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A result in asymmetric Euclidean Ramsey theory

Andrii Arman, Sergei Tsaturian *

University of Manitoba, Canada

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ABSTRACT

It is proved that if the points of the three-dimensional Euclidean space are coloured with red and blue, then there exist either two red points at unit distance, or six collinear blue points with distance one between any two consecutive points.

1. Introduction

The area of Euclidean Ramsey theory is mostly concerned with colouring points of an Euclidean space $\mathbb{E}^n$ and looking for a certain monochromatic point configuration. One of the simplest configurations that can be considered is two points distance one apart. Let $\chi(\mathbb{E}^n)$ be the minimal number of colours needed to colour all of the points of Euclidean space $\mathbb{E}^n$, so that there are no monochromatic points distance one apart. The problem of determining $\chi(\mathbb{E}^n)$ is over seventy years old and is usually referred to as the Hadwiger–Nelson problem (see Soifer [18]). Even the case $n = 2$ remains widely open and the bounds stay unchanged for over fifty years. The best known bounds for the chromatic number of the plane are $4 \le \chi(\mathbb{E}^2) \le 7$. The lower bound is due to Leo Moser and Willy Moser [13] and the upper bound is claimed by Soifer [18] to be due to John Isbell.

In case of three dimensions, $6 \le \chi(\mathbb{E}^3) \le 15$, with lower bound due to Nechushtan [14] and upper bound due to Coulson [2] and to Radoičić and Tóth [15]. The best known general bounds are

$$(1.239\ldots + o(1))^n \le \chi(\mathbb{E}^n) \le (3 + o(1))^n,$$

where the upper bound is due to Larman and Rogers [12] and the lower bound is due to Raigorodskii [16]. For an overview of this and related problems, see Raigorodskii's survey [17].

The main results of this paper are in the area of asymmetric Euclidean Ramsey theory, that deals with the questions of the following type: If F_1 and F_2 are two different configurations of points in an Euclidean space, is it true that for any colouring of the points of an Euclidean space $\mathbb{E}^n$ in red and blue, there always exists a congruent copy of F_1 with all of the vertices red, or a congruent copy of F_2 with all of the vertices blue? In the case of affirmative answer, the "Ramsey arrow" notation is used:

$$\mathbb{E}^n \to (F_1, F_2).$$

Research in this area was initiated in [4–6] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus. Let ℓ_i denote the configuration of i collinear points with distance one between any two consecutive points. The results of Erdős et al. [4–6] include:

* Corresponding author.
E-mail address: tsaturis@cc.umanitoba.ca (S. Tsaturian).

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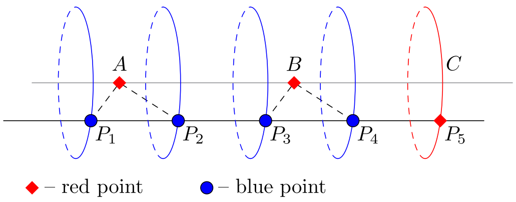

Fig. 1. Proof of Lemma 2.1.

- If T_i is a configuration of i points, then $\mathbb{E}^2 \to (T_2, T_3)$.
- $\mathbb{E}^2 \to (\ell_2, \ell_4)$.
- $\mathbb{E}^4 \to (\ell_2, \ell_5)$.

Juhász [10] proved that if T_4 is any configuration of 4 points, then $\mathbb{E}^2 \to (\ell_2, T_4)$. It was shown by Csizmadia and Tóth [3] that the result of Juhász cannot be generalized to 8 points, because it does not hold for a regular heptagon with its centre. Juhász [11] also informed us that Iván [9] proved that for any configuration T_5 of 5 points, $\mathbb{E}^3 \to (\ell_2, T_5)$, but this result was never published. This result of Iván [9] also follows from results of Szlam [19] and Nechushtan [14].

It was asked by Erdős et al. [5] whether $\mathbb{E}^3 \to (\ell_2, \ell_5)$. The result of Iván [9] implies the positive answer to this question. We present a simple proof of this result, namely, we prove:

Theorem 1.1. *Let the Euclidean space $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance one. Then there exist five blue collinear points with distance one between any two consecutive points.*

We also prove a stronger result, namely:

Theorem 1.2. *Let the Euclidean space $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance one. Then there exist six blue collinear points with distance one between any two consecutive points.*

The existence of a k, such that $\mathbb{E}^3 \not\to (\ell_2, \ell_k)$ follows from a recent result of Conlon and Fox [1], who showed that for all $n \geq 2$, $\mathbb{E}^n \not\to (\ell_2, \ell_{10^{5n}})$. It was shown by Szlam [19] that if $T_{\chi(\mathbb{E}^n)-1}$ is any configuration of $\chi(\mathbb{E}^n) - 1$ points, then $\mathbb{E}^n \to (\ell_2, T_{\chi(\mathbb{E}^n)-1})$, which together with the result of Raigorodskii [16] implies that for high dimensions

$$\mathbb{E}^n \to (\ell_2, \ell_{(1.239+o(1))^n}).$$

For an overview of other results in Euclidean Ramsey theory, see Graham's survey [7].

2. Existence of a blue ℓ_5, proof of Theorem 1.1

To prove Theorem 1.1, it is first proved (in the following two lemmas) that if there is a colouring that forbids red ℓ_2 and blue ℓ_5, there are no two red points at distance 2 or $\sqrt{7}$.

Lemma 2.1. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance 1. If there are no five blue points forming an ℓ_5, then there are no two red points at distance 2.*

Proof. Assume that points A and B are red and $|AB| = 2$. Choose a rectangular coordinate system centred at A so that B has coordinates $(2, 0, 0)$. Then any point at distance 1 to A is blue, otherwise there are two red points at distance 1. In particular, the circles $\{(-\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ and $\{(\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ are blue (see Fig. 1). Similarly, since any point at distance 1 to B is blue, the circles $\{(\frac{3}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ and $\{(\frac{5}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ are blue. Then, for any point P_1 on the first circle, the line ℓ through P_1 parallel to AB intersects the other three circles at some points P_2, P_3, P_4 that together with P_1 form a blue ℓ_4. Therefore the point P_5 of intersection of ℓ with the circle $C = \{(\frac{7}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ is red (otherwise P_1, P_2, P_3, P_4, P_5 form a blue ℓ_5). Since P_1 can be any point on $\{(-\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, the whole circle C is red. Since the radius of C is $\frac{\sqrt{3}}{2}$, C contains two red points at distance 1, which contradicts the assumptions of the lemma. □

Lemma 2.2. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance 1. If there are no five blue points forming an ℓ_5, then there are no two red points at distance $\sqrt{7}$.*

Proof. Assume that points A and B are red and $|AB| = \sqrt{7}$. Choose a rectangular coordinate system so that A has coordinates $(-\frac{\sqrt{7}}{2}, 0, 0)$ and B has coordinates $(\frac{\sqrt{7}}{2}, 0, 0)$. Since A and B are at distance $\sqrt{7}$, there exist points P_1, P_2, P_3, P_4, all collinear, such that $|P_1P_2| = |P_2P_3| = |P_3P_4| = 1$ and the triangles AP_1P_2 and BP_3P_4 are equilateral (see Fig. 2). Let P_1, P_2, P_3, P_4 have coordinates $P_1(-\frac{3}{\sqrt{7}}, \frac{3\sqrt{3}}{2\sqrt{7}}, 0)$, $P_2(-\frac{1}{\sqrt{7}}, \frac{\sqrt{3}}{2\sqrt{7}}, 0)$, $P_3(\frac{1}{\sqrt{7}}, -\frac{\sqrt{3}}{2\sqrt{7}}, 0)$, $P_4(\frac{3}{\sqrt{7}}, -\frac{3\sqrt{3}}{2\sqrt{7}}, 0)$. Then P_1, P_2, P_3, P_4 form a blue ℓ_4,

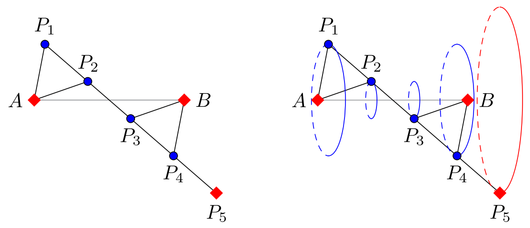

Fig. 2. Proof of Lemma 2.2.

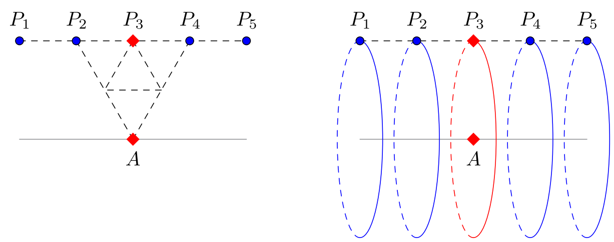

Fig. 3. Proof of Theorem 1.1.

therefore the point $P_5(\frac{5}{\sqrt{7}}, -\frac{5\sqrt{3}}{2\sqrt{7}}, 0)$ is red. When the line $P_1P_2P_3P_4$ is rotated about AB, points P_1, P_2, P_3, P_4 span four blue circles (since every point on the circles is at distance 1 to either A or B), therefore P_5 (when rotated) spans a red circle with radius $\frac{5\sqrt{3}}{2\sqrt{7}} > 1$ that contains two red points at distance 1. □

Proof of Theorem 1.1. Let $\mathbb{E}^3$ be coloured with red and blue in a way that there is no red ℓ_2 or blue ℓ_5. Consider any red point A and a rectangular coordinate system centred at A. By Lemma 2.1, any point at distance 2 to A is blue, in particular, the circles $\{(1, y, z) : y^2 + z^2 = 3\}$ and $\{(-1, y, z) : y^2 + z^2 = 3\}$ are blue (see Fig. 3). Similarly, by Lemma 2.2, the circles $\{(2, y, z) : y^2 + z^2 = 3\}$ and $\{(-2, y, z) : y^2 + z^2 = 3\}$ are blue. Consider any point P_3 on the circle $\{(0, y, z) : y^2 + z^2 = 3\}$. The line through P_3 parallel to $y = z = 0$ intersects the four blue circles at points P_1, P_2, P_4 and P_5 that together with P_3 form an ℓ_5. Since P_1, P_2, P_4, P_5 are blue, P_3 is red. Therefore, the circle $\{(0, y, z) : y^2 + z^2 = 3\}$ is red and has radius $\sqrt{3}$, and so this circle contains two red points at distance 1. □

3. Existence of a blue ℓ_6, proof of Theorem 1.2

The following five lemmas are needed for the proof.

Lemma 3.1. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance 1. If there are no six blue points forming an ℓ_6, then there is no disk with radius $\sqrt{3}$, such that all of its points (including interior) are blue.*

Proof. Suppose that there exists a blue disk D with centre O and radius $\sqrt{3}$. Consider a rectangular coordinate system on the plane containing D, centred at O (then $D = \{x^2 + y^2 \leq 3, x, y \in \mathbb{R}\}$). Let P_4 be any point on the boundary of D (for simplicity, let the coordinates of P_4 be $(\sqrt{3}, 0)$). Then points P_4, $P_3(\sqrt{3} - 1, 0)$, $P_2(\sqrt{3} - 2, 0)$, $P_1(\sqrt{3} - 3, 0)$ belong to D, and therefore are blue (see Fig. 4a). Consider points $P_5(\sqrt{3} + 1, 0)$, $P_6(\sqrt{3} + 2, 0)$, and a point A (say, $(\sqrt{3} + \frac{3}{2}, \frac{\sqrt{3}}{2})$) at distance 1 to both P_5 and P_6. If A is red, then both P_5 and P_6 are blue and $P_1, P_2, P_3, P_4, P_5, P_6$ form a blue ℓ_6. Therefore A is blue. When the point P_4 is rotated around the centre O, the point A (when rotated) spans a blue circle $C = \{x^2 + y^2 = 6 + 3\sqrt{3}, x, y \in \mathbb{R}\}$.

Consider any point Q_6 on C (for simplicity, let the coordinates of Q_6 be $(\sqrt{6 + 3\sqrt{3}}, 0)$; see Fig. 4b). Since $-\sqrt{3} < \sqrt{6 + 3\sqrt{3}} - 5$ and $\sqrt{6 + 3\sqrt{3}} - 2 < \sqrt{3}$, points $Q_4(\sqrt{6 + 3\sqrt{3}} - 2, 0)$, $Q_3(\sqrt{6 + 3\sqrt{3}} - 3, 0)$, $Q_2(\sqrt{6 + 3\sqrt{3}} - 4, 0)$, $Q_1(\sqrt{6 + 3\sqrt{3}} - 5, 0)$ are all inside D, therefore blue. Then the point $Q_5(\sqrt{6 + 3\sqrt{3}} - 1, 0)$ is red (otherwise $Q_1Q_2Q_3Q_4Q_5Q_6$ is a blue ℓ_6). If the point Q_6 is chosen arbitrarily on C, Q_5 spans a red circle with radius $\sqrt{6 + 3\sqrt{3}} - 1 > 1$, that contains two red points at distance 1. □

Lemma 3.2. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance 1. If there are no six blue points forming an ℓ_6, then there are no two red points at distance 2.*

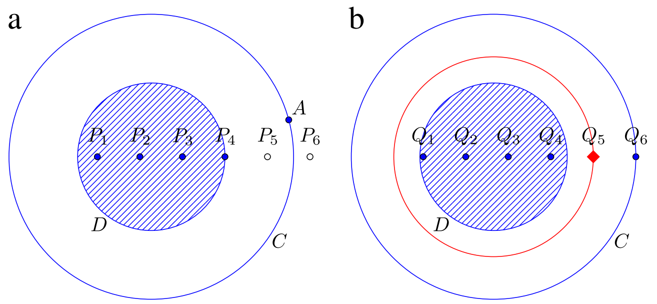

Fig. 4. Proof of Lemma 3.1.

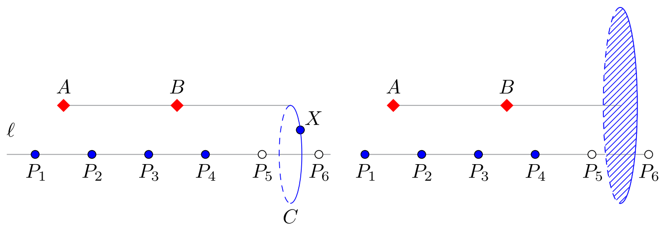

Fig. 5. Proof of Lemma 3.2.

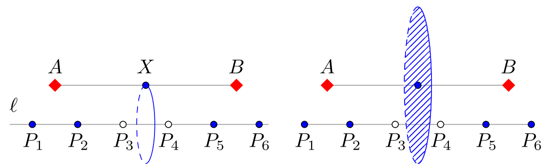

Fig. 6. Proof of Lemma 3.3.

Proof. The setup is the same as in Lemma 2.1. Assume that points A and B are red and $|AB| = 2$. Choose a rectangular coordinate system centred at A so that B has coordinates $(2, 0, 0)$. Then the circles $\{(-\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{3}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ and $\{(\frac{5}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ are blue. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell = \{(x, -\frac{\sqrt{3}}{2}, 0), x \in \mathbb{R}\}$), and let P_1, P_2, P_3, P_4 be the points of intersection (see Fig. 5, in this case $P_1(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_2(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_3(\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_4(\frac{5}{2}, -\frac{\sqrt{3}}{2}, 0)$). Let C be the circle in the plane $x = 4$ with radius $\frac{\sqrt{3}}{2}$ centred at the point on ℓ with x-coordinate 4 (in Fig. 5, the centre has coordinates $(4, -\frac{\sqrt{3}}{2}, 0)$), and let X be any point on C. If X is red, then the points P_5 and P_6 on ℓ with x-coordinates $\frac{7}{2}$ and $\frac{9}{2}$ (in Fig. 5, $P_5(\frac{7}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_6(\frac{9}{2}, -\frac{\sqrt{3}}{2}, 0)$) are both at distance 1 to X. Then P_5 and P_6 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue. When ℓ is rotated around AB, the circle C spans a blue disk with radius $\sqrt{3}$, which contradicts the statement of Lemma 3.1. □

Lemma 3.3. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance 1. If there are no six blue points forming an ℓ_6, then there are no two red points at distance 4.*

Proof. The proof is similar to that of Lemma 3.2. Assume that points A and B are red and $|AB| = 4$. Choose a rectangular coordinate system centred at A so that B has coordinates $(4, 0, 0)$. Then the circles $\{(-\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{1}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$, $\{(\frac{7}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ and $\{(\frac{9}{2}, y, z) : y^2 + z^2 = \frac{3}{4}\}$ are blue. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell = \{(x, -\frac{\sqrt{3}}{2}, 0) : x \in \mathbb{R}\}$), and let P_1, P_2, P_5, P_6 be the points of intersection (see Fig. 6, in this case $P_1(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_2(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_5(\frac{7}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_6(\frac{9}{2}, -\frac{\sqrt{3}}{2}, 0)$). Let C be the circle in the plane $x = 2$ with radius $\frac{\sqrt{3}}{2}$ centred at the point on ℓ with x-coordinate 2 (in Fig. 6, the centre has coordinates $(2, -\frac{\sqrt{3}}{2}, 0)$), and let X be any point on C. If X is red, then the points P_3 and P_4 on ℓ with x-coordinates $\frac{3}{2}$ and $\frac{5}{2}$ (in Fig. 6 $P_3(\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0)$, $P_4(\frac{5}{2}, -\frac{\sqrt{3}}{2}, 0)$) are both at distance 1 to X. Then P_3 and P_4 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue. When ℓ is rotated around AB, the circle C spans a blue disk with radius $\sqrt{3}$, which contradicts the statement of Lemma 3.1. □

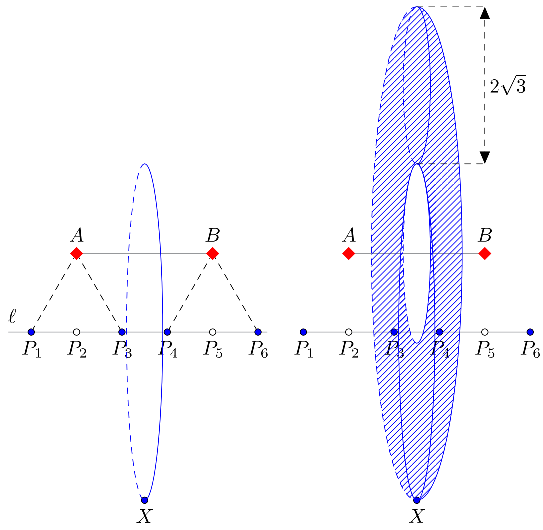

Fig. 7. Proof of Lemma 3.4.

Lemma 3.4. *Let $\mathbb{E}^3$ be coloured with red and blue so that there are no two red points at distance* 1. *If there are no six blue points forming an ℓ_6, then there are no two red points at distance* 3.

Proof. Assume that points A and B are red and $|AB| = 3$. Choose a rectangular coordinate system so that A has coordinates $(1, 0, 0)$ and B has coordinates $(4, 0, 0)$. Then, by Lemma 3.2, any point at distance 2 to A is blue; in particular, the circles $\{(0, y, z) : y^2 + z^2 = 3\}$ and $\{(2, y, z) : y^2 + z^2 = 3\}$ are blue. By the same argument, circles $\{(3, y, z) : y^2 + z^2 = 3\}$ and $\{(5, y, z) : y^2 + z^2 = 3\}$ are blue, since their points are at distance 2 from B. Consider any line ℓ parallel to AB that intersects all the blue circles (for simplicity, let $\ell = \{(x, -\sqrt{3}, 0) : x \in \mathbb{R}\}$), and let P_1, P_3, P_4, P_6 be the points of intersection (see Fig. 7, in this case $P_1(0, -\sqrt{3}, 0)$, $P_3(2, -\sqrt{3}, 0)$, $P_4(3, -\sqrt{3}, 0)$, $P_6(5, -\sqrt{3}, 0)$). Let C be the circle in the plane $x = \frac{5}{2}$ with radius $\frac{\sqrt{55}}{2}$ centred at the point on ℓ with x-coordinate $\frac{5}{2}$ (in Fig. 7, the centre has coordinates $(\frac{5}{2}, -\sqrt{3}, 0)$), and let X be any point on C. If X is red, then the points P_2 and P_5 on ℓ with x-coordinates 1 and 4 (in Fig. 7, $P_2(1, -\sqrt{3}, 0)$, $P_5(4, -\sqrt{3}, 0)$) are both at distance 4 to X. Then, by Lemma 3.3, P_2 and P_5 are blue, and $P_1P_2P_3P_4P_5P_6$ is a blue ℓ_6. Therefore, X is blue, hence the circle C is blue. When ℓ is rotated around AB, the circle C spans the blue annulus bounded by circles of radii $\frac{\sqrt{55}}{2} + \sqrt{3}$ and $\frac{\sqrt{55}}{2} - \sqrt{3}$, which contains a blue disk of radius $\sqrt{3}$. By Lemma 3.1, this leads to a contradiction with the fact that there are no two red points at distance 1. □

Lemma 3.5. *Let $\mathcal{L}$ be a unit triangular lattice on a plane. Let points of $\mathcal{L}$ be coloured with red and blue so that there are no red ℓ_2 and no blue ℓ_6. If $\mathcal{L}$ contains two red points at distance $\sqrt{3}$, then $\mathcal{L}$ does not contain a blue ℓ_5.*

Proof. The following claim is used for the proof of Lemma 3.5.

Claim. *Let A_1 and A_2 be two red nodes of $\mathcal{L}$ at distance $\sqrt{3}$. Then all nodes of $\mathcal{L}$ on the line A_1A_2 are red.*

Proof of Claim. First, it is proved that node A_3, the reflection of A_1 across A_2, is also red. Consider the part of $\mathcal{L}$ depicted in Fig. 8a. Points Q_3 and Q_6 are at distance 3 from A_1, and therefore are blue by Lemma 3.4. Points Q_4 and Q_5 are at distance 1 from A_2, therefore blue. Then point P_1 is blue; otherwise points Q_1 and Q_2 are both blue and form a blue ℓ_6 with points Q_3, Q_4, Q_5, Q_6. Points P_2 and P_6 are at distance 4 from A_1, therefore are blue by Lemma 3.3. Points P_3 and P_5 are at distance 2 from A_2, therefore are blue by Lemma 3.2. Then the point A_3 has to be red in order to prevent the blue $P_1P_2P_3A_3P_5P_6$.

Using the same argument, it can be proved that the node A_4 symmetric to A_2 in A_3 is red, and similarly for any $k \in \mathbb{Z}$ a point on the line A_1A_2 at distance $k\sqrt{3}$ from A_1 is red. This concludes proof of the Claim.

Let A_1 and A_2 be two red nodes of $\mathcal{L}$ at distance $\sqrt{3}$. By the Claim, all nodes of $\mathcal{L}$ on the line A_1A_2 are red. Let A_1, A_2, A_3, A_4, A_5 be five consecutive red nodes on the line A_1A_2. Consider the part of the lattice depicted in Fig. 8b. By Lemmas 3.2, 3.4, 3.3, the points P_1, P_2, P_3, P_4 are blue, since they are at distance 1, 2, 3 and 4 from A_1, respectively. By Lemma 3.4, the point P_6 is blue (since it is at distance 3 from A_4). Then point P_5 is red (otherwise P_1, P_2, P_3, P_4, P_6 form a blue ℓ_6). Similarly, the points Q_1, Q_2, Q_3, Q_6 are blue, hence Q_5 is red. Then P_5 and Q_5 are two red nodes at distance $\sqrt{3}$, which, by Claim, forces every node of $\mathcal{L}$ on the line P_5Q_5 to be red.

If the colouring is expanded further in a similar way, the lattice will be coloured in a unique way, which is shown in Fig. 9. Every ℓ_5 that belongs to $\mathcal{L}$ contains a red point, therefore the colouring does not contain a blue ℓ_5. □

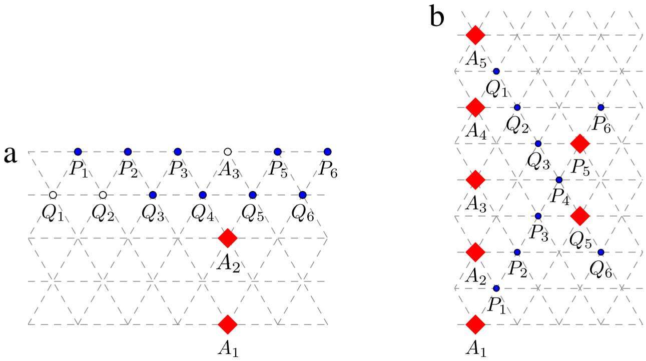

Fig. 8. Proof of Lemma 3.5.

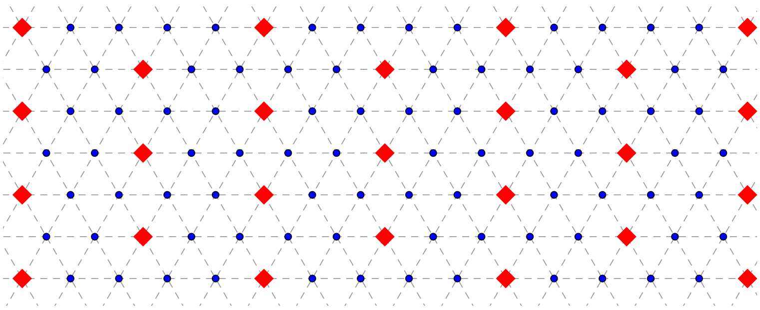

Fig. 9. Proof of Lemma 3.5.

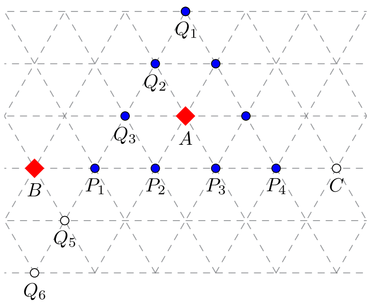

Fig. 10. Proof of Theorem 1.2.

Proof of the Theorem 1.2. Suppose that $\mathbb{E}^3$ is coloured with two colours so that there are no red ℓ_2 and no blue ℓ_6. By Theorem 1.1, there is a blue ℓ_5, say, $X_1X_2X_3X_4X_5$. Consider any unit triangular lattice $\mathcal{L}$ such that X_1, X_2, X_3, X_4, X_5 are nodes of $\mathcal{L}$. Since $\mathcal{L}$ does not contain a blue ℓ_6, there is a red node A in $\mathcal{L}$. Consider the part of $\mathcal{L}$ depicted in Fig. 10. The points P_2 and P_3 are blue, since they are at distance 1 from A. Since $\mathcal{L}$ contains a blue ℓ_5, by Lemma 3.5, there are no two red nodes of $\mathcal{L}$ at distance $\sqrt{3}$, therefore points P_1 and P_4 are blue. Then points B and C cannot be both blue (otherwise a blue ℓ_6 is formed), therefore one of them, say, B, is red. Then points Q_5 and Q_6 are at distance 1 and $\sqrt{3}$ from B, hence blue. The points Q_3, Q_2, Q_1 are at distance 1, 1, $\sqrt{3}$ from A, respectively, therefore blue. Hence, the points $Q_1, Q_2, Q_3, P_1, Q_5, Q_6$ form a blue ℓ_6, which contradicts the initial assumption. □

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Added in proof: Shortly after submission of this paper, Sergei Tsaturian proved that the result of Theorem 1.1 holds in the plane, namely that $\mathbb{E}^2 \to (\ell_2, \ell_5)$ (see https://arxiv.org/abs/1703.10723).

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