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A Fourier-space description of oscillations in an inhomogeneous plasma. Part 1. Continuous Fourier transformation

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Oscillations in inhomogeneous cold plasmas or inhomogeneous magnetofluids are interpreted in terms of the dynamics of their spectra in wavenumber space. By Fourier transforming the basic integro-differential equation of the problem, a generalized wave equation in wavenumber space is derived, thus converting the oscillation and phase-mixing processes in the original x space into processes of dispersive propagation and scattering of the spectrum in wavenumber space. The Barston singular continuum eigenmodes correspond to stationary scattering states of a monochromatic wave in wavenumber space, whereas the damping phenomena in x space correspond to transient 'leaking' phenomena accompanying scattering and dispersive propagation of a wave packet in wavenumber space.

1. Introduction

This paper is the first of a pair examining the nature of phase mixing in inhomogeneous cold plasmas and magnetofluids from a spatial Fourier perspective. These two papers extend some earlier work by the authors (Sedláček 1971*a*, *b*; Cally & Sedláček 1992). In this first paper a continuum approach to an unbounded system is taken by applying a Fourier transform over the spatial co-ordinate along which the inhomogeneity is manifest, whereas in part 2 (Cally & Sedláček 1994) we primarily examine both bounded and periodic cases for which series are appropriate (though the unbounded case is also briefly discussed). A generally consistent picture emerges, though there are some interesting points of difference between the three systems at low mode number. Both papers are written to be essentially self-contained. Notation in the current paper is consistent with that used by Sedláček (1971*a*, *b*), whereas we essentially follow Cally (1991) in part 2.

The initial-value problem for linear electrostatic oscillations of a cold inhomogeneous plasma and its mathematically equivalent counterpart, linear Alfvén oscillations in inhomogeneous incompressible magnetohydrodynamic (MHD) systems, was originally solved by normal-mode analysis (Barston 1964; Sedláček 1971*a*, *b*) and by Laplace-transform and Green-function techniques (Sedláček 1971*a*, *b*).

The normal-mode analysis demonstrated the existence of a continuous spectrum of the system and its principal importance for the character of its solution. The solution was expressed as an integral superposition of singular continuum eigenmodes, and exhibited algebraic damping in time, fully understandable from the mathematical point of view but somewhat puzzling in view of the fact that the physical systems in question are energy-conserving and the operators involved are Hermitian.

With the Laplace-transform approach, the solution was expressed in an alternative form of a contour integral in the complex-frequency plane, which in the end was reduced to a sum over the singularities of the Laplace transform. The singularities comprise (i) branch points on the real axis corresponding to the end points of the intervals of the continuous spectrum and to the singularities of the continuum eigenmodes, and (ii) an infinity of complex poles generated by zeros of a certain analytic function of frequency, which we termed the 'dispersion function', with a complicated structure of the Riemann surface. These poles give rise to exponentially damped terms – a still more puzzling and easily misinterpreted fact, since such solutions are usually encountered in dissipative systems. Some of the complex zeros may be put into correspondence to a 'damped surface wave', although, strictly speaking, a surface wave exists only in a medium with a discontinuous density profile. Equivalency of the normal-mode analysis and the Laplace-transform approach was rigorously proved. This is especially to be stressed, since it shows that the difference between the two methods consists only in different representations of the same solution to the initial-value problem: the normal-mode analysis gives the solution in an integral form, whereas the Laplace transform approach yields an infinite sum. One representation can be changed into the other by formal manipulation, and neither of them involves anything that the other does not.

To summarize: although a rather complete mathematical analysis of the problem exists, a clear and lucid interpretation is lacking.

Recently Cally (1991) performed an analytical and numerical analysis consisting in discretizing the partial differential equation of the system by expanding into a suitable set of functions of the variable along which the system is inhomogeneous. An infinite set of ordinary differential equations for the expansion coefficients thus arose, fully equivalent to the original partial differential equation. By analysing this set, Cally was able to reproduce a number of the main features of the oscillations, such as the nature of the spectrum, and also to derive the exponential decay rate of the surface wave for a steep inhomogeneity. No transform methods were employed. The numerical solutions of the truncated set of differential equations then fully confirmed the analytical results previously obtained. This analysis was later extended by Cally & Sedláček (1992). At that time it also became clear that this approach in fact offers the possibility of an entirely new interpretation of oscillations in inhomogeneous systems by converting the oscillation and phase-mixing processes in the original x space into processes of propagation and scattering of the spectrum of the initial disturbance in the corresponding discrete mode-number space.† This mode-number-space interpretation corresponds exactly to the interpretation of Vlasov–Poisson plasma oscillations in Fourier-transformed

† In the context of shear Alfvén-wave turbulence the idea of propagation in wavenumber space may be found in a paper by Bondeson (1985).

velocity space proposed by Sedláček & Nocera (1992). The correspondence is understandable in view of the analogy between the integro-differential equation for the transverse component of the electric field derived by Sedláček (1971*b*) (see also Uberoi & Sedláček 1992) and the Vlasov–Poisson equation.

In this pair of papers we intend to utilize these ideas to present a rather complete wavenumber-space and mode-number-space description of cold inhomogeneous plasma oscillations. This paper is concerned with the continuum wavenumber-space description generated by the Fourier integral transformation of x space. Discrete mode-number-space descriptions suitable for numerical calculations will be developed in part 2. Also, all the results of numerical calculations will be collected in part 2, although some of them are relevant to both papers.

In §2 we derive the basic equation of the paper, namely the Fourier-transformed integro-differential equation of Sedláček (1971*b*), and discuss approximations to its integral operator. We show that this transformed equation describes dispersive wave propagation in wavenumber space accompanied by scattering. The scattering is governed by the integral operator originating from the plasma inhomogeneity. Its region of influence (the 'scattering region') is well localized in wavenumber space for sufficiently smooth inhomogeneity profiles. In §3 we discuss the dispersion properties of the free propagation in wavenumber space (propagation well outside the 'scattering region'), and use them in §4 to calculate the asymptotic formula for the evolution of a freely propagating wave packet. This formula enables us, among other things, to interpret the conspicuous 'excitation front' propagating in mode-number space that Cally (1991) observed in his computer simulations. In §5 we analyse the most elementary process in wavenumber space: the stationary scattering of a monochromatic wave. We calculate its scattering matrix, prove it to be unitary and demonstrate the absence of reflection for a smooth monotonic density profile. In §6 we treat transient processes in wavenumber space as far as is possible without solving the initial-value problem. A general solution of the basic integro-differential equation is derived, with an exponential time dependence, to show that it is the boundary condition imposed in wavenumber space that determines in a crucial way the character of the resulting solution and makes the whole difference between a singular (monochromatic wave) and regular (wave-packet) solution.

2. Fourier transform of the basic integro-differential equation

The basic integro-differential equation for the transverse component of the electrostatic field derived by Sedláček (1971*b*) and later shown by Uberoi & Sedláček (1992) to be valid also for magnetohydrodynamic oscillations, namely

$$\frac{\partial^2 E(x,t)}{\partial t^2}+\omega_p^2(x)\,E(x,t)=\int_{-\infty}^{\infty} Q(x,x')\,E(x',t)\,dx', \tag{2.1}$$

with
$$Q(x,x')=-\tfrac{1}{2}K\int_{x'}^{\infty} e^{-K|x-u|}\,d\omega_p^2(u) \tag{2.2}$$

and
$$K=(k_y^2+k_z^2)^{\frac{1}{2}}, \tag{2.3}$$

where $\omega_p(x)$ stands for either the plasma frequency or the Alfvén frequency, will be Fourier transformed with respect to the variable x. The transformation is defined as

$$E_q(t) = \int_{-\infty}^{\infty} E(x,t)\, e^{-iqx}\, dx, \tag{2.4}$$

$$E(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_q(t)\, e^{iqx}\, dq, \tag{2.5}$$

with the variable q playing the role of the wavenumber in the x direction. The 'free-oscillation' part, the left-hand side of (2.1), is easily transformed by means of the convolution theorem

$$\int_{-\infty}^{\infty} \omega_p^2(x)\, E(x,t)\, e^{-iqx}\, dx = \int_{-\infty}^{\infty} \Omega_{q-q'}\, E_{q'}(t)\, dq', \tag{2.6}$$

where we have defined

$$\Omega_q = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega_p^2(x)\, e^{-iqx}\, dx. \tag{2.7}$$

Transformation of the integral operator gives

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} Q(x,x')\, e^{-iqx}\, dx\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} E_{q'}(t)\, e^{iq'x'}\, dq'\right] dx' = \int_{-\infty}^{\infty} Q_{qq'}\, E_{q'}(t)\, dq', \tag{2.8}$$

with
$$Q_{qq'} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(x,x')\, e^{-iqx}\, e^{iq'x'}\, dx\, dx'. \tag{2.9}$$

To find $Q_{qq'}$, we write

$$Q(x,x') = -\tfrac{1}{2}K \int_{-\infty}^{\infty} e^{-K|x-u|} H(u-x')\, \frac{d\omega_p^2(u)}{du}\, du \tag{2.10}$$

so that

$$Q_{qq'} = \int_{-\infty}^{\infty} \left(-\tfrac{1}{2}K \int_{-\infty}^{\infty} e^{-K|x-u|}\, e^{-iqx}\, dx\right)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} H(u-x')\, e^{iq'x'}\, dx'\right] \frac{d\omega_p^2(u)}{du}\, du. \tag{2.11}$$

Now
$$-\tfrac{1}{2}K \int_{-\infty}^{\infty} e^{-K|x-u|}\, e^{-iqx}\, dx = -e^{-iqu}\, \frac{K^2}{K^2+q^2} \tag{2.12}$$

and
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H(u-x')\, e^{iq'x'}\, dx' = \frac{1}{2\pi} e^{iq'u} H_{q'}, \tag{2.13}$$

where H_q is the Fourier transform of the Heaviside function (see e.g. Bremermann 1965),

$$H_q = \pi\left[\delta(q) - \frac{i}{\pi} P \frac{1}{q}\right] \tag{2.14}$$

(with P indicating the Cauchy principal value). Thus, finally,

$$Q_{qq'} = -\frac{1}{2\pi} \epsilon_p(q-q')\, a_q\, b_{q'}, \tag{2.15}$$

where
$$\epsilon_p(q) = \int_{-\infty}^{\infty} \frac{d\omega_p^2(u)}{du} e^{-iqu}\, du \tag{2.16}$$
is essentially the Fourier transform of the derivative of the square of the plasma frequency profile and
$$a_q = \frac{K^2}{K^2+q^2}, \tag{2.17}$$
$$b_q = H_q. \tag{2.18}$$

The integral kernel is therefore 'nearly separable' in the sense that as the density profile becomes steeper, approximating the step function,
$$\omega_p^2(x) \to \omega_{p1}^2 + \Delta H(x), \tag{2.19}$$
where as usual ω_{p1} and ω_{p2} are the asymptotic values of the plasma frequency at minus and plus infinity respectively, and $\Delta = \omega_{p2}^2 - \omega_{p1}^2$, its derivative approaches the delta function,
$$\frac{d\omega_p^2(x)}{dx} = \Delta\delta(x), \tag{2.20}$$
so that in the limit
$$\epsilon_p(q) \to \Delta, \tag{2.21}$$
and the integral kernel approaches a simple dyad,
$$Q_{qq'} \to -\frac{1}{2\pi}\Delta a_q b_{q'}. \tag{2.22}$$

To find what corresponds to this dyadic form in the original x representation, we utilize the formula complementary to (2.9),
$$Q(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_{qq'}\, e^{iqx}\, e^{-iq'x'}\, dq\, dq' \tag{2.23}$$
to obtain
$$Q(x,x') = -\Delta a(x)\, b(-x'), \tag{2.24}$$
where
$$a(x) = \tfrac{1}{2}K\, e^{-K|x|}, \tag{2.25}$$
$$b(x) = H(x). \tag{2.26}$$

Essentially the same result has already been found by Sedláček (1971*b*), with somewhat different notation:
$$Q(x,x') = \mathscr{Q}(x)\,\mathscr{R}(x'), \tag{2.27}$$
where
$$\mathscr{Q}(x) = \frac{2}{K}a(x) = e^{-K|x|} \tag{2.28}$$
and
$$\mathscr{R}(x) = -\tfrac{1}{2}K\,\Delta b(-x) = -\tfrac{1}{2}K\,\Delta H(-x). \tag{2.29}$$

It is obvious that the integral kernel $Q_{qq'}$ vanishes at infinity for both q and for q'. The speed with which $Q_{qq'}$ approaches zero is determined by the smoothness properties of the density profile. The smoother the function $\omega_p(x)$, the more rapidly will the Fourier transform of its derivative $\epsilon_p(q)$, vanish at infinity. In the worst case of a discontinuous density profile, $\epsilon_p(q)$ does not vanish at all – it is a constant, and $Q_{qq'}$ approaches zero like q^{-2} and q'^{-1}. In this case the integral kernel $Q_{qq'}$ is very poorly localized. For an infinitely differentiable density profile, $\epsilon_p(q)$ will vanish more rapidly than any negative power of q, and the integral kernel will be very well localized around the origin.

Thus, in the end, the Fourier-transformed integro-differential equation (2.1) becomes

$$\frac{\partial^2 E_q(t)}{\partial t^2} + \int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'}(t)\, dq' = \int_{-\infty}^{\infty} Q_{qq'} E_{q'}(t)\, dq'. \tag{2.30}$$

This equation may be interpreted as an equation describing dispersive wave propagation in wavenumber space accompanied by scattering. The propagation properties are characterized by the 'homogeneous part' of the equation, translation-invariant in wavenumber space, whereas the scattering is caused by the 'inhomogeneous term', an integral operator. The scatterer is localized around the origin of wavenumber space – the better so, the smoother the density profile. Thus, by Fourier transforming with respect to the inhomogeneity variable, we have converted the cold-plasma oscillation process in x space into a classical one-dimensional scattering process in the corresponding wavenumber space: a disturbance propagating dispersively from infinity towards the origin impinges upon the scatterer, is modified inside it and transmitted to the opposite infinity. Since the equation is of second order in time, propagation in either direction is possible, but we shall see that there is no reflection.

The situation is basically similar to what Sedláček & Nocera (1992) have found with the Vlasov–Poisson plasma oscillations in Fourier-transformed velocity space. The Vlasov-plasma case differs from the cold-plasma case in that (i) the Fourier-transformed Vlasov equation is of first order in time, so that the propagation is unidirectional; (ii) there is no dispersion, since the convolution integral simplifies to a first-order derivative; and (iii) the integral operator characterizing the scattering processes is exactly a simple dyad.

3. Free-propagation dispersion relation

For a sufficiently smooth density profile, the Fourier-transformed integral kernel $Q_{qq'}$, (2.15), decays rapidly for q and q' tending to plus or minus infinity. Thus, outside some 'scattering region' around the origin of the q axis, $Q_{qq'}$ becomes negligible, and the solution of the basic integro-differential equation (2.30) is dominated by its 'homogeneous part'

$$\frac{\partial^2 E_q(t)}{\partial t^2} + \int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'}(t)\, dq' = 0. \tag{3.1}$$

This can be viewed as the most general wave equation in the sense that it describes dispersive propagation in wavenumber space, the dispersive properties being determined by a convolution integral (see the discussion in §6). The kernel Ω_p of this integral, given by (2.7), is essentially a Fourier transform of the plasma frequency profile $\omega_p^2(x)$, so the propagation and dispersion is a consequence of the plasma inhomogeneity. Indeed, in a homogeneous plasma we have

$$\Omega_q = \frac{1}{2\pi}\omega_p^2 \int_{-\infty}^{\infty} e^{-iqx}\, dx = \omega_p^2\, \delta(q), \tag{3.2}$$

and, by (2.16), $\epsilon_p(q) = 0$, so that the basic transformed integro-differential equation (2.30) reduces to

$$\frac{\partial^2 E_q(t)}{\partial t^2} + \omega_p^2 E_q(t) = 0. \tag{3.3}$$

The general solution of this equation is

$$E_q(t) = C_1(q)\, e^{-i\omega_p t} + C_2(q)\, e^{i\omega_p t}, \tag{3.4}$$

with $C_1(q)$ and $C_2(q)$ arbitrary functions of q. This is a standing wave solution in q space.

To analyse the solution of (3.1), which we shall call, by analogy with the Vlasov–Poisson equation, the 'free propagation' solution, we first note that it admits monochromatic wave solutions

$$E_q(t) = C\, e^{-i(\omega t + \kappa q)}, \tag{3.5}$$

where κ is the wavenumber in wavenumber space (q space). Substitution into (3.1) gives the free-propagation dispersion relation

$$\omega^2 - \omega_p^2(\kappa) = 0, \tag{3.6}$$

which can be decomposed into two branches

$$\omega = \omega_p(\kappa), \tag{3.7}$$

$$\omega = -\omega_p(\kappa), \tag{3.8}$$

corresponding, for $\kappa > 0$, to propagation in the direction of the negative and positive q axis respectively. The general solution to (3.1), describing dispersive propagation of a wave packet along the q axis, is obtained as an integral superposition of these monochromatic wave solutions:

$$E_q(t) = \int_{-\infty}^{\infty} C_1(\kappa)\, e^{-i[\omega_p(\kappa)\, t + \kappa q]}\, d\kappa + \int_{-\infty}^{\infty} C_2(\kappa)\, e^{-i[-\omega_p(\kappa)\, t + \kappa q]}\, d\kappa. \tag{3.9}$$

This formula enables us to solve the initial-value problem for (3.1), i.e. to find the arbitrary expansion coefficients $C_1(\kappa)$ and $C_2(\kappa)$ such that for $t = 0$, (3.9) gives a prescribed initial value of the Fourier transform of the electric field, $E_q(0)$, and its time derivative, $\dot{E}_q(0)$. To achieve this, we set $t = 0$ in (3.9) to obtain

$$E_q(0) = \int_{-\infty}^{\infty} [C_1(\kappa) + C_2(\kappa)]\, e^{-i\kappa q}\, d\kappa, \tag{3.10}$$

differentiate (3.9) with respect to t and again set $t = 0$:

$$\dot{E}_q(0) = -\int_{-\infty}^{\infty} i\omega_p(\lambda)\, [C_1(\kappa) - C_2(\kappa)]\, e^{-i\kappa q}\, d\kappa. \tag{3.11}$$

These two equations are equivalent to

$$C_1(\kappa) = \frac{1}{2}\left[E(\kappa, 0) + i\frac{\dot{E}(\kappa, 0)}{\omega_p(\kappa)}\right], \tag{3.12}$$

$$C_2(\kappa) = \frac{1}{2}\left[E(\kappa, 0) - i\frac{\dot{E}(\kappa, 0)}{\omega_p(\kappa)}\right]. \tag{3.13}$$

Equation (3.9) also shows that κ can be interpreted as the co-ordinate x. The inverse Fourier transform of $E_q(t)$ gives the usual formula for the solution of the initial-value problem in x space:

$$E(x,t) = E(x,0)\cos\omega_p(x)\,t + \frac{\dot{E}(x,0)}{\omega_p(x)}\sin\omega_p(x)\,t, \tag{3.14}$$

which is of course identical with what was found by Sedláček (1971 *b*, equation (3.3)) if only the free-oscillation part

$$\chi_{K\sigma}(x) = \delta(x - x_p(\sigma)) \tag{3.15}$$

of the full electric field continuum eigenmode is retained (here $x_p(\sigma)$ is the solution of the equation $\sigma^2 = \omega_p^2(x)$ with respect to x).

4. Free-propagation asymptotic evolution

A wave packet propagating in wavenumber space far from the scattering region undergoes dispersion. It is easy to analyse this effect asymptotically for large times. The present problem is very similar to that of waves on water of constant depth (Whitham 1974; Lighthill 1978). Indeed, the group velocity in wavenumber space can be defined as usual by

$$\frac{d\omega}{d\kappa} = \frac{d\omega_p(\kappa)}{d\kappa}. \tag{4.1}$$

For monotonic density profiles like those analysed by Cally & Sedláček (1992), the group velocity has a single maximum at $\kappa = 0$, and we can expand around this point:

$$\omega_p(\kappa) = \omega_p(0) + c_0\,\kappa - \gamma\kappa^3 + \dots, \tag{4.2}$$

where

$$c_0 = \left.\frac{d\omega_p(\kappa)}{d\kappa}\right|_{\kappa=0} \tag{4.3}$$

is the maximum group velocity and

$$\gamma = \frac{1}{3!}\left.\frac{d^3\omega_p(\kappa)}{d\kappa^3}\right|_{\kappa=0}. \tag{4.4}$$

Assume further a very narrow initial wave packet in wavenumber space corresponding to the initial condition

$$E_q(0) = \delta(q). \tag{4.5}$$

This means that the initial condition in the original x space is very wide:

$$E(x,0) = \frac{1}{2\pi}, \tag{4.6}$$

$$\dot{E}(x,0) = 0. \tag{4.7}$$

The evolution of the wave packet generated by this initial condition and propagating in the direction of the positive q axis is described by

$$E_q(t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i[-\omega_p(\kappa)\,t + \kappa q]}\,d\kappa \approx \frac{1}{4\pi}e^{i\omega_p(0)\,t}\int_{-\infty}^{\infty} e^{-i\kappa(q - c_0 t) - i\gamma\kappa^3 t}\,d\kappa. \tag{4.8}$$

The substitution $s = -(3\gamma t)^{\frac{1}{3}}\kappa$ converts this integral into the standard Airy integral

$$E_q(t) = e^{i\omega_p(0)t} \frac{1}{2(3\gamma t)^{\frac{1}{3}}} \mathrm{Ai}\left[\frac{q - c_0 t}{(3\gamma t)^{\frac{1}{3}}}\right]. \tag{4.9}$$

Thus, after a sufficiently long time, an initially narrow wave packet far from the scattering region will be dispersed into a steadily widening oscillatory formation with a steep exponentially increasing front, slowly decreasing tail and steadily diminishing maximum amplitude. Both the widening and the damping are governed by the $t^{-\frac{1}{3}}$ law. This is the remarkable 'excitation front' propagating in mode-number space that Cally (1991) observed in his numerical calculations. Further discussion and numerical analysis of this result in the discrete context will be found in §2 of part 2, where graphical comparison is made with exact solutions for a specific plasma frequency profile. The corresponding is excellent.

5. Scattering states of a monochromatic wave in wavenumber space

Again being led by the analogy with the Vlasov–Poisson equation case (Sedláček & Nocera 1992), we shall now show that the Fourier transform of the electric-field continuum eigenmode of the original basic integro-differential equation (2.1) corresponds to the scattering states of a monochromatic wave launched in wavenumber space.

Sedláček (1971*b*) has shown that a general solution of (2.1) can be expressed in terms of the electric field continuum eigenmode $\chi_{K\sigma}(x)$ as a Fourier integral over the continuous spectrum $\mathscr{S}$ (consisting of the two intervals $-\omega_{p2} < \sigma < -\omega_{p1}$ and $\omega_{p1} < \sigma < \omega_{p2}$)

$$E(x,t) = \int_{\mathscr{S}} A(\sigma)\,\chi_{K\sigma}(x)\cos\sigma t\,d\sigma + \int_{\mathscr{S}} \frac{\mathring{A}(\sigma)}{\sigma}\chi_{K\sigma}(x)\sin\sigma t\,d\sigma \tag{5.1}$$

with the expansion coefficients $A(\sigma)$ and $\mathring{A}(\sigma)$ being expressible in terms of the initial condition for the electric field $E(x,0)$ and its time derivative $\dot{E}(x,0)$. We Fourier transform this expression with respect to x and interchange the order of integration:

$$E_q(t) = \int_{\mathscr{S}} d\sigma \left[A(\sigma)\cos\sigma t + \frac{\mathring{A}(\sigma)}{\sigma}\sin\sigma t\right]\int_{-\infty}^{\infty} dx\,\chi_{K\sigma}(x)\,e^{-iqx}. \tag{5.2}$$

With the monochromatic-wave assumption

$$A(\sigma) = A\delta(\sigma - \sigma_0), \tag{5.3}$$

$$\mathring{A}(\sigma) = \mathring{A}\delta(\sigma - \sigma_0), \tag{5.4}$$

where $\sigma_0 \in \mathscr{S}$, we get

$$E_q(t) = \frac{1}{2}\left(A + \frac{\mathring{A}}{i\sigma}\right)e^{i\sigma t}\int_{-\infty}^{\infty} \chi_{K\sigma}(x)\,e^{-iqx}\,dx + \frac{1}{2}\left(A - \frac{\mathring{A}}{i\sigma}\right)e^{-i\sigma t}\int_{-\infty}^{\infty} \chi_{K\sigma}(x)\,e^{-iqx}\,dx \tag{5.5}$$

(the zero subscript on σ has been dropped). The electric-field continuum eigenmode $\chi_{K\sigma}(x)$ (the derivative of the Barston eigenmode for the potential) has been calculated by Sedláček (1971*a*, *b*):

$$\chi_{K\sigma}(x) = D_1(\sigma)\,\delta(x - x_p(\sigma)) + P\frac{\phi(x,\sigma)}{\sigma^2 - \omega_p^2(x)}, \tag{5.6}$$

where $\phi(x,\sigma)$ stands for the solution of the regular integral equation of the first kind

$$\phi(x,\sigma) = -\int_{-\infty}^{\infty} H(x,x',\sigma)\,\phi(x',\sigma)\,dx', \tag{5.7}$$

$$H(x,x',\sigma) = \frac{Q(x,x')-Q(x,x_p(\sigma))}{\sigma^2-\omega_p^2(x')}, \tag{5.8}$$

and

$$D_1(\sigma) = -P\int_{-\infty}^{\infty} \frac{\phi(x,\sigma)}{\sigma^2-\omega_p^2(x)}\,dx. \tag{5.9}$$

With a general plasma frequency profile $\omega_p(x)$, it is not possible to calculate the Fourier transform of the continuum eigenmode (5.6) in closed form. Nevertheless, utilizing Lighthill's method (theorem 19 of Lighthill 1958), it is possible to find the asymptotic behaviour of the Fourier transform for $|q|\to\infty$, which is sufficient to calculate the transmission and reflection coefficients. To apply this theorem, we write

$$\int_{-\infty}^{\infty} \chi_{K\sigma}(x)\,e^{-iqx}\,dx = D_1(\sigma)\int_{-\infty}^{\infty} \delta(x-x_p(\sigma))\,e^{-iqx}\,dx$$
$$+P\int_{-\infty}^{\infty} \frac{1}{2\omega_p(x)}\,\frac{\phi(x,\sigma)}{\sigma-\omega_p(x)}\,e^{-iqx}\,dx - P\int_{-\infty}^{\infty} \frac{1}{2\omega_p(x)}\,\frac{\phi(x,\sigma)}{\sigma+\omega_p(x)}\,e^{-iqx}\,dx. \tag{5.10}$$

The first integral is readily calculated, yielding

$$\int_{-\infty}^{\infty} \delta(x-x_p(\sigma))\,e^{-iqx}\,dx = e^{-ix_p(\sigma)\,q}. \tag{5.11}$$

Assume for definiteness $\omega_{p1} < \sigma < \omega_{p2}$. Then the integrand of the third integral is an ordinary function of x, absolutely integrable in the infinite interval, so that, by the Riemann–Lebesgue lemma, it contributes zero asymptotically. It therefore remains to calculate the asymptotic behaviour of the second integral. By Lighthill's theorem, for a monotonic density profile, this is determined by the only singularity of the integrand at the point $x_1 = x_p(\sigma)$. To apply the theorem, we must find a function $F_1(x)$ such that the function

$$P\frac{1}{2\omega_p(x)}\,\frac{\phi(x,\sigma)}{\sigma-\omega_p(x)} - F_1(x) \tag{5.12}$$

has an absolutely integrable nth derivative in an interval including the singular point x_1, $F_1(x)$ being a linear combination of elementary generalized functions, for which Lighthill gives their Fourier transforms. We construct $F_1(x)$ easily by expanding the denominator of the integrand around the singular point x_1,

$$\sigma-\omega_p(x) = -\omega_p'(x_p(\sigma))\,[x-x_p(\sigma)]+\dots, \tag{5.13}$$

and putting

$$F_1(x) = -\frac{\phi(x_p(\sigma),\sigma)}{2\omega_p(x_p(\sigma))\,\omega_p'(x_p(\sigma))}P\frac{1}{x-x_p(\sigma)} = -\frac{x_p'(\sigma)}{2\sigma}\,\phi(x_p(\sigma),\sigma)\,P\frac{1}{x-x_p(\sigma)}, \tag{5.14}$$

where we have used the identities $\omega_p(x_p(\sigma)) = \sigma$ and $\omega_p'(x_p(\sigma)) = 1/x_p'(\sigma)$ (valid for $\sigma > 0$). It is not difficult to show that the function defined by (5.12) with this

$F_1(x)$ has at least its first derivative absolutely integrable in an interval including the singular point x_1. Since, further, the Fourier transform of $F_1(x,\sigma)$ is proportional to

$$P\int_{-\infty}^{\infty}\frac{1}{x-x_p(\sigma)}e^{-iqx}\,dx = -i\pi\,e^{-ix_p(\sigma)\,q}\,\mathrm{sgn}\,(q), \tag{5.15}$$

we finally obtain by Lighthill's theorem the asymptotic formula

$$\int_{-\infty}^{\infty}\chi_{K\sigma}(x)\,e^{-iqx}\,dx \rightarrow e^{-ix_p(\sigma)\,q}D^{\pm}(\sigma)+o(q^{-1}) \tag{5.16}$$

as $q\rightarrow\pm\infty$. Here we have used notation consistent with Sedláček (1971*a*, *b*):

$$D^{\pm} = D_1(\sigma)\pm iD_2(\sigma), \tag{5.17}$$

where $D_1(\sigma)$ is given by (5.9) and

$$D_2(\sigma) = \pi\frac{x'_p(\sigma)}{2\sigma}\phi(x_p(\sigma),\sigma). \tag{5.18}$$

Sedláček (1971*b*) has shown that

$$D^{\pm}(\sigma) = -\int_{-\infty}^{\infty}\frac{\phi(x,\sigma)}{(\sigma\pm i0)^2-\omega'_p(x)}dx. \tag{5.19}$$

These are the boundary values on the real axis from above and from below of an analytic function $D(\omega)$ of the complex frequency ω,

$$D(\omega) = -\int_{-\infty}^{\infty}\frac{\phi(x,\omega)}{\omega^2-\omega_p^2(x)}dx, \tag{5.20}$$

which we have identified with the 'dispersion function' (cf. Sedláček 1971*a*, *b*, 1973; Cally & Sedláček 1992).

We can now utilize these results to evaluate the asymptotic behaviour of $E_q(t)$ as expressed by (5.5):

$$E_q(t)\rightarrow\frac{1}{2}\left(A+\frac{\mathring{A}}{i\sigma}\right)D^{\pm}(\sigma)\,e^{-i[-\sigma t+x_p(\sigma)\,q]}$$
$$+\frac{1}{2}\left(A-\frac{\mathring{A}}{i\sigma}\right)D^{\pm}(\sigma)\,e^{-i[\sigma t+x_p(\sigma)\,q]} \tag{5.21}$$

as $q\rightarrow\pm\infty$. The first term represents a monochromatic wave propagating in wavenumber space in the direction of the positive q axis, and the second term one in the direction of the negative q axis. Thus, by starting from appropriate boundary conditions, we can launch either a right-propagating or a left-propagating wave, according to whether

$$A-\frac{\mathring{A}}{i\sigma} = 0 \tag{5.22}$$

or

$$A+\frac{\mathring{A}}{i\sigma} = 0. \tag{5.23}$$

We see that, for a monotonic density profile, there is no reflection – only transmission without change of amplitude. It is obvious from the foregoing discussion that the same remains true also for an arbitrary non-monotonic density profile $\omega_p(x)$. The integrand in (5.10) may then have several singularities corresponding to zero points $x_p(\sigma)$ of $\sigma - \omega_p(x) = 0$. Each of these zero points will contribute to the asymptotic formula (5.21) its own pair of monochromatic waves distinguished by their specific wavenumbers $\omega_p(\sigma)$. All these individual monochromatic waves will remain virtually uncoupled.

The transmission properties of the scattering region can most succinctly be characterized by a scattering matrix. Since the two possible modes propagating in the system are uncoupled, it is sufficient to consider a 1×1 matrix, i.e. a number, separately for each mode. To define this quantity, take first the right-propagating wave and calculate its amplitude at $q = -l$, l much larger than the characteristic dimension of the scattering region,

$$E_{-l}(t) = \left(A + \frac{\mathring{A}}{i\sigma}\right) D^{-}(\sigma)\, e^{i\sigma t}\, e^{ix_p(\sigma)\, l}; \tag{5.24}$$

then calculate its amplitude at $q = l$,

$$E_{l}(t) = \left(A + \frac{\mathring{A}}{i\sigma}\right) D^{+}(\sigma)\, e^{i\sigma t}\, e^{-ix_p(\sigma)\, l}. \tag{5.25}$$

The scattering matrix is defined by the ratio of these two amplitudes,

$$S = \frac{E_l(t)}{E_{-l}(t)} = e^{-2ix_p(\sigma)\, l} \frac{D^{+}(\sigma)}{D^{-}(\sigma)}, \tag{5.26}$$

and similarly for the left-propagating wave,

$$S = \frac{E_{-l}(t)}{E_l(t)} = e^{2ix_p(\sigma)\, l} \frac{D^{-}(\sigma)}{D^{+}(\sigma)}. \tag{5.27}$$

The scattering matrix is unitary,

$$S(\sigma)\, S^*(\sigma) = |S(\sigma)|^2 = 1, \tag{5.28}$$

just as in any instance of classical or quantum-mechanical real-space energy-conserving scattering (Nussenzveig 1972; Livshits 1973). Thus the transmitted wave leaves the scattering region with an unaltered amplitude, having undergone, in comparison with free propagation, only a change in phase.

6. Metastable states in wavenumber space and exponential damping

The scattering state of a monochromatic wave in wavenumber space is a continuum eigenmode corresponding to the continuous spectrum $\mathscr{S}$ of the general eigenmode equation

$$\omega^2 E_{q\omega} - \int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'\omega}\, dq' = -\int_{-\infty}^{\infty} Q_{qq'} E_{q'\omega}\, dq' \tag{6.1}$$

obtained from (2.30) by substituting $E_q(t) = E_{q\omega} e^{-i\omega t}$ ('stationary solutions'). The scattering state of a monochromatic wave need not, of course, be the only possible state of the system. In analogy with quantum mechanics or electromagnetic scattering, the existence of discrete eigenmodes ('bound states') or solutions exponentially decaying in time ('metastable states') may be expected. We shall try to uncover such possible solutions by constructing a general solution to the eigenmode equation (6.1), then imposing appropriate boundary conditions.

Such an analysis is difficult to perform for a general density profile, but simplifies enormously in case of a steep density profile. Then the integral kernel $Q_{qq'}$ degenerates into a simple dyad (2.22), so that (6.1) reads

$$\omega^2 E_{q\omega} - \int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'\omega}\, dq' = \frac{1}{2\pi} \Delta a_q \int_{-\infty}^{\infty} b_{q'} E_{q'\omega}\, dq'. \tag{6.2}$$

Following the standard procedure (Friedman 1956), regard, for a moment, the integral on the right-hand side of (6.2) as a known quantity

$$\beta_\omega = \int_{-\infty}^{\infty} b_{q'} E_{q'\omega}\, dq', \tag{6.3}$$

and rewrite (6.2) correspondingly as

$$\omega^2 E_{q\omega} - \int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'\omega}\, dq' = \frac{1}{2\pi} \Delta \beta_\omega a_q. \tag{6.4}$$

This equation can be treated as an ordinary non-homogeneous differential equation of infinite order. To see this, expand $E_{q'\omega}$ into a Taylor series around the point $q' = q$,

$$E_{q'} = E_q - \frac{1}{1!}\frac{\partial E_q}{\partial q}(q-q') + \frac{1}{2!}\frac{\partial^2 E_q}{\partial q^2}(q-q')^2 + \ldots + (-1)^n \frac{1}{n!}\frac{\partial^n E_q}{\partial q^n}(q-q')^n + \ldots, \tag{6.5}$$

and insert this into the convolution integral in (6.4):

$$\int_{-\infty}^{\infty} \Omega_{q-q'} E_{q'}\, dq' = M_0 E_q - \frac{1}{1!} M_1 \frac{\partial E_q}{\partial q} + \frac{1}{2!} M_2 \frac{\partial^2 E_q}{\partial q^2} + \ldots + (-1)^n \frac{1}{n!} M_n \frac{\partial^n E_q}{\partial q^n} + \ldots, \tag{6.6}$$

where

$$M_n = \int_{-\infty}^{\infty} \Omega_q q^n\, dq, \tag{6.7}$$

$n = 0, 1, 2, \ldots$, are the moments of Ω_q with respect to the origin. Equation (6.4) can therefore be written as

$$(\omega^2 - M_0) E_{q\omega} + \frac{1}{1!} M_1 \frac{\partial E_{q\omega}}{\partial q} - \frac{1}{2!} M_2 \frac{\partial^2 E_{q\omega}}{\partial q^2} + \ldots$$
$$- (-1)^n \frac{1}{n!} M_n \frac{\partial^n E_{q\omega}}{\partial q^n} + \ldots = \frac{1}{2\pi} \Delta \beta_\omega a_q, \tag{6.8}$$

a differential equation of infinite order. Heuristically, the general solution of such an equation is given by the sum of the general solution of the homogeneous equation and a particular solution of the complete non-homogeneous equation. In §3 the general solution of the homogeneous equation was found to be

$$E_q = C_\omega e^{-ix_p(\omega) q}, \tag{6.9}$$

where $x_p(\omega)$ is the solution of the dispersion relation (3.6) with respect to x, and C_ω is arbitrary. A particular solution of the non-homogeneous equation is easily found by returning to the original x representation:

$$[\omega^2-\omega_p^2(x)]E_\omega(x) = \frac{1}{2\pi}\Delta\beta_\omega a(x). \tag{6.10}$$

Thus in this representation a particular solution is

$$E_\omega(x) = \frac{1}{2\pi}\Delta\beta_\omega \frac{a(x)}{\omega^2-\omega_p^2(x)}. \tag{6.11}$$

In the q representation this corresponds to

$$E_{q\omega} = \frac{1}{2\pi}\Delta\beta_\omega \int_{-\infty}^{\infty} \frac{a(x)}{\omega^2-\omega_p^2(x)} e^{-iqx}\,dx, \tag{6.12}$$

so that finally the general solution of (6.4) is

$$E_{q\omega}^{\text{gen}} = C_\omega e^{-ix_p(\omega)q} + \frac{1}{2\pi}\Delta\beta_\omega \int_{-\infty}^{\infty} \frac{a(x)}{\omega^2-\omega_p^2(x)} e^{-iqx}\,dx. \tag{6.13}$$

To find β_ω, we multiply this equation by b_q and integrate over the infinite interval. This makes it possible to express β_ω in terms of $a(x)$, $b(x)$ and C_ω,

$$\beta_\omega = \frac{2\pi b(-x_p(\omega))}{1-\Delta\displaystyle\int_{-\infty}^{\infty} a(x)\,b(-x)\,[\omega^2-\omega_p^2(x)]^{-1}\,dx} C_\omega, \tag{6.14}$$

and finish the calculation of the general solution (3.9):

$$E_{q\omega}^{\text{gen}} = C_\omega\left\{\left[1-\Delta\int_{-\infty}^{\infty}\frac{a(x)\,b(-x)}{\omega^2-\omega_p^2(x)}dx\right]e^{-ix_p(\omega)q} + \Delta b(-x_p(\omega))\int_{-\infty}^{\infty}\frac{a(x)}{\omega^2-\omega_p^2(x)}e^{-iqx}\,dx\right\}, \tag{6.15}$$

where we have utilized the arbitrariness of C_ω. This result can be simplified as follows. By (2.28) and (2.29), we have

$$1-\Delta\int_{-\infty}^{\infty}\frac{a(x)\,b(-x)}{\omega^2-\omega_p^2(x)}dx = 1+\int_{-\infty}^{\infty}\frac{\mathscr{Q}(x)\,\mathscr{R}(x)}{\omega^2-\omega_p^2(x)}dx$$
$$= 1+\int_{-\infty}^{\infty}\mathscr{Q}(x)\frac{\mathscr{R}(x)-\mathscr{R}(x_p(\omega))}{\omega^2-\omega_p^2(x)}dx + \mathscr{R}(x_p(\omega))\int_{-\infty}^{\infty}\frac{\mathscr{Q}(x)}{\omega^2-\omega_p^2(x)}dx. \tag{6.16}$$

Sedláček (1971*b*) found that in the limit of a steep density profile the integral kernel $H(x,x',\omega)$ defined by (5.8),

$$H(x,x',\omega) = \mathscr{Q}(x)\frac{\mathscr{R}(x')-\mathscr{R}(x_p(\omega))}{\omega^2-\omega_p^2(x')}, \tag{6.17}$$

becomes equal to $-\tfrac{1}{2}K\mathscr{Q}(x)$. Hence in this limit

$$\frac{\mathscr{R}(x)-\mathscr{R}(x_p(\omega))}{\omega^2-\omega_p^2(x)} = -\tfrac{1}{2}K. \tag{6.18}$$

Since, further, by (2.28),

$$\int_{-\infty}^{\infty} \mathscr{Q}(x)\, dx = \frac{2}{K}, \tag{6.19}$$

we obtain

$$1-\Delta \int_{-\infty}^{\infty} \frac{a(x)\, b(-x)}{\omega^2-\omega_p^2(x)} dx = \mathscr{R}(x_p(\omega)) \int_{-\infty}^{\infty} \frac{\mathscr{Q}(x)}{\omega^2-\omega_p^2(x)} dx, \tag{6.20}$$

so that finally

$$E_{q\omega}^{\text{gen}} = C_\omega \left[-e^{-ix_p(\omega)\, q} \int_{-\infty}^{\infty} \frac{\mathscr{Q}(x)}{\omega^2-\omega_p^2(x)} dx + \int_{-\infty}^{\infty} \frac{\mathscr{Q}(x)}{\omega^2-\omega_p^2(x)} e^{-iqx}\, dx \right], \tag{6.21}$$

where we have again utilized the arbitrariness of C_ω.

The formula (6.21) comprises all possible solutions to (6.2). No boundary condition has been imposed as yet, so that all ω, real or complex, are still admissible. To restrict the class of admissible solutions, we subject the general solution (6.21) to boundary conditions at infinity.

Let us first analyse those solutions that vanish for both $q \to -\infty$ and $q \to +\infty$. This boundary condition corresponds to discrete eigenmodes. If the corresponding discrete eigenvalue ω is not embedded in the continuous spectrum $\mathscr{S}$, the second integral in (6.21) is absolutely integrable in the infinite interval, so that, by the Riemann–Lebesgue lemma, the integral vanishes for $q \to \pm\infty$. In order that the first term in (6.21) may also vanish for both $q \to -\infty$ and $q \to +\infty$, the relation

$$\int_{-\infty}^{\infty} \frac{\mathscr{Q}(x)}{\omega^2-\omega_p^2(x)} dx = 0, \tag{6.22}$$

must be valid; this is the so-called 'dispersion relation', which has already been derived in this form but by another procedure by Sedláček (1971*b*) and Cally & Sedláček (1992). In another form, though for a specific density profile, it was derived by Sedláček (1971*a*). In these papers we stressed that there are in general no zeros in the principal analytic branch of the dispersion function as defined directly by the Cauchy-type integral (6.22) in the complex ω plane cut along the 'spectral cuts' (cuts identical with the intervals of the continuous spectrum).† Thus there are, in general, no discrete eigenmodes satisfying the boundary condition that they vanish simultaneously for $q \to -\infty$ and $q \to +\infty$. The only solutions that satisfy this boundary condition are non-stationary wave packets propagating in wavenumber space, unobtainable of course from (6.1).

Sedláček (1971*a, b*) has also shown that the dispersion function has an infinity of analytic branches generated by analytic continuation of the principal (physical) branch defined directly by the Cauchy-type integral in (6.22). The corresponding Riemann surface is infinitely multi-sheeted. There are complex zeros of these unphysical analytic branches, of which those located near the spectral cuts are related to the surface wave of a plasma with a discontinuous density profile. For these complex values of ω, the second term in (6.21) will again vanish for both $q \to -\infty$ and $q \to +\infty$, but the first term will not because

† There may be real zeros embedded in the continuous spectrum (Sedláček 1971*a*), but they are rather exceptional.

(6.22) does not hold in this case. Since for complex ω, $x_p(\omega)$ is also complex, $e^{-ix_p(\omega)q}$ may vanish at most at one infinity. At the opposite infinity $E^{\text{gen}}_{q\omega}$ will exponentially blow up: an 'exponential catastrophe' occurs (for a discussion of this phenomenon see e.g. Adam 1986). This demonstrates in an illuminating way that no acceptable eigenfunctions correspond to complex zeros of the dispersion function off the principal sheet, as is sometimes incorrectly claimed. The true role of these complex zeros can only be disclosed by solving the initial-value problem via the Laplace transform (Sedláček 1971*a*).

Finally, if the boundary condition is imposed that the solution be non-zero but finite at infinity, the formula (6.21) generates for real $\omega = \sigma$ a special class of singular (square non-integrable) solutions describing the scattering states of a monochromatic wave in wavenumber space as dealt with in §5. Indeed, if we calculate the boundary values on the real axis of $E^{gen}_{q\omega}$, (6.21), from above and from below, we obtain

$$E^{\pm}_{q\sigma} = C_\sigma \int_{-\infty}^{\infty} \left[-\delta(x-x_p(\sigma)) \int_{-\infty}^{\infty} \frac{\mathscr{Q}(x')}{(\sigma \pm i0)^2 - \omega_p^2(x')} dx' + \frac{\mathscr{Q}(x)}{(\sigma \pm i0)^2 - \omega_p^2(x)} \right] e^{-iqx}\, dx. \tag{6.23}$$

This result can be shown to be the Fourier transform of the electric-field continuum eigenmode $\chi_{K\sigma}(x)$, (5.6), whose alternative boundary-value form $\chi^{\pm}_{K\sigma}(x)$ was calculated by Sedláček (1971*a*, *b*),

$$\chi^{\pm}_{K\sigma}(x) = D^{\pm}(\sigma)\, \delta(x-x_p(\sigma)) + \frac{\phi(x,\sigma)}{(\sigma \pm i0)^2 - \omega_p^2(x)}, \tag{6.24}$$

with $\phi(x,\sigma)$ again being given by the integral equation (5.7), and

$$D^{\pm}(\sigma) = -\int_{-\infty}^{\infty} \frac{\phi(x,\sigma)}{(\sigma \pm i0)^2 - \omega_p^2(x)} dx. \tag{6.25}$$

In the limit of a steep density profile the solution of (5.7) is very simple, in view of (6.17),

$$\phi(x,\sigma) = \mathscr{Q}(x), \tag{6.26}$$

so that in this limit

$$\chi^{\pm}_{K\sigma}(x) = -\delta(x-x_p(\sigma)) \int_{-\infty}^{\infty} \frac{\mathscr{Q}(x)}{(\sigma \pm i0)^2 - \omega_p^2(x)} dx + \frac{\mathscr{Q}(x)}{(\sigma \pm i0)^2 - \omega_p^2(x)}, \tag{6.27}$$

and (6.23) can therefore be written as

$$E^{\pm}_{q\sigma} = C_\sigma \int_{-\infty}^{\infty} \chi^{\pm}_{K\sigma}(x)\, e^{-iqx}\, dx. \tag{6.28}$$

It is easy to show that $\chi^{\pm}_{K\sigma}(x) = \chi_{K\sigma}(x)$ (Sedláček 1971*b*), so that

$$E_{q\sigma} = C_\sigma \int_{-\infty}^{\infty} \chi_{K\sigma}(x)\, e^{-iqx}\, dx, \tag{6.29}$$

the Fourier transform of the continuum eigenmode.

7. Conclusions

The principal aim of this first paper was to show that the phase-mixing phenomena encountered in conservative dynamical systems with continuous spectra of eigenfrequencies (like inhomogeneous cold plasmas or inhomogeneous magnetofluids), and especially the puzzling damping phenomena that accompany them, find their natural interpretation and explanation in wavenumber space, where they become phenomena of propagation, dispersion and scattering of the Fourier transform of the excitation in x space. In fact, we believe that a complete and consistent interpretation of continuum damping in conservative systems is impossible outside this conceptual framework, especially if one wants to understand clearly its distinction from the genuine damping in dissipative systems. Our new interpretation, analogous to the interpretation of metastable (or radioactive) states in quantum mechanics, is at the same time simple and transparent.

Although we have not attempted to give a general solution of the initial-value problem corresponding to the behaviour of a general wave packet in wavenumber space, the analysis performed in this part of the paper enables us to outline a sufficiently complete picture of the processes unfolding in wavenumber space after an initial excitation has been launched. Suppose first that the initial excitation is concentrated within the scattering region – that is to say, around the origin of wavenumber space. In other words, we assume that only low-wavenumber modes are excited, which in x space corresponds to a field structure close to the surface wave. Such cases were studied by Cally (1991). The excitation may be thought of as being trapped in the scattering region, the trapping mechanism being provided by the feedback action of the integral kernel in (2.30). The trapping is not perfect (the scattering centre 'leaks'), and the excitation is slowly released (radiated) into infinity. Thus inside the scattering region the excitation decays exponentially, and the time scales of this decay are given by the roots of the dispersion equation (6.22). Far from the scattering region, the behaviour of the excitation is characterized by the dispersive free propagation. As the analysis of §§3 and 4 shows, the time behaviour of a freely propagating wave packet is distinctly non-exponential. These conclusions are excellently confirmed by the results of numerical calculations of Cally (1991) and Cally & Sedláček (1992), in which the distinction between the time behaviour of low and high modes is clearly manifest. There is no damping in the true sense, only an intricate redistribution of the amplitudes of the excitation in wavenumber space. An enormous advantage of the wavenumber-space description is that it completely unfolds this redistribution process, separating various decay modes and clearly showing their causes. In the original x representation all this is hopelessly convolved, without any possibility of resolution.

Suppose next that in wavenumber space the system is excited predominantly outside the scattering region. In this case we have a true scattering process. The disturbance splits into two, one part propagating to the left, the other to the right. The part propagating away from the scattering region disperses into infinity. The part transmitted through the scattering region undergoes distortion, and is partly trapped but not reflected. The trapped part behaves as described above, while the transmitted part again disperses into infinity, drawing a decaying tail from the scattering region.

It is obvious that if we follow the time evolution of the excitation in a fixed point of wavenumber space then damping is observed. Returning to the x representation, we can speak only about damping of certain functionals of the solution. But in this sense of the word, damping is observed with any propagation, as described for example by the classical wave equation: if we observe a propagating wave packet at a fixed point of space, its amplitude decays.

Exactly this kind of behaviour has been found by Sedláček, Adam & Roberts (1986) with an inhomogeneous string consisting of a finite low-mass segment attached to a massive infinite string. This problem has an enormous advantage in that it admits an exact closed-form solution of the initial-value problem both by Laplace transformation and by normal-mode (Fourier integral) analysis. The results are obtained in closed form and are easy to interpret. If the finite segment is excited, the disturbance slowly leaks along the infinite string. The complete system has only a continuous spectrum, and pure discrete eigenmodes characteristic of oscillations of finite strings do not exist. Instead, transient solutions are found, consisting of an almost-standing-wave pattern within the finite segment, exponentially damped, and a disturbance propagating along the string to infinity. In this case there is no dispersion, so that the leaking disturbance propagates without any change of form and with an undiminished amplitude. Again, a dispersion relation is found, with zeros off the physical sheet of its Riemann surface. 'Eigenmodes' corresponding to these zeros suffer from exponential catastrophe. Thus, to analyse the behaviour of the system, an initial-value problem must be solved. If this is done via the Laplace transform, an expansion is obtained in terms of the residues corresponding to the poles generated by the zeros of the dispersion function. Continuum eigenmodes also exist, interpreted again as scattering states of a monochromatic wave incoming and outgoing along the attached infinite string. One can thus regard the inhomogeneous infinite string as an archetype for the interpretation of phenomena like cold inhomogeneous plasma oscillations in wavenumber space.

In the present context it also becomes clear why the continuous spectrum and damping phenomena always occur together. First, we note that there are a number of physical systems with a continuous spectrum for which a description in terms of an equation analogous to (2.1) is known. Such is the case with the Vlasov–Poisson equation (van Kampen 1955; Case 1959), with a general set of Vlasov–Maxwell equations (Sedláček 1972), with cold inhomogeneous plasma oscillations (or, equivalently, inhomogeneous MHD oscillations) (Sedláček 1971*b*; Uberoi & Sedláček 1992), with MHD oscillations of a tokamak plasma (Ye, Sedláček & Mahajan 1993), with inhomogeneous fluids (Rosencrans & Sattinger 1966; Kamp 1991, 1992) and with neutron transport theory (Sattinger 1966*a*, *b*). We therefore dare to conjecture that whenever a continuous spectrum exists, the problem is describable by an equation consisting of a multiplication operator, which generates the continuous spectrum, perturbed by an integral operator.† Such a description may not be apparent at first sight, but a suitable transformation to this form or to its multi-dimensional analogue should always be feasible. Fourier transformation converts such an equation into a generalized wave equation describing propagation and scattering phenomena. In fact, Fourier transformation of the multiplication operator generates a convolution, and, as we have seen in §6, this can be expanded into an infinite series of derivatives of higher and higher order. This part of the transformed equation represents propagation phenomena. The Fourier trans-

† A detailed mathematical analysis of equations of this kind is given by Friedrichs (1948).

formation of the integral operator creates an inhomogeneity for the propagating disturbance, giving rise to scattering and trapping. If the trapping is imperfect, accompanied by leaking (or, in other words, by radiation out of the scattering region), exponential damping phenomena will occur. Perfect trapping would create a discrete eigenmode (bound state). If the propagation is dispersive, there may be various non-exponential decay processes connected with the distortion of the disturbance travelling to infinity.

Finally, it should be noted that the theory of open systems (Livshits 1973), based essentially on the theory of non-Hermitian operators, provides an excellent conceptual framework for the analysis of any propagation problem. This theory has already been utilized by Sedláček & Nocera (1992) to interpret Vlasov–Poisson plasma oscillations in the Fourier transform of velocity space.

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