(e) \notin G \{M_2, M_3\}$, a contradiction. Therefore, $F$ has type $2Z_1C_6$ or $2Z_1B_6$, and $F \in G \{Q_2, Q_4\}$ follows from Lemma 6.3.

Finally, we prove the claim of the proposition. First, suppose $|E| = 8$. If $E = D_8$ is dihedral with $Z = Z(E)$, then $E$ has a characteristic subgroup $C = 4$ containing $Z$, and it follows that $N_G(E) \leq N_G(C) \in G \{M_2, M_3\}$.

If $E = Q_8$ and $N_G(E)$ is maximal-proper 2-local and $Z < E$, then $N_G(E) = G \{M_4\}$ or $N_G(E) = G \{M_5\}$ as shown above. Now suppose $E$ is extraspecial with $|E| > 8$ and $Z(E) = Z$. Note that $D_8 \times D_8 \cong Q_8 \times Q_8$; by [17, Satz III.13.8], we can write $E = R_1 \times R_2$ where $R_1 = Q_8$ and $R_2$ is extraspecial with $|R_2| > 8$. If $R_1$ is not of type $2Z_1B_6$, then there is $x \in R_1 \cap 2C$, hence $R_2 \leq C_G(x) = ((q - \varepsilon) \omega_3, E_6(q))$. This implies that if $e \in R_2$ has order 4, then $e \in E_6(q)$, a contradiction to $e^2 = z = x^2 \in q - \varepsilon$. Now suppose $R_1$ is of type $2Z_1B_6$, and write $R_1 = \langle x, y \rangle$ with $x^4 = y^4 = 1$. By Lemma 6.3, we can assume $R_1 = Q_4$, hence $C_G(R_1) = 2 \times P\Omega^+_5(q).2^2$ Now, as before, if $e \in R_2$ has order 4, then $e^2 \in P\Omega^+_5(q).2^2 \leq C_G(R_1)$, a contradiction to $e^2 = z = x^2$. □
Lemma 9.1. Up to conjugacy, $C_G(E_3^2) = \{(q - \varepsilon)^2 \circ_{34} \text{SL}_6(q)\}.3_c$. $g$ has three subgroups of type $2^4$ containing $Z$, namely, $E_2^1$, $E_4^1$, and $E_5^1$ with

\[
C_G(E_2^1) = (q - \varepsilon)^3 \circ_2 (\text{SL}_2(q) \times \text{SL}_4(q)).2^2 \quad \text{and} \quad N_G(E_2^1) = C_G(E_2^1).S_4
\]

\[
C_G(E_4^1) = 2^3 \times \text{Sp}_6(q) \quad \text{and} \quad N_G(E_4^1) = C_G(E_4^1).S_4(2)
\]

\[
C_G(E_5^1) = 2^3 \times \text{Sp}_6(q) \quad \text{and} \quad N_G(E_5^1) = C_G(E_5^1).H
\]

with reducible $H \leq \text{SL}_3(2)$ satisfying $D_8 \leq H \leq S_4$. We can assume that $E_1^i, E_3^i \leq E_2^i$ and $E_2^i, E_3^i \leq E_4^i$; moreover, $E_1^i \not\leq E_2^i$ if $i \in \{4, 5\}$. We have $N_G(E_2^i), N_G(E_3^i) \leq M_1$, and $N_G(E_5^i)$ is maximal-proper 2-local.

Proof. Recall that $g \in (L_1 \circ_2 L_2) \setminus L_2$ in an involution, acting as $\iota : \iota : \gamma$ with $C_{\text{SL}_6^2(q)}(g) = \text{Sp}_6(q)$.

Up to conjugacy, $\text{SL}_6^2(q)$ has two non-central involutions with representatives $w_1$ and $w_2$, conjugate over $GF(q^2)$ to diag$(−1, 1_4)$ and diag$(1_2, −1_4)$, respectively; we can assume $w_2 = wyw_1$ where $y = −1_6 \in E_2^i \cap \text{SL}_6^2(q)$. In particular, $C_{\text{SL}_6^2(q)}(w_i) = (q - \varepsilon)^2 (\text{SL}_2(q) \times \text{SL}_4(q)).2$ where $(q - \varepsilon)$ is generated by diag$(a^2, 1_2, 1_4)$ with $a$ of order $q - \varepsilon$, and the rightmost 2 is generated by diag$(-1, 1_4, -1)$. Thus, if we define $E_2^i = \langle E_2^i, w_1 \rangle = \langle E_2^i, w_2 \rangle$,

\[
E_2^i = \langle E_2^i, y \rangle, \quad \text{then} \quad C_G(E_2^1) = (q - \varepsilon)^3 \circ_2 (\text{SL}_2(q) \times \text{SL}_4(q)).2^2; \text{note that} \quad w_i \text{ is centralised by a conjugate of } 3_c \leq C_G(E_2^i), \text{which acts on } \text{SL}_6^2(q) \text{ as a diagonal automorphism. We may suppose } T \leq C = C_G(E_2^1), \text{ hence } 2 \times S_4 \leq N_C(T)/C(T) \leq C_W(E_2^1), \text{ and Table II proves that } C_W(E_2^1) = 2 \times 2 \times S_4 \text{ and } N_W(E_2^1) = C_W(E_2^1).S_4.\]

By Lemma 4.3, we have $N_G(E_2^i) = C_G(E_2^i).S_4$. A direct computation in $2^7.W$ shows that $E_3^1$ contains a conjugate of $E_1^i$, thus we can assume $E_1^i = \langle E_3^1, E_5^1 \rangle$.

Conversely, if $E = \langle E_3^2, y \rangle = 2^4$ with $y \in O^+(C_G(E_2^2)) = \text{SL}_6^2(q)$, the above arguments imply $E = G\cdot E_4^2$.

We now consider the subgroups $E = \langle E_3^2, y \rangle = 2^4 \leq C_G(E_2^2)$ with $y \notin O^+(C_G(E_3^2)) = \text{SL}_6^2(q)$. Note that $g$ and $3_c$ induce a non-trivial graph involution and an outer-diagonal automorphism of order $3_c$ on $\text{SL}_6^2(q)$; we can assume that $g$ inverts $3_c$. Thus, if we define $E_3^i = \langle E_2^i, y \rangle$,

\[
E_3^i = \langle E_2^i, g \rangle,
\]

then $C_G(E_3^1) = 2^3 \times \text{Sp}_6(q)$. Recall that $E_3^i = \langle z, z_A, v \rangle$ and $E_5^i = \langle E_3^i, g \rangle$; by construction, $v = cx_1$ and $g = dy$, where $c, x_1, d, y_1$ were defined via the subgroups $Q_i = \langle x_i, y_i \rangle \leq L_2$ and $\langle c, d \rangle = Q_8 \leq L_1$; recall that $Z(Q_1) = \{1, z_A\}$, $Z(Q_2) = \{1, z\}$, and $Q_1$ and $Q_2$ commute. Thus, $Q_2 \leq C_{\text{G}_G(E_2^i)}(g) = C_G(E_2^1)$, and $z_C \in C_G(E_3^1)$ since $Q_2$ has type $2Z_4.C_6$. Therefore we can assume that $E_5^1 \leq C_G(zC) = ((q - \varepsilon)^2 \circ_3 E_6^2(q)).3_c$.

By the proof of Proposition 7.1c(iii), there is $y \in C_G(E_3^2) \setminus (L_1 \circ_2 L_2)$ with $E_3^2 = \langle E_2^2, y \rangle$. We now define $E_1^i = \langle E_3^2, y \rangle$,

\[
E_1^i = \langle E_3^2, g \rangle,
\]

so that $C_G(E_1^i) = 2^4 \times \text{Sp}_6(q)$ and $E_3^i$ contains both $E_1^i$ and $E_3^i$; this implies that $\text{Out}_G(E_1^i) = \text{Out}_{G/Z}(E_3^1/Z) < \text{SL}_3(2)$. Recall that $C_G(E_2^1) = (L_1, (v_1) \circ_2 L_2(v_2)).2.g = (((q - \varepsilon)^2 \circ_3 \text{SL}_6^2(q)).3_c).g$.

and both $g \in L_1 \circ_2 L_2$ and $y \in M_1 \setminus (L_1 \circ_2 L_2)$ have order 2 and act as $\iota : \iota : \gamma$ on $(q - \varepsilon)^2 \circ_3 \text{SL}_6^2(q)$. In particular, $gy^{-1} \in (q - \varepsilon)^2$, so $y = ug$ for some $u \in (q - \varepsilon)^2$. If $t \in (q - \varepsilon)^2$, then $gt^−1 = t^2g$, so there are at most two non-conjugate involutions of type $xg$ in $C_G(E_3^2)$ with $x \in (q - \varepsilon)^2$. Up to conjugacy, these are $y = ug$ and $g$.

Now suppose $E = \langle E_3^2, h \rangle = 2^4$ with $h \in C_G(E_2^2) \setminus O^+(C_G(E_2^2))$. If $h$ induces a graph automorphism on $O^+(C_G(E_2^2)) = \text{SL}_6^2(q)$, then, since $|h| = 2$, the discussion in the proof of Lemma 6.4 shows that $C_{\text{SL}_6^2(q)}(h) = \text{Sp}_6(q)$. Thus, up to conjugacy, $h$ can be written as $h = cg$ for some $c \in (q - \varepsilon)^2$. As shown above, either $h = G \circ g$ or $h = \gamma g = yg$, hence $E = G\cdot E_4^i$ with $i \in \{4, 5\}$ follows. If $h$ does not induce a graph automorphism, then $h \in (q - \varepsilon)^2 \circ_3 \text{SL}_6^2(q)$. Since $\Omega_1(2 \circ (q - \varepsilon)^2) \leq E$, we can suppose $h \in \text{SL}_6^2(q)$; but then $E = G\cdot E_4^i$.

Thus, up to conjugacy, $C_G(E_2^2)$ has three subgroup $E = 2^4$ with $E_3^i \leq E$, namely, $E_1^i, E_3^i, \text{ and } E_5^i$.

Finally, we normaliser structure. We can suppose $E_3^i \leq C_G(zC)$ and $\Omega_1(2 \circ (C_G(zC))) = \langle 1, z \rangle = Z \leq E_5^i$, thus $E_3^i = Z \times X$ for some $X = 2^3 \leq E_6^2(q) = O^+(C_G(zC))$. We now investigate the possible structure of $X$. By [16, Table 4.5.2], up to conjugacy, $K = E_6^2(q)$ has two classes of inner involutions
where the rightmost 2 is a conjugate of $C$; it follows that if $A$ is not 2b-pure in $K$, then $C_K(X)$ contains a maximal torus $T_X = (q^\epsilon)^6$ of $K$, hence $C_G(Z(C_G(z_G))) = T_X = (q^\epsilon)^7$; this is a contradiction to $T \not\leq G$. $C_G(E_4^4)$, and it follows that $X$ is 2b-pure in $K$. By the proof of [4, Theorem 4.1, up to conjugacy, $K$ has two 2b-pure subgroups $(2b^3)_1$ and $(2b^3)_2$ with $Out_K((2b^3)_1) = SL_3(2)$ and $Out_K((2b^3)_2) = D_8$. In addition, $(2b^3)_1$ is $(2b^2)_\epsilon$-pure, which means that every subgroup $2^3$ of $(2b^2)_\epsilon$ and $(2b^2)_\gamma$ contains both $(2b^2)_\epsilon$ and $(2b^2)_\gamma$. For $i = 1, 2, \{z, (2b^3)_1\}$.

As shown above, $Out_G(E_4^4) \leq SL_3(2)$, which forces $E_2^4 = G$ and $Out_G(E_4^4) = SL_3(2)$. Thus, all subgroups $2^3$ of $E_2^4$ containing $Z$ are conjugate, which implies that $E_2^4 \not\leq E_3^4$.

We have shown that $E_1^4 = G$, hence $D_8 \leq Out_G(E_4^4)$. As seen above, $Out_G(E_4^4) < SL_3(2)$; note that $E_4^4$ contains both $E_3^4$ and $E_3^6$. The structure of $SL_3(2)$ implies that $Out_G(E_4^4) \in \{D_8, S_4\}$. In both cases, $Out_G(E_4^4)$ acts reducibly on a line $L/Z \leq E_4^4$ where $L = 2^2$, thus $N_G(E_4^4) \leq N_G(L) = G$. Clearly, $N_G(E_4^4)$ is maximal-proper 2-local since $Out_G(E_4^4) \leq SL_3(2)$ acts irreducibly.

Finally, suppose $E_3^4 \leq E_1^4$, so there exists $h \in L_2$ with $E_1^4 = \langle E_3^4, h \rangle$ and $\langle z, z_H, h \rangle = E_3^4$. Since $E_1^4, E_2^4 \leq L_1 \circ L_2$, it follows from $h \in L_2$ that $E_2^4 \leq L_1 \circ L_2$; this is impossible since $y \in E_4^4 \setminus \{L_1 \circ L_2\}$. □

Now let $E = 2^4$ with $Z \leq E$ and $E \not\leq C_G(E_4^4)$. Recall $E_4^4 = \langle E^2, b \rangle$ with $b \in M_1 \setminus \{L_1 \circ L_2\}$, and $C_G(E_4^4) = ((q + \epsilon)^2 \cdot _L \gamma, SL_6^-(q))^3 \cdot _L \gamma, g$, where $g \in L_1 \circ L_2$ has order 2, inducing $i = \iota : \gamma$ with $C_{SL_6^-(q)}(\gamma) = Sp_6(q)$. □

Lemma 9.2. If $E = 2^4 \leq G$ contains $E_3^4$, then $E = G$, $E_4^4$ as in Lemma 9.1, or $E = G, E_4^4 = \langle E_3^4, E_2^4 \rangle$ with $C_G(E_4^4) = ((q + \epsilon)^3 \cdot _L \gamma, SL_2(q) \times SL_4^-(q)) \cdot _L \gamma$, and $N_G(E_4^4) = C_G(E_4^4).S_4$.

In addition, $E = G, E_3^4$ if and only if $E \leq (q + \epsilon)^2 \cdot _L \gamma, SL_6^-(q)$. Also, $N_G(E_3^4) \leq G$. □

Proof. The proof is similar to the proof of Lemma 9.1. First, up to conjugacy, $SL_6^-(q)$ has two involutions $w_1 = \text{diag}(-1, 2, 14)$ and $w_2 = -w_1$; they satisfy

$$C_{SL_6^-(q)}(w_1) = (q + \epsilon)^2 \cdot _L \gamma, SL_2(q) \times SL_4^-(q).$$

and $\Omega_1(O_2(C_{SL_6^-(q)}(w_1))) = \Omega_1(O_2(C_{SL_6^-(q)}(w_2))) = 2^2$. We define

$$E_4^4 = \langle E_3^4, w_1 \rangle = \langle E_3^4, w_2 \rangle,$$

so that $C_G(E_4^4) = ((q + \epsilon)^3 \cdot _L \gamma, SL_2(q) \times SL_4^-(q))^2$; the local structure implies that $E_4^4 \neq G$ with $i \in \{2, 4, 5\}$. Conversely, if $E = (E_3^4, y)$ with $y \in (q + \epsilon)^2 \cdot _L \gamma, SL_6^-(q)$, then we may suppose $y \in SL_6^-(q)$, implying $E = G, E_4^4$.

We may suppose $T^\ast = (q + \epsilon)^\gamma \leq C_G(E_4^4)$, thus $S_4 \leq Out_G(E_4^4)$ by Table II; more precisely, up to conjugacy, $E_4^4$ is the unique subgroup of $T^\ast$ with $C_W(E_4^4) = 2 \times 2 \times S_4$. A computation in $2^7.W$ shows that $E_4^4$ contains a subgroup of type $E_2^4$, hence $E_4^4 \leq E_3^4$, and $Out_G(E_4^4) = S_4$ follows.

Suppose $E = (E_3^4, h) = 2^4$ with $h \in C_G(E_4^4) \setminus ((q + \epsilon)^2 \cdot _L \gamma, SL_6^-(q))$. Since $E_3^4 = \langle z, z_H, b \rangle$ and $h \in C_M(b)$, we must have $h \notin L_2$; otherwise $h \in (q + \epsilon)^2 \cdot _L \gamma, SL_6^-(q) \leq C_M(b)$ yields a contradiction to our assumption. Thus $h \in (L_1 \circ L_2) \setminus L_2$ or $h \in M_1 \setminus (L_1 \circ L_2)$; in the former case, $E \leq C_G(E_4^4)$, which is covered by Lemma 9.1 (note that $E_4^4$ is $E_3^4$-pure, so $E = G, E_4^4$). In the latter case, $hb \in (L_1 \circ L_2) \setminus L_2$, hence, again, $E \leq C_G(E_4^4)$.

Note that $E = G, E_4^4$ is not possible since $E_4^4 \leq L_1 \circ L_2$ but $b \in M_1 \setminus (L_1 \circ L_2)$. Finally, $N_G(E_4^4) \leq G$ follows since $S_4 < SL_3(2)$ acts reducibly. □

Finally, suppose $Z < E = 2^4 < G$ contains $E_4^4 = \langle E^2, y \rangle = \langle z, z_H, y \rangle$ with $y \in L_2$. Recall that

$$C_G(E_4^4) = ((SL_2(q) \circ_{(z_H)} (Spin_4^+(q) \circ_{(zz_H)} Spin_6(q))) \cdot _L \gamma, 2,$$

where the rightmost 2 is a conjugate of $x_A \in M_1 \setminus (L_1 \circ_{(z_H)} L_2)$. □
Lemma 9.3. If $E = 2^4 \leq G$ contains $E_3^4$, then $E \in G \{E_3^4, E_3^5\}$ as in Lemmas 9.1 and 9.2, or $E = G E_1^4$ is $E_1^4$-pure with $N_G(E_1^4) = C_G(E_1^4) \cdot \text{SL}_2(3)$ and $C_G(E_1^4) = (\text{SL}_2(q) \circ \text{Spin}_4^+ (q) \circ \text{Spin}_4^+ (q)) \circ \text{Spin}_4^+ (q))^2$.

Proof. By [16, Table 4.5.2], up to conjugacy, the group $\text{Spin}_8^+(q)$ has a unique non-central involution $u$; it satisfies $C_{\text{Spin}_8^+(q)}(u) = (\text{Spin}_4^+ (q) \circ \text{Spin}_4^+ (q))^2$. We define

$$E_1^4 = \langle E_1^3, u \rangle,$$

so that $C_G(E_1^4)$ is as given in the lemma. Recall that $\text{Spin}_4^+(q) = \text{SL}_2(q) \times \text{SL}_2(q)$, hence $C_G(E_1^4)$ contains both $T = (q - \varepsilon)^2$ and $T^* = (q + \varepsilon)^2$, and $C_W(E_1^4) = N_G(E_1^4)(T)/T \geq 2^7$. Now Table II implies that $C_W(E_1^4) = 2^7$ and $N_W(E_1^4) = \text{SL}_3(2)$; Lemma 4.3 shows that $N_G(E_1^4) = C_G(E_1^4) \cdot \text{SL}_3(2)$, in particular, $E_1^4$ is $E_1^4$-pure. If $E = \langle E_1^3, h \rangle = 2^4$ with $h \in \text{Spin}_8^+(q)$, then $E = G E_1^4$.

Now suppose $E = \langle E_1^3, h \rangle = 2^4$ with $h \in C_G(E_1^4) \setminus \text{Spin}_8^+(q)$. Note that if $h = h_1 : h_2 \in L_1 \circ L_2$ with both $h_2^2 = z_A$, then $h \notin L_2$ and $\langle E^2, h \rangle \in G \{E_2^3, E_3^4\}$; then $E \leq C_G(E_1^4)$ with $i \in \{2, 3\}$, which is covered by Lemmas 9.1 and 9.2. Since $E_1^3 = \Omega_2(Z(C_G(E_1^4)))) \leq E$ and $\text{Spin}_8^+(q)$ has no non-central involution, we can suppose $h \in C_{L_2}(y) \setminus \text{Spin}_8^+(q) \circ \text{Spin}_8^+(q)$. In both cases, $h$ induces inner-diagonal automorphism on $\text{Spin}_8^+(q)$ and $\text{Spin}_8^+(q)$, thus, by [16, Table 4.5.2], the centraliser $C_{L_2}(y, h)$ contains a maximal torus $(q - \varepsilon)^6$ or $(q + \varepsilon)^6$. In particular, $T \leq C_G(E_1^4)$ or $T^* \leq C_G(E_1^4)$, say $T \leq C_G(E_1^4)$. Thus, we can suppose $E \leq T$. Recall that $E_1^4$, $E_3^4$, and $E_2^3$ are, up to conjugacy, the unique subgroups of order 24 contained in $T$ or $T^*$, thus $E \in G \{E_1^4, E_2^3, E_3^4\}$. Note that if $E_1^4 = \langle E_1^3, u \rangle$ as above, then $O^*(C_{L_2}((y, h))) \neq O^*(C_{L_2}((y, u)))$; this implies $E \in G \{E_2^3, E_3^4\}$. □

10. Elementary abelian subgroups III

We complete the proofs of Theorem 2.1 and 2.2 and classify the subgroups $2^5$, $2^6$, and $2^7$ of $G$ containing $Z$. The strategy is as before: if $E = 2^{i+1} \leq G$ with $Z < E$, then we can suppose $E \leq C_G(E) \leq C_G(E_i^4)$ for some $i$; we then make a case distinction on $i$. Again, the proof is straightforward, but technical and tedious.

Lemma 10.1. If $E = 2^5 \leq G$ with $Z < E$, then $E \in G \{E_1^5, \ldots, E_5^5\}$ as in Theorem 2.1.e; the group $N_G(E_5^5)$ is maximal-proper 2-local if and only if $j \in \{1, 6\}$.

Proof. Let $E = 2^5 \leq G$; we may suppose $E_1^5 \leq E$ for some $i \in \{1, \ldots, 5\}$, hence $E \leq C_G(E_1^5)$.

Case $i = 4$. Here $E \leq C_G(E_1^4) = 2^4 \circ \text{Sp}_6(q)$, hence $E = \langle E_1^3, y \rangle$ for some non-central involution $y \in \text{Sp}_6(q)$; we can suppose $y \in \{w, w^-\}$ where $w = \text{diag}(-1_2, 1_4)$. Since $-1_6 \in Z(\text{Sp}_6(q)) \leq E_3^4$, we have $E_3^4 = E = \langle E_1^3, w \rangle$, hence

$$C_G(E_1^5) = 2^5 \circ \text{Sp}_2(\text{SL}_2(q) \times \text{Sp}_4(q))$$

and $E_1^5 = Z(C_G(E_1^5)) = \langle E_1^3, w \rangle$. Note that $N_G(E_1^5)$ normalises $M = \langle z, Z(O^*(C_G(E^5_1))) \rangle$, and $M \in \{2^2, 2^4\}$, thus $N_G(E_1^5) \leq G M_1$ or $N_G(E_1^5) \leq G N_G(E_1^5)$ for some $i$. We show $E_5^5 = \langle E_2^3, E_3^4, E_1^5 \rangle$ in Cases $i = 2$ and $i = 3$.

Case $i = 5$. As for $i = 4$, there is a unique $2^5 \leq C_G(E_1^5)$, namely, $E_5^5 = \langle E_1^5, w \rangle$ with $C_G(E_1^5) = 2^5 \circ \text{Sp}_6(q)$; again, $N_G(E_5^5) \leq G M_1$ or $N_G(E_5^5) \leq G N_G(E_5^5)$ for some $i$. We show $E_5^5 = \langle E_2^3, E_3^4, E_1^5 \rangle$ in Case $i = 2$.

Case $i = 2$. Here $E \leq C_G(E_1^2) = ((q - \varepsilon)^2 \circ \text{SL}_2(q) \times \text{SL}_2(q))$, $2^2$ and $E = \langle E_1^2, y \rangle$ for some $y \in C_G(E_1^2) \setminus E_2^4$. We may suppose $E_1^2 = \langle E_1^1, E_2^1 \rangle$, hence $E_1^2 \leq C_G(E_2^1) = ((q - \varepsilon)^2 \circ \text{SL}_4(q), 3e)$. By Lemma 9.1, we can also assume that $E_2^1 = \langle E_2^1, w_1 \rangle = \langle E_2^1, w_2 \rangle$ with $w_2 = -w_1$ and $w_1 = \text{diag}(-1_2, 1_4) \in O^*(C_G(E_1^2)) = \text{SL}_6(q)$. Now define $w_3 = \text{diag}(1, -1_2, 1_3, w_4 = -w_3, w_5 = \text{diag}(-1_4, 1_2)$, and $w_6 = -w_5$, all as elements in $O^*(C_G(E_1^2))$.

Note that each $w_i \in C_{\text{SL}_6(q)}(E_2^1)$.

Recall $E = \langle E_1^2, y \rangle$ for some $y \in C_G(E_1^2) \setminus E_2^4$; if $y \in \text{SL}_6(q)$, then $y \in U = C_{\text{SL}_6(q)}(w_1) = C_{\text{SL}_6(q)}(E_2^1)$ and so $y \not\in U \{w_3, w_4, w_5, w_6\}$. Define $E_1^5 = \langle E_2^1, w_3 \rangle$ and $E_2^5 = \langle E_2^1, w_5 \rangle$, so that

$$C_G(E_1^5) = ((q - \varepsilon)^5 \circ \text{SL}_6(q), 3e) \times 2^3$$

and

$$C_G(E_2^5) = ((q - \varepsilon)^4 \circ \text{SL}_2(q)) \times 3^3 \times 2^3$$

and
Note that $T \preceq G C_G(E_0^i)$ for $i \in \{1, 2\}$, thus Table II and Lemma 4.3 imply $\text{Out}_G(E_0^1) = S_6$ and $\text{Out}_G(E_0^2) = 2^4(S_4 \times S_3)$. Since $\langle \rho, Z(O^+ (C_G(E_0^i))) \rangle \in \{2^3, 2^4\}$, it follows that $N_G(E_0^2) \leq G N_G(E_0^1)$ or $N_G(E_0^2) \leq G N_G(E_0^1)$ for some $i$. We show $E_0^3 = (E_0^3, E_0^3)$ in Case $i = 1$. Since $S_6$ acts irreducibly on $E_2^3/Z$, it follows that $N_G(E_0^1)$ is maximal-proper 2-local. Since $S_6 \leq S_{4}(2)$ acts transitively on 3-dimensional spaces, it follows that $E_0^3$ is $E_2^3$-pure.

Now suppose $E = 2^5 = \langle E_1^3, y \rangle$ with $y \in C_G(E_0^3) \setminus \text{SL}_6^-(q)$. The proof of Lemma 9.1 shows $\langle E_1^3, y \rangle \in G \{E_4^3, E_5^3\}$ and hence $E \leq C_G(E_4^3)$ with $i \in \{4, 5\}$. As shown above, $E \in G \{E_4^3, E_5^3\}$. In particular, $E_1^5 \leq E_1^3$.

Case $i = 3$. Here $E \leq C_G(E_1^3) = (\langle q+e \rangle^3 \text{SL}_2(q) \times \text{SL}_6^-(q) )^{2}$. and $E = \langle E_1^3, y \rangle$ for some $y \in C_G(E_1^3) \setminus E_4^3$; we may suppose $E_1^3 \leq C_G(E_1^3) = (\langle q+e \rangle^3 \text{SL}_6^-(q))^{2}$. First, consider the case $y \in (\langle q+e \rangle^3 \text{SL}_6^-(q))$: we can suppose that $y \in O^+ (C_G(E_0^3)) = \text{SL}_6^-(q)$. A proof similar to Case $i = 2$ shows that $E \in G \{E_3^3, E_6^3\}$ where for $i \in \{3, 5\}$ the local structure of $E_i^3$ is as given in the theorem. Also, $E_0^3$ is $E_3^3$-pure; we show $E_3^3 = (E_1^3, E_3^3)$ in Case $i = 1$.

Now suppose $y \notin (\langle q+e \rangle^3 \text{SL}_6^-(q) )^{2}$. The proof of Lemma 9.2 shows that $\langle E_1^3, y \rangle = G E_1^3$, hence we can assume $E \leq C_G(E_1^3)$, and $E = G E_1^3$. In particular, $E_1^5 \leq E_1^3$.

Case $i = 1$. Here $E \leq C_G(E_1^3) = (\text{SL}_2(q) \times \text{SL}_6^-(q) )^{2}$. and $E = \langle E_1^3, y \rangle$ for some $y \in C_G(E_1^3) \setminus E_4^3$. We may suppose $E_1^3 \leq C_G(E_1^3) = (\text{SL}_2(q) \times \text{SL}_6^-(q))^2$. The construction in the proof of Lemma 9.3 shows that we can assume $E_1^3 = (E_2^3, y)$ where $y \in \text{SL}_6^-(q)$ is, up to conjugacy, the unique non-central involution in $\text{Spin}_6^-(q)$; it satisfies $\text{Spin}_6^+(q)^{(y)} = (\text{Spin}_6^+(q))^2$. In particular, if $y \notin C_G(E_1^3) \setminus \text{SL}_6^-(q)$, then $E_1^3 = G (E_1^3, y) \leq E$ for some $j \in \{2, \ldots, 5\}$; thus, $E \leq C_G(E_4^3)$, which is covered above. Thus, we can suppose $y \in \text{Spin}_6^+(q) \leq C_G(E_1^3) \setminus \text{SL}_6^-(q)$ contains a maximal torus $(q - e)^{5}$ or $(q + e)^5$. Thus, $C_G(E)$ contains a maximal torus $T$ or $T^{*}$, and so $E = G E_1^3$ for some $i \in \{1, 2, 3, 6\}$. Since $\text{SL}_3^+(q) \leq C_G(E)$, we must have $E \in G \{E_3^3, E_6^3\}$. Note that $E_1^3 \leq E$. In fact, since each $E_2^3$ and $E_3^3$ lies in a torus $T$ or $T^{*}$, a direct computation in $2^7$.W shows that $E_3^3 = \langle E_1^3, E_3^3 \rangle$ and $E_3^3 = \langle E_3^3, E_3^3 \rangle$; note that $E_1^3$ is, up to conjugacy, the unique group $2^4$ in $T$ (and $T^{*}$) with $\text{Out}_W(E_1^3) = \text{SL}_3(2)$.

**Lemma 10.2.** If $E = 2^6 \leq G$ with $Z < E$, then $E \in G \{E_1^6, \ldots, E_6^6\}$ as in Theorem 2.1f); no group $N_G(E_0^5)$ is maximal-proper 2-local.

**Proof.** Let $E = 2^6 \leq G$ with $Z \leq E$; we can assume $E_0^5 \leq E$ for some $i \in \{1, 2, 3, 4, 5, 6\}$, hence $E \leq C_G(E_0^5)$.

Case $i = 4$. Here $E \leq C_G(E_4^i) = 2^6 \langle \text{SL}_2(q) \times \text{Sp}_4(q) \rangle$ and $E = \langle E_4^i, y \rangle$ with $y \in \text{Sp}_4(q)$. Up to conjugacy, $\text{Sp}_4(q)$ has two involutions $w = \text{diag}(1, -1, 1, -1)$ and $-I_4$, thus we can suppose $y = w$ or $y = -I_4$. Since $Z(\text{Sp}_4(q)) \leq E_4^5$, see Case $i = 4$ in the proof of Lemma 10.1, we define $E_4^3 = (E_4^i, w)$, so that $C_G(E_4^3) = 2^6 \langle \text{SL}_2(q)^3 \rangle$. Since $\langle \rho, Z(O^+ (C_G(E_4^3))) \rangle \in \{2^3, 2^4\}$, it follows that $N_G(E_4^3) \leq G N_G(E_4^3)$ for some $j \in \{3, 4\}$ and $E_4^3 \leq C_G(E_4^3)$.

Case $i = 5$. As in Case $i = 4$, up to conjugacy, $C_G(E_5^5)$ contains a unique $E_0^6 = \langle E_5^5, w \rangle = 2^6$ with local structure as given in the theorem. Note that $E_5^6 \leq G E_5^5$.

Case $i = 1$. Here $E \leq C_G(E_1^3) = (\langle q - e \rangle^3 \text{SL}_2(q), 3.)^2$ and $E = \langle E_1^3, y \rangle$ for some $y \in C_G(E_1^3) \setminus E_4^3$. We may suppose $E_2^3 \leq E_1^3$, hence $y \in C_G(E_2^3) = (\langle q - e \rangle^3 \text{SL}_2(q), 3.)^2$.

If $y \in O^+ (C_G(E_0^3)) = \text{SL}_3(q)$, then we can suppose $y = \text{diag}(1, -1, 2)$, hence $E = E_1^6 = \langle E_5^5, y \rangle$ with $C_G(E_1^6) = (\langle q - e \rangle^6 \text{SL}_2(q), 2)^2$.

Note that $Z(O^+ (C_G(E_0^3))) = Z(\text{SL}_2(q)) = \langle u \rangle$; we show that $u \neq z$: We can suppose that $C_G(E_1^6) \leq C_G(E_3^3)$ and, by construction, $O^+ (C_G(E_3^3)) \leq \text{SL}_2(q) \leq O^+ (C_G(E_3^3)) = \text{SL}_6^-(q)$. The proof of Proposition 7.1(ii) shows that $Z(\text{SL}_6^-(q)) = \langle z_A \rangle \leq L_2 = \text{Spin}_2^-(q)$ and, using Notation 6.1, we know that $\pi(z_A) = -12$ and $z \notin \text{SL}_6^-(q)$.  


If \( \pi(u) = -112 \), then \( u.a \in \ker \pi = \langle zz_a \rangle \), hence \( z \in S L_6^5(q) \), a contradiction. Thus, \( \pi(u) \neq -112 \), and therefore \( u \neq z \). This yields \( \langle z, Z(O'(C G(E^6_i))) \rangle = G \) \( E^2 \), which proves that \( N_G(E^6_i) \leq G M_1 \). We can assume \( E^6_1 \leq T \), hence \( \text{Out}_G(E^6_i) = 2^4.S_6 \) by Table II and Lemma 4.3.

If \( y \notin O'(C G(E^6_i)) \), then we consider \( y \in C G(E^6_i) = ((q - \varepsilon)^3 \circ q, S L_6^5(q).3) \). If \( y \notin (q - \varepsilon)^2 \circ q, S L_6^5(q) \), then Lemma 9.1 shows that \( Y = \langle E^6_3, y \rangle \in G \{ E^6_3, E^6_4 \} \). Therefore we can suppose \( Y < E^6_5 < E \) for some \( j \in \{ 2, 3, 4, 5 \} \). Note that \( Y \) is not contained in any conjugate of \( T \) or \( T^* \), but each \( E^6_3 \) and \( E^6_4 \) is. Thus \( j \in \{ 3 \} \) is not possible, and therefore \( E \leq C G(E^6_i) \) with \( j \in \{ 4, 5 \} \), which is covered above.

If \( y \in (q - \varepsilon)^2 \circ q, S L_6^5(q) \), then we may suppose \( y \in S L_6^5(q) \). We can suppose that \( E^6_1 = E^3_1 \times X \) for some \( 2^2 = X \leq S L_6^5(q) \); note that the case \( X \leq S L_6^5(q).2 \setminus S L_6^5(q) \) yields \( x \in S L_6^5(q) \), and \( k \in \{ 4, 5 \} \), which leads to \( E \leq C G(E^6_i) \) with \( j \in \{ 4, 5 \} \) as discussed above. Thus \( E = \langle E^6_3, y \rangle \) with \( y \in S L_6^5(q) \), and \( U = \langle (x, y) = 2^3 \rangle \) is an elementary abelian 2-subgroup of \( S L_6^5(q) \). The outer 2 in \( C G(E^6_i) \) inverts each element of the radical \( O(C G(E^6_i)) = (q - \varepsilon)^2 \); similarly, the outer 2 in \( C G(E^6_i) \) acts as inversion on \( O(C G(E^6_i)) = (q - \varepsilon)^5 \). Now \( E^6_1 = E^3_1 \times X \) implies that \( C G(E^6_1) = C C G(E^6_3)(X) \), and so \( S L_6^5(q)(X) = (q - \varepsilon)^3 \circ q, S L_6^5(q).3 \). Since \( y \in C S L_6^5(q)(X) \), we may suppose \( y \in S L_6^5(q) \) – which contradicts our assumption \( y \notin O'(C G(E^6_i)) \).

**Case** \( i = 6 \). Here \( E \leq C G(E^6_6) = ((q + \varepsilon)^5 \circ q, S L_6^5(q).3) \). and \( E = \langle E^6_3, y \rangle \) for some \( y \in C G(E^6_6) \). Analogously to Case \( i = 1 \), the only new group we obtain is \( E^6_3 = \langle E^6_6, y \rangle \) where \( y = \text{diag}(1, -1, 2) \in O'(C G(E^6_6)) \), and

\[ C G(E^6_6) = ((q + \varepsilon)^5 \circ q(2) \cdot 2) \cdot 2. \]

Moreover, \( E^6_3 \leq T^* \leq C G(E^6_i) \), hence \( \text{Out}_G(E^6_3) = 2^4.S_6 \).

**Case** \( i = 2 \). Here \( E \leq C G(E^6_2) = ((q - \varepsilon)^3 \circ q(2) \cdot 2 \cdot 2) \cdot 2 \). and \( E = \langle E^6_3, y \rangle \) for some \( y \in C G(E^6_2) \). We may suppose \( E^6_3 \leq E^6_2 \), hence \( y \in C G(E^6_2) = ((q - \varepsilon)^3 \circ q(2) \cdot 2 \cdot 2) \cdot 2 \). We proceed as in Case \( i = 1 \).

If \( y \notin (q - \varepsilon)^2 \circ q, S L_6^5(q) \), then \( Y = \langle E^6_3, y \rangle \in G \{ E^6_3, E^6_4 \} \), hence \( E \leq C G(Y) \) and \( E^6_2 < E \) for some \( j \in \{ 4, 5 \} \); but \( E \leq C G(E^6_i) \) is covered above. If \( y \notin (q - \varepsilon)^2 \circ q, S L_6^5(q) \), then \( E = g E^6_1 \).

**Case** \( i = 3 \). Here \( E \leq C G(E^6_3) = ((q + \varepsilon)^5 \circ q(S L_6^5(q))^3) \cdot 2^3 \). and \( E = \langle E^6_3, y \rangle \) for some \( y \in C G(E^6_3) \). As in Case \( i = 2 \), we obtain no new group; the proof is similar to Case \( i = 6 \). □

**Lemma 10.3.** If \( E = 2^7 \leq G \) with \( Z < E \), then \( E \in G \{ E^7_1, E^7_2 \} \) as in Theorem 2.1f); both \( N_G(E^7_i) \) are maximal-proper 2-local.

**Proof.** Let \( E = 2^7 \leq G \) with \( Z \leq E \); we can assume \( E^6 \leq E \) for some \( i \in \{ 1, 2, 3, 4 \} \), hence \( E \leq C G(E^6_i) \).

**Case** \( i = 2 \). Here \( E \leq C G(E^6_2) = 2^6 \circ q(S L_6^5(q))^3 \). and \( E = \langle E^6_3, y \rangle \) for some \( y \in C G(E^6_2) \). Since \( E^6_3 = Z(C G(E^6_3)) \), we may suppose \( y \in O'(C G(E^6_2)) \) \( Z(O'(C G(E^6_2))) \); but this is impossible since \( O'(C G(E^6_2)) = (S L_6^5(q))^3 \) has no non-central involution.

**Case** \( i = 3 \). Here \( E \leq C G(E^6_3) = 2^6 \circ q(S L_6^5(q))^3 \), which is impossible, cf. Case \( i = 2 \).

**Case** \( i = 1 \). Here \( E \leq C G(E^6_1) = ((q - \varepsilon)^3 \circ q(S L_6^5(q))^3) \cdot 2^2 \). and \( E = \langle E^6_3, y \rangle \) with \( y \in C G(E^6_1) \). We may suppose \( E^6_3 \leq E^6_2 \leq E^6_i \), so that

\[ C G(E^6_1) \leq C G(E^6_2) = ((q - \varepsilon)^3 \circ q(S L_6^5(q).3)_z).2. \]

If \( E^6_3 \leq Z(C G(E^6_3))O'(C G(E^6_3)) = ((q - \varepsilon)^3 \circ q(S L_6^5(q)), then there is \( h \in E^6_1 \cap C G(E^6_3) \). \( ((q - \varepsilon)^3 \circ q(S L_6^5(q)), \)

and Lemma 9.1 shows that \( E^6_3 = \langle E^6_3, h \rangle \leq E^6_1 \) for some \( j \in \{ 4, 5 \} \). Thus \( C G(E^6_3) \leq C G(E^6_1) = 2^3 \times S p_6(q), \)

which is impossible. This proves \( E^6_1 \leq (q - \varepsilon)^2 \circ q(S L_6^5(q). Similarly, if \( y \in C G(E^6_2) \). \( ((q - \varepsilon)^2 \circ q(S L_6^5(q), then \)

\( E \leq C G(E^6_1) = 2^3 \times S p_6(q) \) for some \( j \in \{ 4, 5 \} \). By the proofs of parts d) and e), we can assume \( E^6_3 \leq E^6_i \leq E \) for some \( k \in \{ 2, 3 \} \); but \( E \leq C G(E^6_k) \) is not possible as shown in Cases \( i = 2 \) and \( i = 3 \). In conclusion, \( E = (q - \varepsilon)^2 \circ q(S L_6^5(q), and so \( E = 2^3 \times X \) for some \( 2^5 = X = S L_6^5(q) \). Since \( E \) is good for \( S L_6^5(q), the abelian group \( X \) is contained in a maximal torus \( (q - \varepsilon)^5 \) of \( S L_6^5(q) \) with \( S L_6^5(q)(X) = (q - \varepsilon)^5 \). We can assume that \( E \leq T \leq C G(E) \); by Table II and Lemma 4.3, it follows that \( C G(E) = T.2 \) and \( N_G(E) = T.W \). In particular, \( E = \Omega_1(O_2(T)) \).
Case \( i = 4 \). Here \( E \leq C_G(E_6^2) = (q + \varepsilon)^6 \circ_2 (\text{SL}_2(q), 2) \) and \( E = \langle E_6^2, y \rangle \) with \( y \in C_G(E_6^2) \setminus E_4^2 \). As in Case \( i = 1 \), we obtain only \( E_2^2 = \Omega_1(O_2(T^*)) \) as new group.

References

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