As an application, one can, in the presence of Schur indices, look at all fields of minimal degree affording ρ. Experimental evidence suggests that in this situation one can always find both field where ρ can be made integral over as well as fields where this cannot be done. In particular, we have a large number of explicit fields and representations that cannot be made integral.

A short survey on Coclass Graphs

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Leedham-Green & Newman [11] defined the coclass of a p-group of order p^n and nilpotency class c as $r = n - c$. The investigation of the p-groups of a fixed coclass led to deep results in p-group theory (see the book of Leedham-Green & McKay [10]), applications (see for example [1, 12]), and generalisations to other algebraic objects (see for example [5, 8]). In the last decade, the focus in coclass theory is on the investigation of the coclass graph $\mathcal{G}(p,r)$ associated with the finite p-groups of coclass r. It is conjectured that this infinite graph can be described by a finite subgraph and several "periodic patterns". The aim of this talk is to give a survey on the known periodicity results, the outstanding problems, and a recent new result [4] for the graph $\mathcal{G}(p,1)$. Some details are given below.

1. Coclass graphs

The coclass graph $\mathcal{G}(p,r)$ has as vertices the isomorphism type representatives of the finite p-groups of coclass r, and there is an edge $H \to G$ if and only if H is isomorphic to $G/\gamma(G)$ where $\gamma(G)$ is the last non-trivial term in the lower central series of G. It is a deep result that $\mathcal{G}(p,r)$ can be partitioned into a finite subgraph and finitely many so-called coclass trees, which are infinite trees having exactly one infinite path starting at their root. Let $\mathcal{T}$ be such a coclass tree with maximal infinite path $S_t, S_{t+1}, \ldots$ where S_n has order p^n. The n-th branch $\mathcal{B}_n$ of $\mathcal{T}$ is the finite subtree of $\mathcal{T}$ induced by all descendants of S_n which are not descendants of S_{n+1}; clearly, the structure of these branches determines $\mathcal{T}$. For a positive integer k let $\mathcal{B}_n(k)$ be the pruned subtree of $\mathcal{B}_n$ induced by the groups in $\mathcal{B}_n$ of distance at most k to the root S_n of $\mathcal{B}_n$.

2. Periodicity results

Motivated by computational work of Newman & O'Brien, it has been proved by du Sautoy [14] and Eick & Leedham-Green [6] that for every coclass tree $\mathcal{T}$ and every positive integer k there exist integers $f = f(\mathcal{T}, k)$ and $d = d(\mathcal{T})$ such that the pruned branches $\mathcal{B}_n(k)$ and $\mathcal{B}_{n+d}(k)$ are isomorphic for all $n \geq f$. This shows that the pruned tree $\mathcal{T}_{(k)}$ with branches $\mathcal{B}_t(k), \mathcal{B}_{t+1}(k), \ldots$ has a periodic structure and can be described by a finite subgraph. There exists $k > 0$ such that $\mathcal{T} = \mathcal{T}_{(k)}$ for all coclass trees in $\mathcal{G}(p,r)$ if and only if $p = 2$ or $(p,r) = (3,1)$; in these cases the structure of $\mathcal{G}(p,r)$ is already determined by a finite subgraph. However, for all other values of p and r there exist coclass trees with $\mathcal{T} \neq \mathcal{T}_{(k)}$ for all k, and the periodic pattern proved in [6, 14] is not able to describe the structure of such a coclass tree completely. In other words, it remains to describe the growth (in depth and width) of the branches in such coclass trees. Computer experiments and results for $\mathcal{G}(5,1)$ and $\mathcal{G}(3,2)$ suggest that this can be done by using a second periodic pattern, see [2, 3, 9, 13] for some conjectural descriptions. Most recently, Conjecture W in [7] suggests another construction of $\mathcal{T}$ from a finite subgraph. In [4] we give the first explicit evidence in support of Conjecture W in the context of coclass trees whose branches grow in depth and width. More precisely, we consider the (unique) coclass tree in $\mathcal{G}(p,1)$ with $p \geq 7$ and define $\mathcal{S}_n^*$ to be the subtree induced by the so-called skeleton groups in $\mathcal{B}_n(n-2p+8)$ with automorphism group order divisible by $p-1$. We show that Conjecture W holds for these subtrees $\mathcal{S}_n^*$; we refer to [4] for details and a report on further computational evidence.

References

[1] M. Couson. *Character degrees of finite p-groups by coclass.* J. Algebra **418**, 91–109 (2014).

[2] H. Dietrich, B. Eick, and D. Feichtenschlager. *Investigating p-groups by coclass with GAP.* Contemp. Math. AMS, Providence, RI, Vol. **470**, 45–61 (2008).

[3] H. Dietrich. *A new periodic pattern in the graph of p-groups of maximal class.* Bull. London Math. Soc., **42**, 1073–1088 (2010).

[4] H. Dietrich and B. Eick. *Finite p-groups of maximal class with 'large' automorphism groups.* To appear in J. Group Theory (2016).

[5] A. Distler. *Finite nilpotent semigroups of small coclass.* Comm. Algebra **42**, 1136–1150 (2014).

[6] B. Eick and C. R. Leedham-Green. *On the classification of prime-power groups by coclass.* Bull. London Math. Soc. **40**, 274–288 (2008).

[7] B. Eick, C. R. Leedham-Green, M. F. Newman, and E. A. O'Brien. *On the classification of groups of prime-power order by coclass: The 3-groups of coclass 2*, Internat. J. Algebra Comput. **23**, 1243–1288 (2013).

[8] B. Eick and T. Moede. *Nilpotent associative algebras and coclass theory.* J. Algebra **434**, 249–260 (2015).

[9] C. R. Leedham-Green and S. McKay. *On the classification of p-groups of maximal class.* Quart. J. of Math. Oxford **35**, 293–304 (1984).

[10] C. R. Leedham-Green and S. McKay. The structure of groups of prime power order. London Mathematical Society Monographs. Oxford Science Publications, 2002.

[11] C. R. Leedham-Green and M. F. Newman. *Space groups and groups of prime-power order I.* Archiv der Mathematik **35** (1980), 193–203.

[12] P. Moravec. *On the Schur multipliers of finite p-groups of given coclass.* Israel J. Math. **185**, 189–205 (2011).

[13] M. F. Newman. *Groups of prime-power order* Groups-Canberra 1989, Lecture notes in math., **1456**, Springer, 49-62 (1990).

[14] M. du Sautoy. *Counting p-groups and nilpotent groups.* Inst. Hautes Etudes Sci. Publ. Math. **92**, 63–112 (2001).