ENUMERATION OF LABELLED GRAPHS II: CUBIC GRAPHS WITH A GIVEN CONNECTIVITY

NICHOLAS C. WORMALD

Abstract

Labelled cubic graphs and labelled connected cubic graphs were first counted by Read. In this paper, a new derivation of Read's recursive formulae for the numbers of these graphs is obtained by taking into account graphs which are cubic except for one or two points of degree 2. The same methods are then used to count labelled 2-connected cubic graphs. Labelled 3-connected cubic graphs are counted by using the partial differential equation satisfied by the exponential generating function for labelled 3-connected graphs, which was derived in the first paper in this series. As no cubic graph is 4-connected, these results determine the number of labelled k-connected cubic graphs with p points for all p and k.

1. *Introduction*

Except where otherwise specified, all graphs are labelled. The first paper in this series [**8**] was devoted to developing a method of enumerating 3-connected graphs. The results obtained therein are applied directly to the enumeration of 3-connected cubic graphs in the present paper, a cubic graph being one whose points have degree 3. This leads naturally to consideration of the more general problem of enumerating cubic graphs with any given connectedness.

Cubic graphs and connected cubic graphs were counted by Read in two essentially different ways, in [**5**] as an application of his superposition theorem and in [**6**] by deriving a recurrence relation satisfied by the number of cubic graphs with $2n$ points. As each cubic graph has an even number of points, the recurrence solution is complete. It was obtained by considering various interrelationships of the cubic pseudographs, and is by far the preferred method of the two when exact results are desired.

In Section 2, this recurrence relation for the number of cubic graphs is derived using an approach simpler than that of Read, in that loops and multiple lines are not introduced. The same methods are then applied to counting connected cubic graphs in Section 3 and 2-connected cubic graphs in Section 4, although in the latter case a result of [**11**] is also used. The 3-connected cubic graphs are counted in Section 5 by making use of the fact that they are precisely those 3-connected graphs whose ratio of points to lines is 2/3. A recurrence relation for these graphs is obtained from the partial differential equation satisfied by the exponential generating function for 3-connected graphs which was derived in [**8**]. This technique was pointed out to the author by E. M. Wright, and is more elegant than an extension of the methods of the earlier sections. As no cubic graph is 4-connected, the number of cubic graphs with $2n$ points and any given connectedness can now be found. These numbers are available from the author for n up to 25.

Finally, some problems related to those considered in the present paper are mentioned in Section 6. Graph theoretic notation not defined here can be found in the book by Harary [**2**].

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2. *Enumeration of cubic graphs*

We shall consider graphs which are cubic except for one or two points of degree 2. After finding relationships between the numbers of these sorts of graphs, we derive a recurrence relation for the number q_n of cubic graphs on $2n$ points.

A *closely cubic* graph, is a graph in which just one point is of degree 2 and the rest are of degree 3. A *fairly cubic* graph, is one which has just two points of degree 2 and the rest of degree 3. An *almost cubic* graph, is a fairly cubic graph in which the two points of degree 2 are non-adjacent. The *removal of a point* v from a graph G consists of the deletion of v from the point set of G and all lines incident with v from the line set of G. The *suppression* of a point v of degree 2 consists of removing v and joining the two points formerly adjacent to v.

Let $h = h(x)$ be the exponential generating function (e.g.f.) for cubic graphs; that is, the coefficient of x^p is $1/p!$ times the number h_p of cubic graphs with p points. Similarly define $c = c(x)$ to count closely cubic graphs, $f = f(x)$ to count fairly cubic graphs, and $a = a(x)$ to count almost cubic graphs, and let c_p, f_p and a_p denote the numbers of these graphs with p points. We follow the convention that the null-graph is a disconnected cubic graph, so that the constant term in h is 1.

Suitable operations must be found by which the above classes of graphs are converted from one type to one or more others. The simplest operation consists of removing a line from a cubic graph with p points, which produces an almost cubic graph. There are $\frac{3}{2}p$ possibilities for the line to be removed, and each almost cubic graph can be formed in a unique way from this operation. Hence $2ap = 3ph_p$, and so

$$2a = 3xh', \qquad (2.1)$$

where h' denotes the derivative of h with respect to x.

The next operation is the removal of the point of degree 2 from a closely cubic graph with p points, which forms an improperly labelled fairly cubic graph with $p-1$ points. This graph in turn can be converted into a properly labelled fairly cubic graph with $p-1$ points by compressing its point labels from the range $1, \ldots, p$, with one missing, to the range $1, \ldots, p-1$, whilst preserving their order. The compression process is p-to-one, and it follows that $c_p = pf_{p-1}$. Hence

$$c = xf. \qquad (2.2)$$

A slightly more complex operation can be applied to any unlabelled fairly cubic graph G. This consists of three sub-operations. Let u and v be the points of degree 2 in G. Firstly, if u and v are non-adjacent, then G is just an unlabelled almost cubic graph and is left unaltered. Secondly, if u and v are two points in a 3-cycle, then the removal from G of all points in this 3-cycle produces an unlabelled closely cubic graph. Thirdly, in all other cases the removal of u and v produces an unlabelled fairly cubic graph. By analysing this operation in the same fashion as those before, one obtains

$$f = a+\tfrac{1}{2}x^3c+x^2f. \qquad (2.3)$$

A *network* is a graph in which two points are distinguished as positive and negative poles, denoted $(+)$ and $(-)$. A fairly cubic network is *special* if its points of degree 2 are the poles. This can be arranged in two ways, so the e.g.f. for special almost cubic networks is $2a$. The positive pole of an unlabelled special almost cubic network can be removed in one of five ways as follows. Let u and v denote the two points adjacent to (+). Firstly, if u and v are non-adjacent, then the suppression of (+) produces an unlabelled closely cubic graph. Secondly, if u and v are adjacent and (+) is the only point adjacent to both u and v, then removal of the line $\{u, v\}$ and subsequent suppression of u and v produces an unlabelled special fairly cubic network. In each of the remaining three cases, u and v are adjacent and there is some point w other than (+) adjacent to both u and v. If w is (−), then removal of (+), u, v and (−) produces an unlabelled cubic graph, which may be the null-graph. If w is adjacent to (−), then removal of (+), u, v, w and (−) produces an unlabelled closely cubic graph. In the final case, w is adjacent to a point t other than u and v, where t is not (−). If we remove (+), u, v and w and regard t as a positive pole, the result is an unlabelled special fairly cubic network. Enumerative analysis of these operations yields

$$2a = \tfrac{3}{2}x^2c' - \tfrac{5}{2}xc + 2x^2f + \tfrac{1}{2}x^4h + \tfrac{1}{2}x^5c + x^4f. \tag{2.4}$$

A linear second-order differential equation in h is found by solving equations (2.1) through (2.4). This can be written as

$$6x^2(2-2x-x^2)q''(x) - (x^5+6x^4+6x^3-32x+8)q'(x) + \tfrac{1}{6}x(2-2x-x^2)^2q(x) = 0, \tag{2.5}$$

where $q(x^2) = h(x)$. This is essentially the equation obtained by Read in **[6]**, where it was derived using different arguments, although the general idea of both methods is the same: an examination of the result of removing a line from a cubic graph. The method used in the present section, however, is more easily adapted to the problems which are considered in the following sections.

Equating coefficients of x^n in (2.5) establishes the recurrence relation

$$\begin{aligned} q_{n+1} &= (2n+1)(3n^2+5n)q_n - \binom{2n+1}{3}(18n^2-54n+35)q_{n-1} \\ &\quad -20\binom{2n+1}{5}(9n^2-36n+38)q_{n-2} - 6\times 7!\binom{2n+1}{8}q_{n-3} \\ &\quad -6\times 7!\binom{2n+1}{9}(3n-14)q_{n-4} + 330\times 7!\binom{2n+1}{11}q_{n-5}. \end{aligned} \tag{2.6}$$

This can be used to compute the q_n given $q_0 = 1$ and $q_n = 0$ for $n < 0$.

3. *Connected cubic graphs*

Let $\hat{h}(x)$ be the exponential generating function for connected cubic graphs. Since a graph is cubic if and only if its connected components are cubic, one can apply the well-known relation between the e.g.f. for a class of labelled connected graphs and that for the labelled graphs whose components are members of this class **[1]**. In this case, the relation is

$$h(x) = e^{\hat{h}(x)}. \tag{3.1}$$

We could therefore obtain a differential equation for $\hat{h}$ by substituting $h(x) = q(x^2)$ and (3.1) into (2.5). What is done instead in this section is to derive an equivalent set of differential equations for $\hat{h}$ directly by modifying the arguments in Section 2.

Let $\hat{a} = \hat{a}(x)$ be the e.g.f. for connected almost cubic graphs, and similarly define $\hat{f}$ and $\hat{c}$ to count fairly cubic and closely cubic connected graphs, respectively.

The removal of a line from a connected cubic graph produces either a connected almost cubic graph, or a graph consisting of an unordered pair of connected closely cubic graphs. By the Labelled Counting Lemma [**3**; p. 8] the e.g.f. for the latter graph is $\frac{1}{2}\hat{c}^2$. Corresponding to (2.1), we therefore have

$$2\hat{a}+\hat{c}^2 = 3x\hat{h}'. \tag{3.2}$$

Equations (2.2), (2.3) and (2.4) are readily modified for connected graphs in a similar way, becoming

$$\hat{c} = x\hat{f}+\tfrac{1}{2}x\hat{c}^2, \tag{3.3}$$

$$\hat{f} = \hat{a}+x^2\hat{f}+\tfrac{1}{2}x^2\hat{c}^2+\tfrac{1}{2}x^3\hat{c}, \tag{3.4}$$

and

$$2\hat{a} = \tfrac{3}{2}x^2\hat{c}'-\tfrac{5}{2}x\hat{c}+2x^2\hat{f}+\tfrac{1}{2}x^4+\tfrac{1}{2}x^5\hat{c}+x^4\hat{f}. \tag{3.5}$$

Putting $\hat{q}(x^2) = \hat{h}(x)$ and $\hat{p}(x^2) = \hat{c}(x)/x$, we find from (3.2), (3.3) and (3.4) that

$$\hat{p}(x) = 3x(1-x-\tfrac{1}{2}x^2)^{-1}\hat{q}'(x), \tag{3.6}$$

and thus $\hat{q}$ can be found after $\hat{p}$ is obtained. Equations (3.3) through (3.6) combine to yield

$$6x^2\hat{p}'(x)+(x^3+4x^2+6x-4)\hat{p}(x)+x(2-2x-x^2)\hat{p}^2(x)+x^2 = 0. \tag{3.7}$$

This is the equation obtained (with a misprint) by Read in [**6**] after he made the convenient substitution (3.6), apparently without noticing the combinatorial interpretation of $\hat{p}(x)$. The standard method of converting (3.7) into a linear second-order differential equation entails the substitution (3.6), followed by the substitution (3.1), and it is not surprising that the result is equivalent to (2.5). This was the path followed by Read in deriving equation (2.5).

For $n \geqslant 0$ let $\hat{s}_n$ be 2^n times the coefficient of x^n in $\hat{p}(x)$. Then equating coefficients of x^n in (3.7) leads to the recurrence

$$\hat{s}_n = 3n\hat{s}_{n-1}+4\hat{s}_{n-2}+2\hat{s}_{n-3}+\sum_{i=2}^{n-3}\hat{s}_i(\hat{s}_{n-1-i}-2\hat{s}_{n-2-i}-2\hat{s}_{n-3-i}) \tag{3.8}$$

for $n \geqslant 3$, with initial conditions $\hat{s}_0 = \hat{s}_1 = 0$ and $\hat{s}_2 = 1$. The number of connected closely cubic graphs with $2n+1$ points is then $(2n+1)!2^{-n}\hat{s}_n$. Next, for $n \geqslant 0$ define $\hat{r}_n$ to be $3n2^n$ times the coefficient in $\hat{q}(x)$ of x^n. From (3.6),

$$\hat{r}_n = \hat{s}_n-2\hat{s}_{n-1}-2\hat{s}_{n-2}$$

for $n \geqslant 2$. The number $\hat{q}_n$ of connected cubic graphs with $2n$ points is now given by

$$\frac{(2n)!}{3n2^n}\hat{r}_n.$$

A somewhat unexpected consequence of this method of calculating $\hat{q}_n$ is the conclusion that for $n \geqslant 2$, $\hat{q}_n$ is divisible by the integer $(2n)!/(3n2^n)$.

4. 2-*connected cubic graphs*

A *block* is a 2-connected graph or the graph K_2 which consists of a single line. To count 2-connected cubic graphs we combine the method of Sections 2 and 3 with the technique used in [11] to count blocks in general.

Define the e.g.f.'s $\tilde{h}$, $\tilde{a}$, $\tilde{f}$ and $\tilde{c}$ to count just the 2-connected graphs of the types corresponding to h, a, f and c respectively. Also, let $\tilde{g}$ be the e.g.f. for the class $\mathscr{C}$ of special almost cubic networks which are not 2-connected but become 2-connected when a line is added between the poles. The method of Sections 2 and 3 is sufficient to derive the equations

$$2\tilde{a}+\tilde{g} = 3x\tilde{h}', \tag{4.1}$$

$$\tilde{c} = x\tilde{f}+\tfrac{1}{2}x\tilde{g}, \tag{4.2}$$

$$\tilde{f} = \tilde{a}+x^2\tilde{f}+\tfrac{1}{2}x^2\tilde{g} \tag{4.3}$$

and

$$2\tilde{a} = \tfrac{3}{2}x^2\tilde{c}'-\tfrac{5}{2}x\tilde{c}+2x^2\tilde{f}+\tfrac{1}{2}x^4. \tag{4.4}$$

To find a fifth equation we apply a result of [11] to the special almost cubic networks in $\mathscr{C}$. This is, that each network M in $\mathscr{C}$ must consist of a chain of at least two blocks, $G_1, \ldots, G_j$ say, each having a single cutpoint of M in common with each of its neighbours. The block G_1 contains the positive pole and a cutpoint, and G_j contains the negative pole and a cutpoint, whilst every other block contains two cutpoints.

In the present situation, the poles of M have degree 2 but every other point has degree 3, in particular the point shared by G_i and G_{i+1} for $1 \leqslant i \leqslant j-1$. Hence G_i or G_{i+1} has a point of degree 1 and is consequently isomorphic to K_2, as no 2-connected graph has a point with degree less than 2. It follows that j is odd and that G_i is isomorphic to K_2 when i is even, whilst when i is odd G_i is a 2-connected fairly cubic graph whose points of degree 2 are poles or cutpoints of M. Conversely, each chain of G_i satisfying these criteria will correspond to a graph in $\mathscr{C}$.

Each graph G_i can be considered a network by regarding the point in common with G_{i-1} as a positive pole and the point in common with G_{i+1} as a negative pole. The e.g.f. for G_i is thus $2\tilde{f}$ if i is odd and x^2 if i is even. As demonstrated in [11], choosing $G_1, \ldots, G_k$ simultaneously has the effect of multiplying the corresponding e.g.f.'s together, whilst identifying the negative pole of G_i with the positive pole of G_{i+1} has the effect of dividing the corresponding e.g.f. by x each time the identification process is applied. Hence, the e.g.f. for those networks in $\mathscr{C}$ which possess $j-1 = 2m$ cutpoints is $(2\tilde{f})^{m+1}(x^2)^m x^{-2m}$. It follows that

$$\tilde{g} = \sum_{m=1}^{\infty} (2\tilde{f})^{m+1} = 4\tilde{f}^2/(1-2\tilde{f}).$$

Combining this with equations (4.1) through (4.4) produces

$$12x^2\tilde{y}(x)\tilde{y}'(x)+6x^2\tilde{y}'(x)+4x\tilde{y}^2(x)+(2x^2+6x-4)\tilde{y}(x)+x^2 = 0 \tag{4.5}$$

and

$$3x\tilde{q}'(x) = (1-x)\tilde{y}(x), \tag{4.6}$$

where $\tilde{y}(x^2) = \tilde{c}(x)/x$ and $\tilde{q}(x^2) = \tilde{h}(x)$.

Let $\tilde{s}_n$ be 2^n times the coefficient of x^n in $\tilde{y}(x)$, and let $\tilde{r}_n$ be $3n2^n$ times the coefficient of x^n in $\tilde{q}(x)$. Equating coefficients of x^n in (4.5) and (4.6) yields the relations

$$\tilde{s}_n = 3n\tilde{s}_{n-1} + 2\tilde{s}_{n-2} + (3n-1)\sum_{i=2}^{n-3} \tilde{s}_i \tilde{s}_{n-1-i} \tag{4.7}$$

and

$$\tilde{r}_n = \tilde{s}_n - 2\tilde{s}_{n-1}, \tag{4.8}$$

where (4.7) applies for $n \geqslant 3$ with initial conditons $\tilde{s}_1 = 0$ and $\tilde{s}_2 = 1$, and (4.8) applies for $n \geqslant 2$. The number of 2-connected cubic graphs with $2n$ points is given by

$$\frac{(2n)!}{3n2^n}\tilde{r}_n.$$

5. 3-*connected cubic graphs*

Let $d = d(x, y)$ be the e.g.f. counting 3-connected graphs by points and lines. That is, the coefficient of $x^p y^m$ in d is $p!$ times the number of 3-connected graphs with p points and m lines. It was shown in [8] that d satisfies the equation

$$(1+y)d_y - \tfrac{1}{2}x^2 d_{xx} = \frac{x^4 T_x{}^2}{4T_y} - \frac{x^4 y^4}{4(1+xy)^2} \tag{5.1}$$

where suffixes denote partial differentiation and

$$T = \log\,(1+y) - \frac{xy^2}{1+xy} - \frac{2d_y}{x^2}.$$

The 3-connected cubic graphs with $2n$ points are just those 3-connected graphs with $2n$ points and $3n$ lines. So if we substitute $x = zt^{-3}$ and $y = t^2$ into (5.1), simplify, then multiply by t^2 and put $t = 0$, we have an equation in $d(z, 0) = \bar{h}(z)$, where $\bar{h}$ is the e.g.f. for 3-connected cubic graphs. Putting $b(z^2) = \tfrac{1}{2}z\bar{h}'(z)$, then $x = z^2$, this equation becomes

$$9x(b^2(x))' - 6b^2(x) + (\tfrac{1}{2}x - 1)b(x) + \tfrac{3}{2}x^2 b'(x) + \tfrac{1}{12}x^2 = 0. \tag{5.2}$$

As with its connected and 2-connected counterparts, the number of 3-connected cubic graphs with $2n$ points is evidently divisible by $(2n)!/(3n2^n)$. For, if $\bar{r}_n$ is 3×2^n times the coefficient of x^n in $b(x)$, equating coefficients in (5.2) leads to the recurrence relation

$$\bar{r}_n = (3n-2)\left(\bar{r}_{n-1} + \sum_{i=2}^{n-2} \bar{r}_i \bar{r}_{n-i}\right)$$

for $n > 2$, with the initial condition $\bar{r}_2 = 1$. The number $\bar{q}_n$ of 3-connected cubic graphs with $2n$ points is now given by

$$\frac{(2n)!}{3n2^n}\bar{r}_n.$$

6. *Related results*

The problem of counting unlabelled cubic graphs is considerably more complicated than the labelled problem. R. W. Robinson has announced [7] a method of counting unlabelled cubic graphs by first counting all unlabelled homeomorphically irreducible graphs with p points and m lines which have no isolates or endpoints. Then the cubic graphs are just those with $m = 3p/2$. We hope to solve the same problem in a different way, by using techniques related to those of the present paper.

One can also consider the problem of counting regular graphs of degree $r \geqslant 4$. A generalisation of this problem was solved by Read [5; Theorem IV]. When r is 4, a solution which is more efficient computationally can be found by extending the ideas used in Section 2. This will be presented elsewhere. The corresponding problem for unlabelled graphs is unsolved. Similar problems to be considered elsewhere include that of finding the average number of triangles in a cubic graph on $2n$ points [9], and the enumeration of cubic graphs which contain no triangles.

Read [4] showed that the number q_n of cubic graphs on $2n$ points is asymptotic to

$$\frac{(6n)!}{(3n)!288^n} e^{-2} \tag{6.1}$$

as n approaches infinity. It will be shown in a separate paper that almost all cubic graphs are 3-connected, which implies that (6.1) is also asymptotic to the number of k-connected cubic graphs on $2n$ points for $k = 1, 2$ or 3.

Finally, in [10] the fact that $\hat{q}_n$, $\tilde{q}_n$ and $\bar{q}_n$ are all divisible by $(2n)!/(3n2^n)$ is explained in terms of the automorphism group of a connected cubic graph.

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Department of Mathematics,
University of Newcastle,
New South Wales, 2308,
Australia.