Marching in Squares

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Recently, a video of Japanese students performing some truly amazing precision marching (youtu.be/R3rKGfxtXM) inspired us to ponder a very special kind of marching choreography. Our plan is to have two identical squares of marchers, each square performing the usual stunning steps. Then, the grand finale will consist of the squares being merged into one big square.

It would definitely be a showstopper, but first there are details to sort out. The interlacing of the squares will be tricky, requiring planning, practice, and fancy footwork.

We also have to decide the size of squares to use, which would need to consist of a suitable number of marchers. For example, beginning with two tiny $2 \times 2$ squares would not work: That would give us 8 marchers in total, one short of the number required to rearrange into a $3 \times 3$ square. Similarly, beginning with two $3 \times 3$ squares of marchers would mean we have 18 marchers, too many to form a $4 \times 4$ square and insufficient for a $5 \times 5$ square.

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/ucmj.
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Hmm. This will take some figuring, but we should be able to do it. We will try $4 \times 4$, then $5 \times 5$ and so on, and eventually we should have in hand the smallest squares that work.

For now, we will leave that calculation and just assume we have located the smallest squares that can be merged. Then, imagining we have sufficient marchers to occupy those squares, we can plan the marching steps.

We begin with an empty large quadrangle of just the right size to accommodate all our marchers. Then, a stylish approach would be to have the two identical squares of marchers enter the quadrangle from opposite sides.

At this stage the little squares on the upper right and lower left are unoccupied and our two squares of marchers are overlapping. It would be crowded in the middle dark square, but that is not a problem; we can simply arrange for half of the marchers to stand on the shoulders of the others.

Finally, we will have the marchers leap off their comrades’ shoulders and into the empty little squares, a spectacular finish to our merging of the two squares of marchers into the big square. Ta da!

But wait a minute. If, as planned, we have precisely the correct number of leaping marchers to fill the little squares, then this can be represented by the following picture.

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Uh oh. We assumed that we began with the very smallest squares that could be merged to make a larger square. Yet, somehow we created even smaller squares that would work. How can that be?

Simply, it cannot be. For our smallest possible squares to result in even smaller squares that work is a plain logical impossibility. The unavoidable conclusion is that no squares can be merged in the way we had originally contemplated. So much for our plans to impress our audience with a great marching finale. However, perhaps our audience will be impressed by some intriguing mathematics that emerges from our failed attempt.

Of course, what we have outlined above is a proof by contradiction. We began by assuming that certain squares were possible, and that assumption resulted in a logical impossibility. This contradiction proves that our original assumption was wrong and that no such squares can exist.

We can now reconsider this conclusion in terms of numbers rather than squares. What we have demonstrated is that there are no natural numbers $m$ and $n$ that solve the equation $2m^2 = n^2$. Rearranging, it follows that there is no fraction $n/m$ that solves the
equation \((n/m)^2 = 2\). So, there is no fraction whose square is 2, which is exactly the same as saying \(\sqrt{2}\) is not a fraction. That is, as you may have realized a while back, our marching ponderings prove that \(\sqrt{2}\) is irrational.

There are many proofs of the irrationality of \(\sqrt{2}\), however the one underlying our marching square story above is probably our favorite. We learned it from the great John H. Conway, who attributes it to Stanley Tennenbaum [1, 2].

**Time to cheat**

Now all this is good and well, but we suspect that our mathematical ponderings will not impress our square-loving audience. So, instead of completely giving up on the idea of merging squares into larger squares, we consider various ways of bending the rules, suggested by near-miss identities like the following.

- Merging two identical squares of marchers into a larger square with one marcher missing such as \(12^2 + 12^2 = 17^2 - 1\).
- Merging two consecutive squares into a larger square like \(20^2 + 21^2 = 29^2\).

First, what happens when we merge two 12-squares into a 17-square as suggested by the first identity?

Two light 5-squares and one dark 7-square materialize and \(5^2 + 5^2 = 7^2 + 1\), of course, another instance of two squares almost adding up to another square. One difference: The identity we started with has two squares adding up to a square with one marcher missing, whereas this new identity features an excess marcher. In any case, what happens when we start with our new near-miss and merge two 5-squares into a 7-square?

From \(12^2 + 12^2 = 17^2 - 1\), our merging maneuver first yields \(5^2 + 5^2 = 7^2 + 1\) and now \(2^2 + 2^2 = 3^2 - 1\), the two smallest examples of two squares differing by 1 from another square. In fact, it is easy to see that we can generate all instances of two squares differing by 1 from another square by successively “unmerging.” For example, to generate the next larger example beyond \(12^2 + 12^2 = 17^2 - 1\) just arrange two 12-squares and one 17-square as follows.
Now the two new highlighted squares are 29- and 41-squares. We only note that the pairs of integers corresponding to these near-misses combine into the best rational approximations of \(\sqrt{2}\), namely, \(3/2, 7/5, 17/12, 41/29, \ldots \) [3].

Very neat, but will our audience be impressed? After all, we promised to merge two perfect squares into another perfect square and a missing or excess marcher may end up sticking out too much.

But not to worry, there are many other cheats that almost do what we set out to do originally. As a second example, consider cheats based on two consecutive squares adding up to another square like \(20^2 + 21^2 = 29^2\). Identities like this are special Pythagorean triples called Pythagorean twins.

Merging a 20-square and a 21-square into a 29-square,

we arrive at a new 12-square in the middle. However, the light lower left and upper right configurations are only “near-squares” of dimensions 8 \(\times\) 9. This suggests another way of cheating: two identical near-squares adding up to a square. But staying on track, what happens when we merge the two near-squares inside the square?
Aha, $3^2 + 4^2 = 5^2$, an old friend and the smallest Pythagorean twin. As in our first example, we can run things backward and in this way generate all Pythagorean twins and all instances of two near-squares adding up to a square.

Other examples of near-miss identities to which our merging trick applies include two identical squares adding up to a near-square like $6^2 + 6^2 = 8 \cdot 9$ and two identical near-squares adding up to a near square like $14 \cdot 15 + 14 \cdot 15 = 20 \cdot 21$.

The latter are of particular interest to us because they translate into ways to merge two equilateral triangles into a larger equilateral triangle since the triangle numbers $(14 \cdot 15)/2$ and $(20 \cdot 21)/2$ are the numbers of marchers in these triangles. Maybe our audience also likes equilateral triangles? We sure hope it does because we have already choreographed a pretty way of merging the two triangles in practice which also has a few more mathematical insights to offer; watch a preview at http://youtu.be/rjHFkx6eTL8.

Starting with our geometric merging trick it is straight-forward to derive algebraic formulas for the numbers that correspond to the different types of near-misses we have considered. However, the truth is that these formulas have been known for a long time and can be found in many places, such as the sequences A001652, A001653, A001844, A005408, A046090, A046092, A095861, A001108, A115598, and A001542 of [5].

**A little perspective on marching squares**

This article contains much that is very well known. Let us explain our motivations.

The first highlight of our story is the clever covering argument in Tennenbaum’s proof that $\sqrt{2}$ is an irrational number. We believe that this very beautiful proof deserves to be more widely known. However, there is a problem.

Tennenbaum’s is a proof by contradiction and people with little mathematical background, especially students being led into the wonders of mathematics, almost invariably struggle with the basic line of reasoning in such proofs. We have attempted to
present Tennenbaum’s lovely proof so that the contradiction aspect is not even noticeable until the final summing up, when we announce that a proof by contradiction is what we have just done. We have been pleased with the success with this story approach in public lectures, school, and university presentations, and even in a column in an Australian newspaper. Of course, selling tricky mathematics with stories is hardly new. There is even an entire, very good book about $\sqrt{2}$ that treads similar ground [3].

In the second part of the article we do something different and apparently new. We take the “destructive” argument within a proof by contradiction and turn it around to construct examples of interesting and very pretty near-misses. We have also identified a few other proofs by contradiction that can be flipped in this surprising manner; examples include some covering arguments which are similar in spirit to Tennenbaum’s proof [2, 4]. Many other “infinite descent” geometric proofs may also bear fruit.

Finally, if somewhat optimistically, we hope that we have provided some food for thought for anyone in the business of choreographing marching bands and parades.

Summary. We begin with a new take (in the context of precision marching choreography) on a beautiful proof by contradiction. Then we turn around the destructive part of this proof to construct some very pretty near-misses.

References