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USING CROOKED LINES FOR THE HIGHER ACCURACY IN SYSTEM OF INTEGRAL EQUATIONS

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ABSTRACT. The numerical solution to the linear and nonlinear and linear system of Fredholm and Volterra integral equations of the second kind are investigated. We have used crooked lines which includ the nodes specified by modified rationalized Haar functions. This method differs from using nominal Haar or Walsh wavelets. The accuracy of the solution is improved and the simplicity of the method of using nominal Haar functions is preserved. In this paper, the crooked lines with unknown coefficients under the specified conditions change the system of integral equations to a system of equations. By solving this system the unknowns are obtained and the crooked lines are determined. Finally, error analysis of the procedure are considered and this procedure is applied to the numerical examples, which illustrate the accuracy and simplicity of this method in comparison with the methods proposed by these authors.

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1. Introduction

Integral equations of various type play an important role in many fields of science and engineering. The most frequently investigated integral equations are Volterra and Fredholm equations. In the recent years, many kinds of basic functions have been used for numerical solutions of integral equations, such as Fourier functions [1], Chebyshev polynomials [3] and wavelets. Begining from 1991 the wavelet method has been applied for solving integral equations. Some of these approaches have weak accuracy such as Harr wavelets, some others like Daubechies wavelets need a high capacity of computations.

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In [4] and [5], the authors applied the rationalized Haar wavelets to solve differential equations. Also, Fredholm and Volterra integral equations of the second kind using rationalized Haar wavelets has been considered in [6], [7], [9] and [10]. Although, using the rationalized Haar scaling function or the corresponding wavelet for solving integral equations has simplified the method and it's applications, but the approximation does not have a good degree of accuracy. Therefore, we decided to state a new method by the same simplification but further accuracy.

In this paper, we use crooked lines containing a set of lines $l_k(x) = a_k x + b_k$ under some specific regulations for solving linear and nonlinear Fredholm and Volterra integral equations of the second kind, where a_k and b_k are unknown coefficients. We also have considered a linear system of Fredholm and Volterra integral equations. In Section 2 the coefficients a_k and b_k are computed with respect to diadic points $x_l = \frac{l}{2^J}, l = 0, 1, \cdots, 2^J$. In Section 3 we consider the computation of diadic points for a known function f. To evaluate the diadic points, first, approximate function f by rationalized Haar functions. Since the main is computing diadic points, then this approximation should be done by a specific scheme and not according [6], [7], [9] and [10]. As the main purpose of this paper is to compute the approximate solution for unknown function f which satisfies in the Fredholm or Volterra integral equations of second kind, in Sections 4 and 5, we consider the linear and non-linear forms of these equations. Finally at the last Sections we introduce error analysis and numerical examples for estimating the accuracy of the procedure.

The main advantage of the present method is the high accuracy of the solution in comparison with the previous methods using rationalized Haar functions.

2. Function approximation

Any function $f \in L^2[0,1]$ can be approximated at resolution J by crooked lines $l_k(x) = a_k x + b_k,\ k = 0, 1, \ldots, 2^J - 1$. First, we divide the interval $[0,1]$ into 2^J subintervals of equal length, and then, the function f can be approximated at the subinterval $[\frac{k}{2^J}, \frac{k+1}{2^J})$ by crooked line $l_k(x)$. The end point of crooked line $l_k(x)$ in the interval $[\frac{k}{2^J}, \frac{k+1}{2^J})$ accords to the beginning point of the function $l_{k+1}(x)$ in the interval $[\frac{k+1}{2^J}, \frac{k+2}{2^J})$, therefore the function approximation is well defined. So we have

$$f(x) = \sum_{k=0}^{2^J-1} (a_k(x) + b_k)\varphi_{J,k}(x)\ ,\ x \in [0,1]\ , \tag{1}$$

where $\varphi_{J,k}(x) = \varphi(2^J x - k)$ and the function $\varphi(x)$ is defined as

$$\varphi(x) = \begin{cases} 1\ , & x \in [0,1)\ , \\ \\ 0\ , & otherwise\ . \end{cases}$$

The function $l_k(x)$ is used for approximation of function f in the interval $[\frac{k}{2^J}, \frac{k+1}{2^J})$, so the ordered pair $(\frac{k}{2^J}, f(\frac{k}{2^J}))$ is the beginning point of the function and the ordered pair $(\frac{k+1}{2^J}, f(\frac{k+1}{2^J}))$ is the end point of this function in the same subinterval. Therefore, these points of $l_k(x)$ determines the coefficients a_k and b_k. By solving the corresponding equations we get

$$a_k = 2^J[f(\frac{k+1}{2^J}) - f(\frac{k}{2^J})] \ , \tag{2}$$

$$b_k = (k+1)f(\frac{k}{2^J}) - kf(\frac{k+1}{2^J}) \ . \tag{3}$$

It is obtained from Eqs.(2) and (3) that by computing $f(\frac{l}{2^J}), l = 0, 1, \cdots, 2^J$, the coefficients a_k and b_k can be obtained. By replacing these coefficients in Eq.(1), the function approximation will be achieved. Thus, the convergency of the method is related to the diadic points.

It should be noted that the main target of paper is finding an approximation for the unknown function f which satisfies in the given integral equation. Therefore, $f(\frac{l}{2^J}), l = 0, 1, \cdots, 2^J$ should be computed for this unknown function, but $f(\frac{l}{2^J}), l = 0, 1, \cdots, 2^J - 1$ can be directly computed by rationalized Haar functions, then $f(1) = f(\frac{2^J}{2^J})$ should only be considered.

3. Properties of rationalized Haar functions

3.1.Rationalized Haar functions. Let φ be the Haar scaling function defined as

$$\varphi(x) = \begin{cases} 1 \ , & x \in [0, 1) \ , \\ 0 \ , & otherwise \ , \end{cases} \tag{4}$$

then, we define rationalized Haar functions as

$$\varphi_{J,k}(x) = \varphi(2^J x - k) \ , \ J = 0, 1, 2, \cdots \ , \tag{5}$$
$$k = 0, 1, 2, \cdots, 2^J - 1 \ .$$

Any function $f \in L^2[0, 1]$ can be approximated by rationalized Haar function series at resolution J as

$$f(x) = \sum_{k=0}^{2^J-1} c_k \varphi_{J,k}(x) \ , \ x \in [0, 1] \ , \tag{6}$$

or

$$f(x) = \Phi^T(x) C \ , \tag{7}$$

where $\Phi(x) = [\varphi_{J,0}(x), \varphi_{J,1}(x), \cdots, \varphi_{J,2^J-1}(x)]^T$, $C = [c_0, c_1, \cdots, c_{2^J-1}]^T$.

Here, we deal with the computation of coefficients c_k for $f \in L^2[0,1]$. Suppose the values of f are known at diadic points $x_l = \frac{1}{2^J}$, $l = 0, 1, 2, \cdots, 2^J - 1$. Substituting $x = x_l$ in Eq.(6), gives

$$f(\frac{l}{2^J}) = \sum_{k=0}^{2^J-1} c_k \varphi(l-k) \ , \tag{8}$$

and by substituting $l = 0, 1, 2, \cdots, 2^J - 1$ in Eq.(8) finally we get

$$F = IC \ , \tag{9}$$

where $F = [f(0/2^J), f(1/2^J), \cdots, f((2^J-1)/2^J)]^T$ and I is the $2^J \times 2^J$ identity matrix. Note that, the target of this section is computing matrix F and eventually obtaining the coefficients a_k and b_k from Eqs.(2) and (3), not approximating the function f by rationalized Haar functions.

Similar to Eq.(6) a function $k \in L^2([0,1] \times [0,1])$ may be approximated at resolution J as

$$k(x,t) = \Phi^T(x) S \Phi(t) \ , \tag{10}$$

where S is a $2^J \times 2^J$ coefficient matrix. Imagining $k(x,t)$ as a one-dimensional function of variable x and t respectively and invoking Eq.(8), we finally have

$$S = IKI = K \ , \tag{11}$$

where K is the $2^J \times 2^J$ kernel matrix with $K_{m,n} = k(\frac{m-1}{2^J}, \frac{n-1}{2^J})$.

3.2.Operation matrix of integration. The orthogonality property for rationalized Haar functions is given by

$$\int_0^1 \varphi_{J,k}(t)\varphi_{I,l}(t)dt = \begin{cases} 2^{-J} \ , & J = I \ , \ k = l \ , \\ 0 \ , & otherwise \ . \end{cases} \tag{12}$$

So, the integration of the product of two function vectors $\Phi(t)$ defined in Eq.(7) is obtained as

$$\int_0^1 \Phi(x)\Phi^T(x)dx = 2^{-J} I \ , \tag{13}$$

where I is the $2^J \times 2^J$ identity matrix. Also, the integration of $\Phi(t)$ can be expanded into Haar series with matrix $P(x)$ as

$$\int_0^x \Phi(t)dt = P(x)\Phi(x) \ , \tag{14}$$

where $P(x)$ is the $2^J \times 2^J$ upper-triangular operational matrix for integration and is given as

$$\mathbf{P(x)} = \begin{pmatrix} x & \frac{1}{2^J} & \frac{1}{2^J} & \dots & \frac{1}{2^J} \\ 0 & x-\frac{1}{2^J} & \frac{1}{2^J} & \dots & \frac{1}{2^J} \\ 0 & 0 & x-\frac{2}{2^J} & \dots & \frac{1}{2^J} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & & x-\frac{2^J-1}{2^J} \end{pmatrix} . \tag{15}$$

In the other words, $P(x)$ is the operational matrix for integration with

$$P_{m,n}(x) = \begin{cases} \frac{1}{2^J} \ , & m < n \ , \\ x - \frac{m-1}{2^J} \ , & m = n \ , \\ 0 \ , & m > n \ . \end{cases}$$

Let $M(x) = P(x)\Phi(x)$. According to the orthogonality of rationalized Haar functions, the integration of the product of two function vectors $\Phi(t)$ gives

$$\int_0^x \Phi(t)\Phi^T(t)dt = \widetilde{M}(x) \ , \tag{16}$$

where

$$\widetilde{\mathbf{M}}(\mathbf{x}) = diag\left(M_{1,1}(x), \quad M_{2,1}(x), \quad \dots, \quad M_{2^J,1}(x) \right) . \tag{17}$$

4. Fredholm integral equations of the second kind

4.1.Linear integral equations. Consider the Fredholm linear integral equation of the second kind

$$f(x) - \int_0^1 k(x,t)f(t)dt = g(x) \ , \tag{18}$$

where $g(x) \in L^2[0,1]$ and $k(x,t) \in L^2([0,1]\times[0,1])$ are known. The problem is to find an unknown function $f(x)$, satisfying Eq.(18). We approximate $f(x)$, $g(x)$ and $k(x,t)$ as

$$f(x) = \Phi^T(x)C \ , \tag{19}$$

$$g(x) = \Phi^T(x)D \ , \tag{20}$$

$$k(x,t) = \Phi^T(x)S\Phi(t) \ , \tag{21}$$

where $\Phi^T(x)$ is given by Eq.(7) and C is an unknown $2^J \times 1$ vector. S is a known $2^J \times 2^J$ matrix given by Eq.(11) and D is a known $2^J \times 1$ vector given by Eq.(7). By substituting Eqs.(19)-(21) in Eq.(18), we get

$$\Phi^T(x)C - \int_0^1 \Phi^T(x)S\Phi(t)\Phi^T(t)Cdt = \Phi^T(x)D \ ,$$

$$\Phi^T(x)C - \Phi^T(x)S(\int_0^1 \Phi(t)\Phi^T(t)dt)C = \Phi^T(x)D \ . \tag{22}$$

Using the orthogonality of rationalized Haar functions stated in Eq.(13) we obtain

$$\Phi^T(x)(I - 2^{-J}S)C = \Phi^T(x)D \ , \tag{23}$$

and by invoking Eqs.(9) and (11) we finally get the following linear system

$$(I - 2^{-J}K)C = G \ , \tag{24}$$

where I is the $2^J \times 2^J$ identity matrix and $G = [g(0/2^J), g(1/2^J), \cdots, g((2^J - 1)/2^J)]^T$ is a known matrix. Eq.(24) is a system of linear equations and can be easily solved for the unknown vector C.

Note that, from Eq.(9), $F = C$ which means by getting matrix C from Eq.(24), matrix F will be resulted automatically.

4.2.Non-linear integral equations. Consider the nonlinear Fredholm integral equation of the second kind

$$f(x) - \int_0^1 k(x,t)h[f(t)]dt = g(x) \ , \tag{25}$$

where $h \in L^2[0,1]$ is known. We can approximate $f(x)$, $g(x)$ *and* $k(x,t)$ by the mentioned method in Section 3 again. Let we imagine $h(f)$ as a function of variable x, suppose $h(f)$ can be approximated as

$$h[f(x)] = h'(x) = \Phi^T(x)C' \ , \tag{26}$$

where $C' = [c'_0, c'_1, \cdots, c'_{2^J-1}]^T$. Similar to Eq.(9), we have

$$H(f) = IC' \ , \tag{27}$$

where I is the $2^J \times 2^J$ identity matrix and $H(f) = [h'(0/2^J), h'(1/2^J), \cdots, h'((2^J - 1)/2^J)]^T$. Substituting Eqs.(19)-(21) and (26) into Eq.(25), we get

$$\Phi^T(x)C - \int_0^1 \Phi^T(x)S\Phi(t)\Phi^T(t)C'dt = \Phi^T(x)D \ ,$$

and then

$$F - 2^{-J}KH(f) = G \ . \tag{28}$$

Eq.(28) is a system of nonlinear equations and can be solved by many methods (*e.g.* Newton iteration method) for the unknown vector F.

4.3.System of Fredholm integral equations. Let us consider the system of linear Fredholm integral equations

$$\mathbf{f}(x) - \int_0^1 \mathbf{k}(x,t)\mathbf{f}(t)dt = \mathbf{g}(x) \ , \tag{29}$$

where

$$\mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_N(x)]^T, \tag{30}$$

$$\mathbf{g}(x) = [g_1(x), g_2(x), \ldots, g_N(x)]^T, \tag{31}$$

$$\mathbf{k}(x,t) = [k_{m,n}(x,t)], \ m,n = 1,2,\ldots,N. \tag{32}$$

For convenience, we consider the m^{th} equation of (29) as

$$f_m(x) - \sum_{n=1}^{N} \int_0^1 k_{m,n}(x,t) f_n(t)dt = g_m(x), \ m = 1,2,\ldots,N. \tag{33}$$

Similar to Eqs.(19)-(21) we have $f_m(x) = \Phi^T(x)C_m$, $g_m(x) = \Phi^T(x)D_m$ and $k_{m,n}(x,t) = \Phi^T(x)S_{m,n}\Phi(t)$, where C_m is an unknown $2^J \times 1$ vector related to the unknown function f_m. D_m is also a known $2^J \times 1$ vector related to the known function g_m. Finally $S_{m,n}$ is a known $2^J \times 2^J$ matrix related to the known function $k_{m,n}(x,t)$. By substituting these equations in Eq.(33) we get

$$\Phi^T(x)C_m - \sum_{n=1}^{N} \int_0^1 \Phi^T(x)S_{m,n}\Phi(t)\Phi^T(t)C_n dt = \Phi^T(x)D_m \ ,$$

and finally

$$C_m - 2^{-J} \sum_{n=1}^{N} K_{m,n}C_n = G_m, \ m = 1,2,\ldots,N. \tag{34}$$

(34) is a linear system of $N \times 2^J$ equations and $N \times 2^J$ unknowns and can be easily solved for the unknown vectors $C_m, \ m = 1,2,\ldots,N$.

5. Volterra integral equations of the second kind

5.1.Linear integral equations. Consider the Volterra linear integral equation of the second kind

$$f(x) - \int_0^x k(x,t)f(t)dt = g(x) \ . \tag{35}$$

We approximate $f(x)$, $g(x)$ and $k(x,t)$ similar to Eqs.(19)-(21) again. It should be note that in Volterra integral equations we have $t \leq x$. In the other words, $k(x,t) = 0$, if $t > x$. Thus, K and hence S are lower-triangular matrices. By substituting Eqs.(19)-(21) in Eq.(35), we get

$$\Phi^T(x)C - \int_0^x \Phi^T(x)S\Phi(t)\Phi^T(t)Cdt = \Phi^T(x)D \ ,$$

$$\Phi^T(x)C - \Phi^T(x)S(\int_0^x \Phi(t)\Phi^T(t)dt)C = \Phi^T(x)D \ . \qquad (36)$$

By using Eqs.(16) and (36) we obtain

$$\Phi^T(x)(I - S\widetilde{M}(x))C = \Phi^T(x)D \ , \qquad (37)$$

and by substituting $x_l = l/2^J \ , \ l = 0, 1, 2, \cdots, 2^J - 1$ in Eq.(37) we finally get the following linear system

$$(I - 2^{-J}K')C = G \ , \qquad (38)$$

where I is the $2^J \times 2^J$ identity matrix, $G = [g(0/2^J), g(1/2^J), \cdots, g((2^J - 1)/2^J)]^T$ is a known matrix and K' is a known $2^J \times 2^J$ matrix with

$$K'_{m,n} = \begin{cases} K_{m,n} \ , & m \neq n \ , \\ \frac{1}{2}K_{m,n} \ , & m = n \ . \end{cases} \qquad (39)$$

Eq.(38) is a system of linear equations and can be easily solved for the unknown vector C and hence F.

5.2.Non-linear integral equations. Consider the nonlinear Volterra integral equation of the second kind

$$f(x) - \int_0^x k(x,t)h[f(t)]dt = g(x) \ . \qquad (40)$$

We can approximate $f(x)$, $g(x)$, $k(x,t)$ and $h[f(t)]$ by the mentioned method in Section 3 similar to Eqs.(19)-(21) and (26). By substituting these equations into Eq.(40) we get

$$\Phi^T(x)C - \int_0^x \Phi^T(x)S\Phi(t)\Phi^T(t)C'dt = \Phi^T(x)D \ ,$$

and then

$$F - 2^{-J}K'H(f) = G \ . \qquad (41)$$

Eq.(41) is a system of nonlinear equations and can be solved for the unknown vector F.

5.3.System of Volterra integral equations. Now, we consider the system of linear Volterra integral equations

$$\mathbf{f}(x) - \int_0^x \mathbf{k}(x,t)\mathbf{f}(t)dt = \mathbf{g}(x) \ , \tag{42}$$

where $\mathbf{f}, \mathbf{g}$ and $\mathbf{k}$ are defined in Eqs.(30)-(32). Similar to Section 4.3 we get the following linear system of $N \times 2^J$ equations and $N \times 2^J$ unknowns and can be solved for the unknown vectors $C_m, \ m = 1, 2, \ldots, N$.

$$C_m - 2^{-J} \sum_{n=1}^{N} K'_{m,n} C_n = G_m, \ m = 1, 2, \ldots, N. \tag{43}$$

Remark: Computation method of matrix F and hence $f(\frac{l}{2^J}), l = 0, 1, 2, \cdots, 2^J - 1$ was stated in Sections 4 and 5. To economize the computation we don't suggest to compute the $f(1) = f(\frac{2^J}{2^J})$. Because, in order to compute $f(1)$, the method should be repeated again. Instead, we approximate the function $f(x)$ in the small subinterval $[f(\frac{2^J-1}{2^J}), f(1)]$ by the ordinary rationalized Haar functions and we have applied this procedure in our numerical examples in Section 7. However computation of $f(1)$ by this method, needs $\varphi(x)$ be replaced in all of the Eqs.(6)-(43) by $\phi(x)$ defined as

$$\phi(x) = \begin{cases} 1 \ , & x \in (0,1] \ , \\ \\ 0 \ , & otherwise \ . \end{cases}$$

Then, the matrices F, G and $H(f)$ get the following values

$$F = [f(1/2^J), f(2/2^J), \cdots, f(2^J/2^J)]^T \ ,$$

$$G = [g(1/2^J), g(2/2^J), \cdots, g(2^J/2^J)]^T \ ,$$

$$H(f) = [h'(1/2^J), h'(2/2^J), \cdots, h'(2^J/2^J)]^T \ ,$$

and K is the $2^J \times 2^J$ kernel matrix with $K_{m,n} = k(\frac{m}{2^J}, \frac{n}{2^J})$.

6. Error analysis

As we mentioned in Section 2, the convergency of the method by crooked lines is related to the diadic points and they also are related to convergency of the method using Haar functions. Hence, there is a bilateral relationship between the convergency of both methods. It can be shown that for each J, the sequence $\{\varphi_{J,k}(x)\}_{k=0}^{2^J-1}$ is an orthogonal system in $L^2[0,1]$ and for each $f \in C^2[0,1]$ the series $\sum_{k=0}^{2^J-1} c_k \varphi_{J,k}(x)$ converges uniformly to f when $J \to \infty$ [15].

Suppose $\widetilde{f}$ is the approximate function of f obtained from Eq.(1), where $f \in C[0,1]$. It is relatively straightforward to show

$$\max_{x \in [0,1]} |f(x) - \widetilde{f}(x)| \leq \max_{|x-y| \leq 2^J} | \ f(x) - f(y) \ | \ \ . \tag{44}$$

In the special case, we assume that $f \in C^2[0,1]$. For considering the error analysis in this case, we need the following proposition.

Proposition. Assume $f \in C^{m+1}[0,1]$. Then there exists a ξ_x between $\min_i\{x, x_i\}$ and $\max_i\{x, x_i\}$ such that

$$f(x) - p_n(x) = \frac{w_n(x)}{(n+1)!}\ f^{(n+1)}(\xi_x)\ ,$$

where x_i's are diadic points, $p_n(x)$ is a polynomial of degree less than or equal to n by the conditions $p_n(x_i) = f(x_i)$ and $w_n(x) = \prod_{i=0}^{n}(x - x_i)$.

Proof. See [15].

By using the proposition in above and the structure of the crooked lines, it is straightforward to show that

$$\max_{x\in[0,1]} |f(x) - \widetilde{f}(x)| \le 2^{-(J+3)} \max_{x\in[0,1]} |\ f''(x)\ |\ . \tag{45}$$

7. Numerical examples

In this section, we present some examples and their numerical results to show the high accuracy of the solution obtained by the proposed method. Comparing between this method and the method obtained from Haar wavelets in Example 2, the high accuracy of the proposed method is observable. Also, Figure 1 demonstrate the convergency of the method.

Example 1. Consider the linear Fredholm integral equation

$$f(x) - \int_0^1 \cos(4\pi x + 2\pi t) f(t)dt = \sin(2\pi x)(1 + \cos(2\pi x))\ , \tag{46}$$

with exact solution $f(x) = \sin(2\pi x)$. We applied the method presented in this paper and solved Eq.(46). In Table 1, a comparison is made between the computational results for J=7 and 9 with exact values.

Table 1. Estimated and exact values of $f(t)$ for Example 1

t	J=7	J=9	Exact solution
0	0.000000000	0.000000000	0
0.1	0.587732499	0.587781841	0.5877852524
0.2	0.950877296	0.951045293	0.9510565165
0.3	0.950877295	0.951045294	0.9510565165
0.4	0.587732505	0.587781834	0.5877852524
0.5	0.000000000	0.000000000	0
0.6	-0.587732507	-0.587781895	-0.5877852524
0.7	-0.950877304	-0.951045297	-0.9510565165
0.8	-0.950877300	-0.951045332	-0.9510565165
0.9	-0.587732495	-0.587781861	-0.5877852524
1	-0.049067674	-0.012271538	0

In Table 1, the error of the method at the end point is relatively high respect to previous points, because in the last subinterval the function is approximated by Haar functions, as mentioned previously. If against of Haar functions, we approximate the function by crooked lines then the approximate solution at t=1 with J=7 and 9 is 0.000000000.

Example 2. Consider the nonlinear Fredholm integral equation

$$f(x) + \int_0^1 e^{x-2t} f^3(t) dt = e^{x+1} \ . \tag{47}$$

The exact solution is $f(x) = e^x$. The Newton iteration method is used to solve the nonlinear system of equations which it is obtained from Eq.(28). The initial value of F is given by $F_0 = [1, 1, \cdots, 1]^T_{2^J \times 1}$. After five iterations, the computational results for J=7 and 9 with the exact solution are given in Table 2.

Table 2. Estimated and exact values of $f(t)$ for Example 2

t	J=7	J=9	Exact solution
0	1.000000000	1.000000000	1
0.1	1.105176310	1.105171252	1.105170918
0.2	1.221411704	1.221403246	1.221402758
0.3	1.349868704	1.349859494	1.349858808
0.4	1.491831990	1.491825158	1.491824698
0.5	1.648642548	1.648722476	1.648721271
0.6	1.822127598	1.822119366	1.822118800
0.7	2.013767451	2.013753634	2.013752707
0.8	2.225557210	2.225541955	2.225540928
0.9	2.459615078	2.459604419	2.459603111
1	2.697127991	2.712977866	2.718281828

In Table 3, comparison between our method and the method using nominal Haar wavelets [6] are given for J=5.

Table 3. The comparison between the Haar wavelets method and our method

t	Haar method	Our method	Exact solution
0	1.015747709	1.000000000	1
0.1	1.107217811	1.105257803	1.105170918
0.2	1.218102916	1.221546193	1.221402758
0.3	1.341165462	1.350016667	1.349858808
0.4	1.474918603	1.491940528	1.491824698
0.5	1.667402633	1.649042552	1.648721271
0.6	1.833861053	1.822262052	1.822118800
0.7	2.016679830	2.013989195	2.013752707
0.8	2.217456630	2.225801196	2.225540928
0.9	2.437978177	2.459794079	2.459603111
1	2.676138775	2.634649089	2.718281828

Example 3. Consider the system of linear Volterra integral equations

$$\mathbf{f}(x) - \int_0^x \mathbf{k}(x,t)\mathbf{f}(t) dt = \mathbf{g}(x) \ , \tag{48}$$

where $\mathbf{f}(x) = [f_1(x), f_2(x)]^T$ is a unknown vector and the functions $\mathbf{g}$ and $\mathbf{k}$ defined as

$$\mathbf{g}(x) = \left[e^{-x^2} - \frac{x}{2}(1-e^{-x^2})\ ,\ e^{2x} + \frac{1-e^{x(x+2)}}{x+2}\right]^T,$$

$$\mathbf{k}(x,t) = \begin{bmatrix} xt & 0 \\ 0 & -e^{xt} \end{bmatrix}.$$

The exact solution of this system is $\mathbf{f}(x) = \left[e^{-x^2}, e^{2x}\right]$. We used Eq.(43) and solved this system of linear Volterra integral equations. In Tables 4.1 and 4.2, the comparison is made between the computational results for J=7 and 9 with exact values for $f_1(x)$ and $f_2(x)$.

Table 4.1. *Estimated and exact values of* $f_1(t)$ *for Example* 3

t	J=7	J=9	Exact solution
0	1.000000000	1.000000000	1
0.1	0.989825009	0.989995372	0.990049833
0.2	0.960345805	0.960680878	0.960789439
0.3	0.913274198	0.913768867	0.913931185
0.4	0.851276800	0.851927967	0.852143789
0.5	0.777948170	0.778062812	0.778800783
0.6	0.696382360	0.697352994	0.697676326
0.7	0.611118811	0.612249285	0.612626394
0.8	0.525571884	0.526861515	0.527292424
0.9	0.442922723	0.444373293	0.444858066
1	0.373649892	0.369317871	0.367879441

Table 4.2. *Estimated and exact values of* $f_2(t)$ *for Example* 3

t	J=7	J=9	Exact solution
0	1.000369076	1.000092273	1
0.1	1.221877334	1.221516950	1.221402758
0.2	1.492418972	1.491965022	1.491824698
0.3	1.822844763	1.822290269	1.822118800
0.4	2.226405854	2.225748991	2.225540928
0.5	2.718570088	2.718065240	2.718281828
0.6	3.321406901	3.320427464	3.320116923
0.7	4.056815324	4.055580702	4.055199967
0.8	4.955005813	4.953498621	4.953032424
0.9	6.051998810	6.049213220	6.049647464
1	6.051998810	6.050213220	7.389056099

Example 4. Consider the nonlinear Volterra integral equation of the second kind

$$f(x) - \int_0^x x\sin(2\pi t)f^2(t)dt = \cos(2\pi x) - \frac{x\cos^3(2\pi x)}{6\pi}. \quad (49)$$

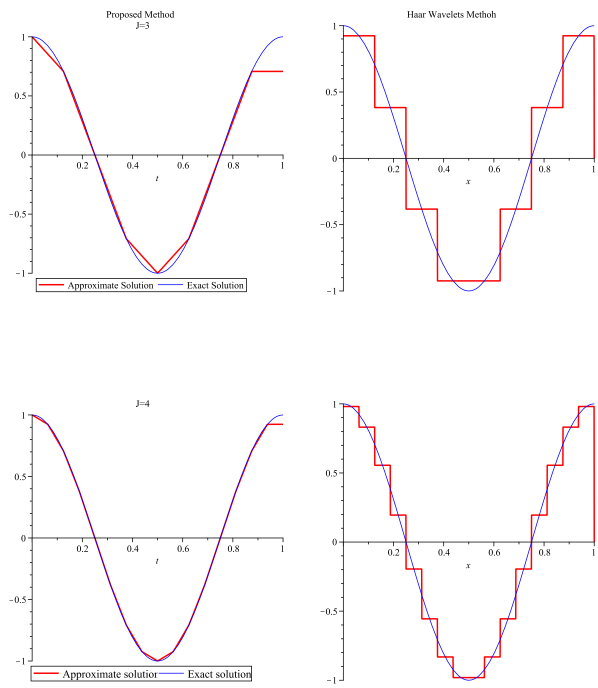
The exact solution is $f(x) = \cos(2\pi x)$. The Newton iteration method is used to solve the nonlinear system of equations has been obtained from Eq.(41). The initial value of F is given by $F_0 = [0, 0, \cdots, 0]^T_{2^J\times 1}$. After five iterations, the computational results for J=7 and 9 with the exact solution are given in Table 5.

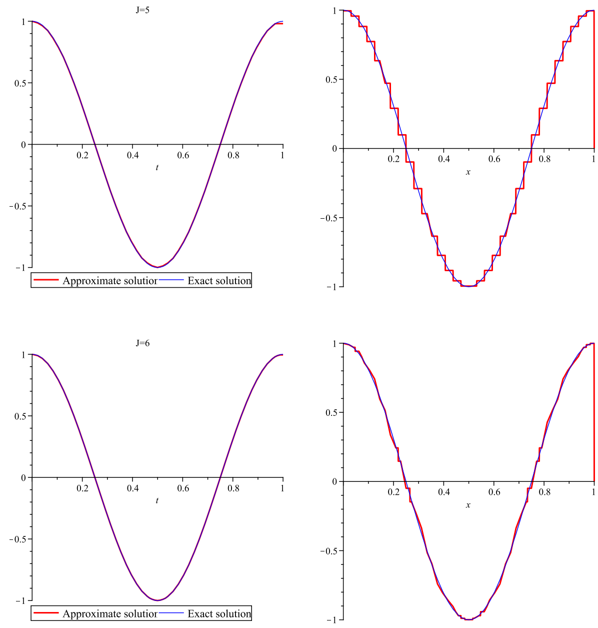
Table 5. Estimated and exact values of $f(t)$ for Example 4

t	J=7	J=9	Exact solution
0	1.000000000	1.000000000	1
0.1	0.808859946	0.809007264	0.8090169943
0.2	0.308926747	0.309011422	0.3090169938
0.3	-0.308926743	-0.309011422	-0.3090169942
0.4	-0.808859947	-0.809007281	-0.8090169945
0.5	-0.999999994	-1.000000000	-1
0.6	-0.808859947	-0.809007228	-0.8090169940
0.7	-0.308926749	-0.309011456	-0.3090169934
0.8	0.308926744	0.309011407	0.3090169946
0.9	0.808859940	0.809007308	0.8090169947
1	0.998795456	0.999924701	1

Figure 1 show the comparison of the proposed method with the method is obtained by using nominal Haar wavelets for Example 4. The convergence of this method can be deduced, too.

Figure 1. The comparison between the Haar wavelets method and our method





8. Conclusion

In this paper we suggested a method based on crooked lines which the nodes were specified by the rationalized Haar functions for the numerical solution of linear and nonlinear integral equations. The accuracy of the solution is improved and the simplicity of the method of using nominal Haar functions is preserved as shown in the given examples. Thus, instead of using nominal Haar wavelets, the proposed method for numerical solution of linear and nonlinear and the system of integral equations of the second kind is recommended.

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