V. I. Korzyuk, N. V. Vinh, N. T. Minh, Conservation law for the Cauchy–Navier equation of elastodynamics wave via Fourier transform, *Tr. Inst. Mat.*, 2016, Volume 24, Number 1, 100–106

Национальная академия наук Беларуси
Труды Института математики. 2016. Том 24. № 1. С. 100–106

УДК 517.956

CONSERVATION LAW FOR THE CAUCHY–NAVIER EQUATION OF ELASTODYNAMICS WAVE VIA FOURIER TRANSFORM

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Поступила 28.04.2016

In this paper, we use the method of Fourier analysis to derive the formula of the total energy for the Cauchy problem of the Cauchy–Navier elastodynamics wave equation describing the motion of an isotropic elastic body. The conservation law of the total energy is obtained and consequently, the global uniqueness of the solution to the problem is implied.

1. Introduction. Methods of Fourier analysis are widely used for studying the theory of partial differential equations (PDEs). An important aspect is to find fundamental and classical solutions of linear PDEs via Fourier transform. By the isometric property of Fourier transform, one can estimate energy functionals of a solution to some concrete mathematical modelling problems in physics. Our main work is to derive the formula of the total energy for the Cauchy problem of the Cauchy–Navier elastodynamics wave equation describing the motion of an isotropic elastic body. For explanations about physical context related to the mathematical theory of linear elasticity, we refer the readers to [4, 5, 6].

Let us resume about the mathematical formulation and some necessary notations. Remark that, the Newton's second law leads to the Cauchy's motion equation of an elastic body, which takes the form

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\mathbf{f}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

where $\boldsymbol{\sigma}=(\sigma_{ij}(\overline{x},t))_{i,j=\overline{1,n}}$ is the Cauchy stress tensor field, $\mathbf{u}=(u_i(\overline{x},t))_{i=\overline{1,n}}$ is the displacement vector field, $\mathbf{f}=(f_i(\overline{x},t))_{i=\overline{1,n}}$ is the vector field of body force per unit volume, ρ is the mass density and $\dfrac{\partial^2\mathbf{u}}{\partial t^2}=\left(\dfrac{\partial^2u_1}{\partial t^2},\dfrac{\partial^2u_2}{\partial t^2},\ldots,\dfrac{\partial^2u_n}{\partial t^2}\right)$. The infinitesimal strain tensor field is given by the equation

$$\boldsymbol{\varepsilon}=\frac{1}{2}[\boldsymbol{\nabla}\mathbf{u}+(\boldsymbol{\nabla}\mathbf{u})^T], \tag{2}$$

where $(\boldsymbol{\nabla}\mathbf{u})^T$ — transformation matrix of $\boldsymbol{\nabla}\mathbf{u}$.

Moreover, the Hooke's law for homogeneous isotropic bodies has the form

$$\boldsymbol{\sigma}=\lambda\,\mathrm{trace}(\boldsymbol{\varepsilon})\,\mathbf{I}+2\mu\,\boldsymbol{\varepsilon}, \tag{3}$$

where $\lambda, \mu > 0$ are Lamé's parameters, $\operatorname{trace}(\boldsymbol{\varepsilon})$ — trace of matrix $(\boldsymbol{\varepsilon})$ and $\mathbf{I}$ is the second-order identity tensor. Substituting the strain-displacement equation (2) and the Hooke's equation (3) into the equilibrium equation (1), we obtain the Cauchy–Navier elastodynamic wave equation

$$(\lambda + \mu)\nabla \operatorname{div}(\mathbf{u}) + \mu\Delta\mathbf{u} + \mathbf{f} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{4}$$

where $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \ldots + \dfrac{\partial^2}{\partial x_n^2}$.

This equation in the Cartesian coordinates has the form

$$(\lambda + \mu)\frac{\partial}{\partial x_k}\left(\sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}\right) + \mu\Delta u_k + f_k = \rho\frac{\partial^2 u_k}{\partial t^2}, \quad k = \overline{1, n}.$$

The Cauchy problem consists in finding a vector function $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, which satisfies (4) and the initial condition as follows

$$\mathbf{u}(\overline{x}, t)|_{t=0} = \boldsymbol{\varphi}_0(\overline{x}), \quad \left.\frac{\partial \mathbf{u}}{\partial t}(\overline{x}, t)\right|_{t=0} = \boldsymbol{\varphi}_1(\overline{x}). \tag{5}$$

It is easy to show that the Cauchy problem (4), (5) has a Cauchy–Kovalevski–Somigliana solution in the following form

$$\mathbf{u} = \left(\frac{\partial^2}{\partial t^2} - \frac{\lambda + 2\mu}{\rho}\Delta\right)\mathbf{w} + \frac{\lambda + \mu}{\rho}\nabla \operatorname{div}(\mathbf{w}), \tag{6}$$

where $\mathbf{w}$ is a solution to the Cauchy problem of the biwave equation

$$\begin{gathered} \left(\frac{\partial^2}{\partial t^2} - \frac{\lambda + 2\mu}{\rho}\Delta\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho}\Delta\right)\mathbf{w} = \frac{\mathbf{f}}{\rho}; \\ \mathbf{w}|_{t=0} = \left.\frac{\partial \mathbf{w}}{\partial t}\right|_{t=0} = 0; \\ \left.\frac{\partial^2 \mathbf{w}}{\partial t^2}\right|_{t=0} = \boldsymbol{\varphi}_0, \quad \left.\frac{\partial^3 \mathbf{w}}{\partial t^3}\right|_{t=0} = \boldsymbol{\varphi}_1. \end{gathered} \tag{7}$$

Note that, in [3], a formula of exact solution to the Cauchy problem of the biwave equation is given by using Fourier transform.

The total energy $E(t)$ of a solution to the equation (4) is defined as the summation of the kinetic energy and the potential energy, where the kinetic energy functional is given by

$$U(t) = \frac{1}{2}\int_{\mathbb{R}^n} \rho\left|\frac{\partial}{\partial t}\mathbf{u}\right|^2 dx = \frac{1}{2}\int_{\mathbb{R}^n} \rho\sum_{k=1}^{n}\left(\frac{\partial u_k}{\partial t}\right)^2 dx,$$

and the potential energy functional or the strain energy has the following form

$$K(t) = \int_{\mathbb{R}^n}\left(\frac{1}{2}\lambda \operatorname{trace}(\boldsymbol{\varepsilon})^2 + \mu \operatorname{trace}(\boldsymbol{\varepsilon}^2)\right) dx.$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of Schwartz functions (seen by in [9] and [10]). Remark that an indefinitely differentiable function ϕ is called Schwartz function when ϕ and all its derivatives are required to be rapidly decreasing, i.e.

$$||\phi||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(\frac{\partial}{\partial x}\right)^\beta \phi(x)\right| < \infty,$$

for every multi-index α and β. The Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}[\phi](\xi) \equiv \widehat{\phi}(\xi) = \int\limits_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\phi(x)\,dx.$$

It is well-known by the Plancherel theorem that the Fourier transform is an isometry in $\mathcal{S}(\mathbb{R}^n)$ with respect to L^2-norm, i.e.

$$\int\limits_{\mathbb{R}^n} |\phi(x)|^2\,dx = \int\limits_{\mathbb{R}^n} |\widehat{\phi}(\xi)|^2\,d\xi$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$. The basic concepts of Fourier analysis and application for solving PDEs and study the conservation law of energy can be found in [7, 8].

By using some Fourier analysis techniques, we will show the exact formula of the total energy, which depends only on the initial data and is independent of time. Consequently, the global uniqueness of the solution to the Cauchy problem of the Cauchy–Navier elastodynamics wave equation is proved.

2. Main results. Let us denote $a^2 = (\lambda + 2\mu)/\rho$, $b^2 = \mu/\rho$ and assume that $\mathbf{f} = 0$, then (4) can be rewritten as the following homogeneous equation

$$\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\mathbf{u} - (a^2 - b^2)\nabla\,\mathrm{div}(\mathbf{u}) = 0, \tag{8}$$

where $\mathrm{div}(\mathbf{u}) = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \ldots + \dfrac{\partial u_n}{\partial x_n}$.

Note that, if $\mathbf{u}(\overline{x}, t) \in C^4(\mathbb{R}^n \times [0,\infty))$ is a solution to the equations (5), (8), then it is also a solution to the homogeneous biwave equation

$$\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\mathbf{u} = 0, \quad \overline{x} \in \mathbb{R}^n, \quad t > 0, \tag{9}$$

together the initial condition

$$\begin{gathered}\mathbf{u}|_{t=0} = \varphi_0, \quad \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \varphi_1,\\ \left.\frac{\partial^2 \mathbf{u}}{\partial t^2}\right|_{t=0} = \varphi_2 = (a^2 - b^2)\nabla(\mathrm{div}\,\varphi_0) + b^2\nabla^2\varphi_0,\\ \left.\frac{\partial^3 \mathbf{u}}{\partial t^3}\right|_{t=0} = \varphi_3 = (a^2 - b^2)\nabla(\mathrm{div}\,\varphi_1) + b^2\nabla^2\varphi_1.\end{gathered} \tag{10}$$

Indeed, taking divergent to the both side of (8), we have

$$\frac{\partial^2}{\partial t^2}\,\mathrm{div}(\mathbf{u}) - a^2\Delta\,\mathrm{div}(\mathbf{u}) = 0. \tag{11}$$

Taking Laplacian and doubly differentiating with respect to t, we obtain that

$$\left(\frac{\partial^2}{\partial t^2}\Delta - b^2\Delta^2\right)\mathbf{u} = (a^2 - b^2)\nabla\Delta\,\mathrm{div}(\mathbf{u}), \tag{12}$$

$$\left(\frac{\partial^4}{\partial t^4} - b^2\Delta\frac{\partial^2}{\partial t^2}\right)\mathbf{u} = (a^2 - b^2)\nabla\frac{\partial^2}{\partial t^2}\,\mathrm{div}(\mathbf{u}). \tag{13}$$

From (11), (12), (13), we have

$$\left(\frac{\partial^2}{\partial t^2}\Delta - a^2\Delta^2\right)\left(\frac{\partial^2}{\partial t^2}\Delta - b^2\Delta^2\right)\mathbf{u} =$$

$$= \left(\frac{\partial^4}{\partial t^4} - b^2\Delta\frac{\partial^2}{\partial t^2}\right)\mathbf{u} - a^2\left(\frac{\partial^2}{\partial t^2}\Delta - b^2\Delta^2\right)\mathbf{u} =$$

$$= (a^2 - b^2)\nabla\left(\frac{\partial^2}{\partial t^2}\operatorname{div}(\mathbf{u}) - a^2\Delta\operatorname{div}(\mathbf{u})\right) = 0.$$

The initial conditions (10) is easily verified from (5), (8). For more important discussions about the biwave equation, we refer to [1, 2] and [3].

Using the above connection, we will derive the formula of the total energy of a solution of (8), (5) via Fourier transform. In the next sequence, we assume that $\mathbf{u} = (u_1, \dots, u_n)$, $\boldsymbol{\varphi}_0 = (\varphi_{0,1}, \dots, \varphi_{0,n})$, $\boldsymbol{\varphi}_1 = (\varphi_{1,1}, \dots, \varphi_{1,n})$ are vectors of Schwartz functions.

Lemma 1. *The Fourier transform of solution to the homogeneous biwave equation (9), (10) has the following form*

$$\hat{\mathbf{u}}(\xi, t) = \frac{\cos(a|\xi|t)}{|\xi|^2}\langle\xi, \hat{\boldsymbol{\varphi}}_0\rangle\xi + \frac{\sin(a|\xi|t)}{a|\xi|^3}\langle\xi, \hat{\boldsymbol{\varphi}}_1\rangle\xi +$$

$$+ \frac{\cos(b|\xi|t)}{|\xi|^2}(|\xi|^2\hat{\boldsymbol{\varphi}}_0 - \langle\xi, \hat{\boldsymbol{\varphi}}_0\rangle\xi) + \frac{\sin(b|\xi|t)}{b|\xi|^3}(|\xi|^2\hat{\boldsymbol{\varphi}}_1 - \langle\xi, \hat{\boldsymbol{\varphi}}_1\rangle\xi). \tag{14}$$

Proof. Taking Fourier transform to the both sides of the equation (9), we obtain that

$$\frac{\partial^4}{\partial t^4}\hat{\mathbf{u}}(\xi, t) + (a^2 + b^2)|\xi|^2\frac{\partial^2}{\partial t^2}\hat{\mathbf{u}}(\xi, t) + a^2b^2|\xi|^4\hat{\mathbf{u}}(\xi, t) = 0.$$

This fourth ODE has the general solution, which takes the form

$$\hat{\mathbf{u}}(\xi, t) = \mathbf{C}_1\cos(a|\xi|t) + \mathbf{C}_2\sin(a|\xi|t) + \mathbf{C}_3\cos(b|\xi|t) + \mathbf{C}_4\sin(b|\xi|t),$$

where vector of parameters $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$ are determined from the initial condition (10) by solving a system of linear equations. After some simplifications, the Fourier transform of $\mathbf{u}$ is calculated as

$$\widehat{\mathbf{u}}(\xi, t) = -\frac{b^2|\xi|^2\widehat{\varphi_0}(\xi) + \widehat{\varphi_2}(\xi)}{(a^2 - b^2)|\xi|^2}\cos(a|\xi|t) - \frac{b^2|\xi|^2\widehat{\varphi_1}(\xi) + \widehat{\varphi_3}(\xi)}{(a^3 - ab^2)|\xi|^3}\sin(a|\xi|t) +$$

$$+ \frac{a^2|\xi|^2\widehat{\varphi_0}(\xi) + \widehat{\varphi_2}(\xi)}{(a^2 - b^2)|\xi|^2}\cos(b|\xi|t) + \frac{a^2|\xi|^2\widehat{\varphi_1}(\xi) + \widehat{\varphi_3}(\xi)}{(a^2b - b^3)|\xi|^3}\sin(b|\xi|t). \tag{15}$$

In the above formula, we note that

$$\begin{aligned} \hat{\boldsymbol{\varphi}}_2 &= -(a^2 - b^2)\langle\xi, \hat{\boldsymbol{\varphi}}_0\rangle\xi - b^2|\xi|^2\hat{\boldsymbol{\varphi}}_0, \\ \hat{\boldsymbol{\varphi}}_3 &= -(a^2 - b^2)\langle\xi, \hat{\boldsymbol{\varphi}}_1\rangle\xi - b^2|\xi|^2\hat{\boldsymbol{\varphi}}_1. \end{aligned} \tag{16}$$

Substituting (16) into (15), we get the formula (14).

Theorem 1. *Let* $\mathbf{u}$ *be a solution of (5), (8). Then the total energy of* $\mathbf{u}$ *takes the form*

$$E(t) = E(0) = \frac{1}{2}\int\limits_{\mathbb{R}^n}\rho\left(a^2\left(\sum_{i=1}^n\frac{\partial\varphi_{0,i}}{\partial x_i}\right)^2 + \sum_{i<j}b^2\left(\frac{\partial\varphi_{0,j}}{\partial x_i} - \frac{\partial\varphi_{0,i}}{\partial x_j}\right)^2 + \sum_{i=1}^n\varphi_{1,i}^2\right)d\xi. \tag{17}$$

Proof. We only need to verify that the total energy of a solution of the biwave equation (9), (10) also has the form (17). Applying Plancherel theorem, the strain energy functional can be rewritten as

$$K(t)=\int\limits_{\mathbb{R}^n}\left(\frac{1}{2}\lambda\,\mathrm{trace}(\hat{\boldsymbol{\varepsilon}})^2+\mu\,\mathrm{trace}(\hat{\boldsymbol{\varepsilon}}^2)\right)d\xi.$$

We note that

$$\hat{\boldsymbol{\varepsilon}}=\frac{1}{2}(\xi\hat{\mathbf{u}}^T+\hat{\mathbf{u}}\xi^T),\quad \hat{\boldsymbol{\varepsilon}}^2=\frac{1}{4}((\xi\hat{\mathbf{u}}^T)^2+(\hat{\mathbf{u}}\xi^T)^2+|\hat{\mathbf{u}}|^2\xi\xi^T+|\xi|^2\hat{\mathbf{u}}\hat{\mathbf{u}}^T).$$

Hence

$$K(t)=\frac{1}{2}\int\limits_{\mathbb{R}^n}\left(\lambda\langle\xi,\hat{\mathbf{u}}\rangle^2+\frac{1}{2}\mu(\langle\xi,\hat{\mathbf{u}}\rangle^2+|\hat{\mathbf{u}}|^2|\xi|^2)\right)d\xi=$$

$$=\frac{1}{2}\int\limits_{\mathbb{R}^n}\rho(a^2\langle\xi,\hat{\mathbf{u}}\rangle^2+b^2(|\hat{\mathbf{u}}|^2|\xi|^2-\langle\xi,\hat{\mathbf{u}}\rangle^2))\,d\xi=\frac{1}{2}\int\limits_{\mathbb{R}^n}\rho(a^2\langle\xi,\hat{\mathbf{u}}\rangle^2+b^2|\xi\times\hat{\mathbf{u}}|^2)\,d\xi=$$

$$=\frac{1}{2}\int\limits_{\mathbb{R}^n}\rho(a^2\langle\xi,\psi\rangle^2+b^2|\xi\times\eta|^2)\,d\xi,$$

where we denote

$$\psi=\cos(a|\xi|t)\hat{\boldsymbol{\varphi}}_0+\frac{\sin(a|\xi|t)}{a|\xi|}\hat{\boldsymbol{\varphi}}_1,\quad \eta=\cos(b|\xi|t)\hat{\boldsymbol{\varphi}}_0+\frac{\sin(b|\xi|t)}{b|\xi|}\hat{\boldsymbol{\varphi}}_1,$$

and for real vectors $\mathbf{a}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+\ldots+a_n\mathbf{e}_n,$ $\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+\ldots+b_n\mathbf{e}_n,$ where $\{\mathbf{e}_j\}_{j=\overline{1,n}}$ is the canonical basic of $\mathbb{R}^n,$ we use the notation of the exterior product

$$\mathbf{a}\times\mathbf{b}=\sum_{i<j}(a_ib_j-a_jb_i)[\mathbf{e}_i\times\mathbf{e}_j]\in\mathbb{R}^{n(n-1)/2}.$$

Note that $\mathbf{a}\times\mathbf{a}=0$ and we have the Lagrange's identity $|\mathbf{a}\times\mathbf{b}|^2=|\mathbf{a}|^2|\mathbf{b}|^2-\langle a,b\rangle^2.$ Applying Plancherel theorem again for the kinetic energy functional, we get that

$$U(t)=\frac{1}{2}\int\limits_{\mathbb{R}^n}\rho\left|\frac{\partial}{\partial t}\hat{\mathbf{u}}(\xi,t)\right|^2d\xi.$$

Moreover, we have

$$\frac{\partial}{\partial t}\hat{\mathbf{u}}(\xi,t)=\left\langle\xi,-a\frac{\sin(a|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_0+\frac{\cos(a|\xi|t)}{|\xi|^2}\hat{\boldsymbol{\varphi}}_1\right\rangle\xi+$$

$$+|\xi|^2\left(-b\frac{\sin(b|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_0+\frac{\cos(b|\xi|t)}{|\xi|^2}\hat{\boldsymbol{\varphi}}_1\right)-\left\langle\xi,-b\frac{\sin(b|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_0+\frac{\cos(b|\xi|t)}{|\xi|^2}\hat{\boldsymbol{\varphi}}_1\right\rangle\xi=$$

$$=\langle\xi,\alpha\rangle\xi+|\xi|^2\beta-\langle\xi,\beta\rangle\xi,$$

where

$$\alpha=-a\frac{\sin(a|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_0+\frac{\cos(a|\xi|t)}{|\xi|^2}\hat{\boldsymbol{\varphi}}_1,$$

$$\beta=-b\frac{\sin(b|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_0+\frac{\cos(b|\xi|t)}{|\xi|^2}\hat{\boldsymbol{\varphi}}_1.$$

Note that $\langle \xi, |\xi|^2\beta - \langle \xi, \beta\rangle\xi\rangle = 0$. Hence

$$\left|\frac{\partial}{\partial t}\hat{\mathbf{u}}(\xi,t)\right|^2 = \langle \xi, \alpha\rangle^2|\xi|^2 + ||\xi|^2\beta - \langle \xi, \beta\rangle\xi|^2 =$$

$$= \langle \xi, \alpha\rangle^2|\xi|^2 + |\xi|^4|\beta|^2 + \langle \xi, \beta\rangle^2|\xi|^2 - 2\langle |\xi|^2\beta, \langle \xi, \beta\rangle\xi\rangle =$$

$$= |\xi|^2(\langle \xi, \alpha\rangle^2 + |\xi|^2|\beta|^2 - \langle \xi, \beta\rangle^2) = |\xi|^2(\langle \xi, \alpha\rangle^2 + |\xi \times \beta|^2).$$

Therefore, the total energy functional is calculated as

$$E(t) = K(t) + U(t) = \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho(a^2\langle \xi, \psi\rangle^2 + b^2|\xi \times \eta|^2 + |\xi|^2(\langle \xi, \alpha\rangle^2 + |\xi \times \beta|^2))\, d\xi.$$

Note that

$$a^2\langle \xi, \psi\rangle^2 + |\xi|^2\langle \xi, \alpha\rangle^2 = \langle \xi, a\psi\rangle^2 + \langle \xi, |\xi|\alpha\rangle^2 =$$

$$= \left\langle \xi, a\cos(a|\xi|t)\hat{\boldsymbol{\varphi}}_0 + \frac{\sin(a|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_1\right\rangle^2 + \left\langle \xi, -a\sin(a|\xi|t)\hat{\boldsymbol{\varphi}}_0 + \frac{\cos(a|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_1\right\rangle^2 =$$

$$= \left(\sum_{i=1}^n \xi_i\left(a\cos(a|\xi|t)\hat{\varphi}_{0,i} + \frac{\sin(a|\xi|t)}{|\xi|}\hat{\varphi}_{1,i}\right)\right)^2 +$$

$$+ \left(\sum_{i=1}^n \xi_i\left(-a\sin(a|\xi|t)\hat{\varphi}_{0,i} + \frac{\cos(a|\xi|t)}{|\xi|}\hat{\varphi}_{1,i}\right)\right)^2 =$$

$$= \sum_{i=1}^n \xi_i^2\left(a^2\hat{\varphi}_{0,i}^2 + \frac{1}{|\xi|^2}\hat{\varphi}_{1,i}^2\right) + 2\sum_{i<j}\xi_i\xi_j\left(a^2\hat{\varphi}_{0,i}\hat{\varphi}_{0,j} + \frac{1}{|\xi|^2}\hat{\varphi}_{1,i}\hat{\varphi}_{1,j}\right) =$$

$$= a^2\left(\sum_{i=1}^n \xi_i\hat{\varphi}_{0,i}\right)^2 + \frac{1}{|\xi|^2}\left(\sum_{i=1}^n \xi_i\hat{\varphi}_{1,i}\right)^2,$$

and

$$b^2|\xi \times \eta|^2 + |\xi|^2|\xi \times \beta|^2 = |\xi \times b\eta|^2 + |\xi \times |\xi|\beta|^2 =$$

$$= \left|\xi \times \left(b\cos(b|\xi|t)\hat{\boldsymbol{\varphi}}_0 + \frac{\sin(b|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_1\right)\right|^2 + \left|\xi \times \left(-b\sin(b|\xi|t)\hat{\boldsymbol{\varphi}}_0 + \frac{\cos(b|\xi|t)}{|\xi|}\hat{\boldsymbol{\varphi}}_1\right)\right|^2 =$$

$$= \sum_{i<j}\left(\xi_i\left(b\cos(b|\xi|t)\hat{\varphi}_{0,j} + \frac{\sin(b|\xi|t)}{|\xi|}\hat{\varphi}_{1,j}\right) - \left(b\cos(b|\xi|t)\hat{\varphi}_{0,i} + \frac{\sin(b|\xi|t)}{|\xi|}\hat{\varphi}_{1,i}\right)\xi_j\right)^2 +$$

$$+ \sum_{i<j}\left(\xi_i\left(-b\sin(b|\xi|t)\hat{\varphi}_{0,j} + \frac{\cos(b|\xi|t)}{|\xi|}\hat{\varphi}_{1,j}\right) - \left(-b\sin(b|\xi|t)\hat{\varphi}_{0,i} + \frac{\cos(b|\xi|t)}{|\xi|}\hat{\varphi}_{1,i}\right)\xi_j\right)^2 =$$

$$= \sum_{i<j}\left(\xi_i^2\left(b^2\hat{\varphi}_{0,j}^2 + \frac{1}{|\xi|^2}\hat{\varphi}_{1,j}^2\right) + \xi_j^2\left(b^2\hat{\varphi}_{0,i}^2 + \frac{1}{|\xi|^2}\hat{\varphi}_{1,i}^2\right) - 2\xi_i\xi_j\left(b^2\hat{\varphi}_{0,j}\hat{\varphi}_{0,i} + \frac{1}{|\xi|^2}\hat{\varphi}_{1,j}\hat{\varphi}_{1,i}\right)\right) =$$

$$= \sum_{i<j}\left(b^2(\xi_i\hat{\varphi}_{0,j} - \xi_j\hat{\varphi}_{0,i})^2 + \frac{1}{|\xi|^2}(\xi_i\hat{\varphi}_{1,j} - \xi_j\hat{\varphi}_{1,i})^2\right).$$

So we obtain that

$$E(t) = \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho\left(a^2\left(\sum_{i=1}^n \xi_i\hat{\varphi}_{0,i}\right)^2 + \frac{1}{|\xi|^2}\left(\sum_{i=1}^n \xi_i\hat{\varphi}_{1,i}\right)^2 +\right.$$

$$\left. + \sum_{i<j}\left(b^2(\xi_i\hat{\varphi}_{0,j} - \xi_j\hat{\varphi}_{0,i})^2 + \frac{1}{|\xi|^2}(\xi_i\hat{\varphi}_{1,j} - \xi_j\hat{\varphi}_{1,i})^2\right)\right)d\xi =$$

$$= \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho\left(a^2\left(\sum_{i=1}^{n} \xi_i \hat{\varphi}_{0,i}\right)^2 + \sum_{i<j} b^2(\xi_i\hat{\varphi}_{0,j} - \xi_j\hat{\varphi}_{0,i})^2 + \sum_{i=1}^{n} \hat{\varphi}_{1,i}^2\right) d\xi =$$

$$= \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho\left(a^2\left(\sum_{i=1}^{n} \frac{\partial\varphi_{0,i}}{\partial x_i}\right)^2 + \sum_{i<j} b^2\left(\frac{\partial\varphi_{0,j}}{\partial x_i} - \frac{\partial\varphi_{0,i}}{\partial x_j}\right)^2 + \sum_{i=1}^{n} \varphi_{1,i}^2\right) d\xi = E(0).$$

Corollary. Assume that $\boldsymbol{\varphi}_0$, $\boldsymbol{\varphi}_1$, and $\mathbf{f}$ are vectors of Schwartz functions. Then, the solution to the elastodynamics wave equation (4) with the initial condition (5) is unique.

Proof. Suppose that $\mathbf{u}$ and $\mathbf{u}'$ are two solutions of (4), (5). It is clear that $\mathbf{w} = \mathbf{u} - \mathbf{u}'$ is a solution of the following homogeneous equation

$$\left(\frac{\partial^2}{\partial t^2} - b^2\Delta\right)\mathbf{w} - (a^2 - b^2)\nabla\,\mathrm{div}(\mathbf{w}) = 0,$$

$$\mathbf{w}(\overline{x}, t)|_{t=0} = \frac{\partial \mathbf{w}}{\partial t}(\overline{x}, t)\bigg|_{t=0} = 0.$$

Applying Theorem 1, we obtain that the energy of $\mathbf{w}$ is equal to $E(0) = 0$. It implies that $\mathbf{w} = 0$ a.e., so we have completed the proof.

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Conservation law for the Cauchy–Navier equation of elastodynamics wave via Fourier transform

Summary

In this paper, we use the method of Fourier analysis to derive the formula of the total energy for the Cauchy problem of the Cauchy–Navier elastodynamics wave equation describing the motion of an isotropic elastic body. The conservation law of the total energy is obtained and consequently, the global uniqueness of the solution to the problem is implied.