Completely Positive Divisibility Does Not Mean Markovianity

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In the classical domain, it is well known that divisibility does not imply that a stochastic process is Markovian. However, for quantum processes, divisibility is often considered to be synonymous with Markovianity. We show that completely positive divisible quantum processes can still involve non-Markovian temporal correlations, that we then fully classify using the recently developed process tensor formalism, which generalizes the theory of stochastic processes to the quantum domain.

No system is fully isolated from its surroundings. This is especially true for quantum processes, where along with the surrounding environment, the act of observation can disturb the system [1]. The field of open system dynamics attempts to develop methods that describe the dynamics of systems, quantum and classical, away from isolation [2]. These tools become crucial in analyzing a whole host of problems, from strong coupling thermodynamics [3] to error correction in quantum technologies [4]. An important consideration for describing open dynamics is the size and length of memory that the surroundings possess about the system's past [5,6]. In general, the future states of the system depend nontrivially on its own past, leading to complex joint measurement statistics in time [7,8]. A process where the environment has *no* memory is called Markovian, and the complexity of describing such dynamics scales only as the Hilbert-space dimension of the system [9], while the complexity of a non-Markovian process can scale exponentially in the number of times considered [10,11].

Markovianity plays an important role in fields ranging far beyond the physical sciences. This is both for the fact that many processes in nature are approximated sufficiently well by memoryless dynamics, and the computational and simulation intricacy that arises once memory effects are taken into account [2,12]. As experimental control over complex quantum systems becomes increasingly sophisticated, the ability to directly determine whether a Markovian description is applicable is becoming ever more important [13–17]. Consequently, in recent years, a large body of work dedicated to the characterization of temporal correlations in quantum systems has arisen [18–21]. However, strictly testing for the presence of memory effects is an intractable task in general [10], and a large number of non-Markovianity witnesses has emerged in the past two decades [22–36].

The concept that underpins *all* of these witnesses is completely positive (CP) divisibility, a frequently used proxy for Markovianity. While experimentally accessible [37–40], this criterion lacks a clear, quantifiable link to Markovianity, which casts its implications for potential memory effects, and the interpretation of all the memory witnesses derived thereof, into doubt. In this Letter, we first demonstrate that, *a priori*, there are inequivalent experimental definitions of CP divisibility. After clearing up these ambiguities, we close the fundamental gap in the understanding of CP divisibility and comprehensively derive its quantitative relationship to Markovianity. While the difference between CP divisibility and Markovianity has been pointed out before [9,41], our results yield both a quantifiable delineation between them, as well as a comprehensive characterization of the temporal correlations CP divisibility is sensitive to. This, in turn, provides a meaningful way forward for experimentalists looking to definitively characterize noise in their devices by means of memory witnesses. To motivate the relation of Markovianity and divisibility we first briefly review them in the context of classical processes.

Markovianity and divisibility.—Mathematically, a classical process is called Markovian if the current state conditionally only depends on the last one, and not the whole history:

$$\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_0, t_0) = \mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}). \quad (1)$$

A generalization of this condition to quantum theory has recently been achieved [9].

From Eq. (1) it is clear that Markovianity is a statement about *multitime* correlations and checking for it requires an exponentially large set of conditions to be satisfied. A simpler criterion that follows from Markovianity, but is not sufficient to define it, is *divisibility*. This requires the conditional probabilities, for any *three* times $t > s > r$, to factorize according to the Chapman-Kolmogorov equation: $\mathbb{P}(x, t | z, r) = \sum_y \mathbb{P}(x, t | y, s) \mathbb{P}(y, s | z, r)$ for all states x, y, z,

where each $\mathbb{P}$ is a probability distribution. The natural quantum generalization of a conditional probability distribution (with a single argument) is a completely positive map, i.e., one which preserves the positivity of even correlated density operators [42], and the Chapman-Kolmogorov equation generalizes to the condition for CP divisibility [43]:

Definition 1: CP divisibility.—A quantum dynamical process of a system on an interval $[0, \mathcal{T}]$ is CP divisible if (i) the dynamical map from r to t acting on the system of interest can be broken up at s such that

$$\Phi_{t:r} = \Phi_{t:s} \circ \Phi_{s:r} \quad \forall \ \mathcal{T} \geq t \geq s \geq r \geq 0, \quad (2)$$

and (ii) each map $\Phi_{x:y}$ is completely positive.

Intuitively, the connection between this definition and Markovianity is that CP maps describe dynamics without initial system-environment correlations [2], and Eq. (2) suggests that the dynamics between intermediate times are independent of the past. Together, these properties could be taken to imply the absence of memory. However, while mathematically well defined [44], *a priori*, neither the operational meaning of the family of maps $\{\Phi_{t:s}\}$, nor its relation to prevalent memory effects are clear. That is, in an experimental setting, what exact *quantum process tomography* procedure [45] is required to determine CP divisibility, and what exactly does this property imply?

There are (at least) two nonequivalent ways to address these questions. In what follows, we first motivate and define two types of CP divisibility and show their nonequivalence. We then give a full characterization of the non-Markovian temporal correlations that may hide in a divisible process, thus providing a clear connection and delineation between Markovianity and CP divisibility. Throughout this Letter, we will only consider systems with finite Hilbert-space dimension d.

CP divisibility by inversion.—Consider an experimental setup where one is allowed to prepare any desired state at the initial time $r = 0$ and perform measurements on the system at any later time s. Within these experimental constraints, using the standard method of quantum process tomography, one can construct a family of maps $\lambda_0 := \{\Lambda_{s:0}\}$ that describe the dynamics from time $r = 0$ to time s, see Fig. 1(a). Assuming that all the maps of this family are invertible, we obtain the following definition, which is the one that most frequently appears in the literature [22,46,47]:

Definition 2: ICP divisibility.—A process is CP-divisibility by inversion (ICP divisible) if for any two maps $\Lambda_{s:0}, \Lambda_{t:0} \in \lambda_0$ with $\mathcal{T} \geq t > s \geq 0$ the map

$$\Phi_{t:s} := \Lambda_{t:0} \circ \Lambda_{s:0}^{-1} \quad (3)$$

is completely positive.

Here, we choose a convention where experimentally accessible maps are denoted by Λ. Notably, if all elements
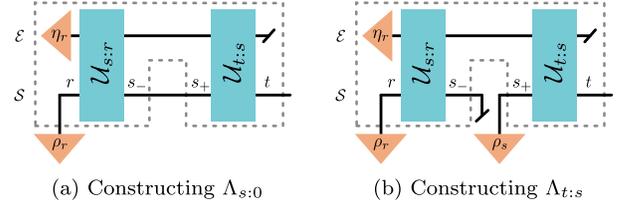

FIG. 1. Circuits for checking ICP and OCP divisibility. (a) To construct the maps $\Lambda_{s:0}$ and $\Lambda_{t:0}$ we may set $r = 0$ and measure the system at s or t respectively. (b) To construct $\Lambda_{t:s}$, the system is discarded at s_- and a fresh state is fed in at s_+. The dotted line encapsulates the three-time process tensor $T_{t:s:r}$.

of λ_0 are invertible then each $\Phi_{t:s}$ constructed according to Eq. (3) is well defined, and can be obtained computationally from λ_0.

Operational divisibility.—While ICP divisibility is well defined, it leaves the operational meaning of the inferred maps $\Phi_{t:s}$ open [48]; particularly, these maps do not necessarily relate to anything that could actually be measured at intermediate times. Additionally, due to their nonoperational definition, it is not possible to straightforwardly characterize the memory effects that ICP divisibility is blind to. A more operationally motivated definition of CP divisibility based on experimentally reconstructed maps alone is thus desirable.

To this end, let us consider a scenario where an experimenter can manipulate the system at *any* time $s \in [0, \mathcal{T}]$, which we split infinitesimally into s_- and s_+ as shown in Fig. 1(b). At time s_- the system is discarded and, at s_+, replaced with a fresh one in state ρ_s. Subsequently, the experimenter measures the system at time t. With this procedure they can experimentally reconstruct maps $\lambda := \{\Lambda_{t:s}\}$ as

$$\Lambda_{t:s}[\rho_s] = \text{tr}_{\mathcal{E}}[\mathcal{U}_{t:s}(\rho_s \otimes \eta_s)], \quad (4)$$

where η_s is the reduced state of the environment at time s and $\mathcal{U}_{t:s}(x_s) := U_{t:s} x_s U_{t:s}^{\dagger} = x_t$ is the unitary system-environment map. We can thus define OCP divisibility:

Definition 3: OCP divisibility.—A process is operationally CP divisible (OCP divisible), if for any $\mathcal{T} \geq t > s > r \geq 0$

$$\Lambda_{t:r} = \Lambda_{t:s} \circ \Lambda_{s:r} \quad (5)$$

holds, where the maps above belong to set λ and are defined in Eq. (4).

Importantly, complete positivity of the respective maps is guaranteed, as system-environment correlations are discarded for their reconstruction. Formally, Eq. (5) resembles Eq. (2), with the important distinction that here each map has a clear operational meaning.

Still, there is a level of ambiguity in the reconstruction procedure of the maps $\Lambda_{t:s}$. In principle, they could depend on preparations at any previous time r, which would imply

non-Markovianity [50]. However, if there are at least two different states ρ_r and $\tilde{\rho}_r$, such that the corresponding maps $\Lambda_{t:s}$ and $\tilde{\Lambda}_{t:s}$ differ, then OCP divisibility is not uniquely defined. Thus, Definition 3 implicitly requires that the intermediate maps are independent of any earlier state preparations. This independence constitutes a nonsignaling condition [51–53], as we discuss formally in the Supplemental Material [54].

Importantly, this nonsignaling requirement is a *conditional* one; for OCP-divisible dynamics, there is no signaling from r to t given that the system state was discarded at s_-. Equivalently, this condition can be thought of as follows: consider an experiment where one part of a correlated state $\rho_{rr'}$ is fed into the process at time r. At time s_- the system is discarded and a fresh state prepared at s_+, and the experimenter looks for correlation in the resulting state $\rho_{tr'}$ at time t. If $\rho_{tr'} \neq \rho_t \otimes \rho_{r'}$ then we have conditional signaling from r to t.

Conditional nonsignaling is, for example, satisfied if the system interacts only once, between any two times, with a part of the environment that is discarded afterwards. However, while conditional nonsignaling is necessary for OCP divisibility, it is not sufficient, see Supplemental Material [54] and also [55].

The absence of signaling is reminiscent of the concept of no information backflow attributed to CP-divisible processes [23]. Here, however, in contrast to the increase of trace distance between trajectories, signaling is a genuine multitime statement. Now, before further discussing their relationship to Markovianity, we show that ICP and OCP divisibility do not coincide.

OCP divisibility $\neq$ ICP divisibility.—Despite their superficial resemblance, ICP and OCP divisibility differ in their experimental reconstruction and the meaning of the maps they comprise. Consequently, the relationship between them is *a priori* unclear. First, note that ICP divisibility is only defined if all elements of the set λ_0 are invertible. This limitation does not apply to OCP divisibility. Focusing on the invertible case, we find that OCP divisibility implies ICP divisibility by direct application of Eq. (5).

To see that the converse does not hold, we construct an ICP-divisible dynamics that is conditionally signaling, and thus not OCP divisible. Consider the two circuits in Fig. 1, where both the system and the environment are qubits, and let the initial environment state be maximally mixed, i.e., $\eta_r = \mathbb{1}_2/2$. The system-environment dynamics is given by the partial swap $U_{s:r} = \exp(-i\omega \mathbf{S} u) = \cos(\omega u)\mathbb{1}_4 - i \sin(\omega u)\mathbf{S}$, where $\mathbf{S}|ij\rangle = |ji\rangle$, and $u := s - r$. We show in the Supplemental Material [54] that the resulting dynamics on the system is ICP divisible for $\omega t \leq (\pi/2)$. On the other hand, if we discard the state of the system at s_- and insert a fresh state at s_+ we will find that the state at t depends on ρ_r due to the partial swap. In other words we have signaling, and therefore the process is not OCP divisible.

(a) Constructing $\Lambda_{s:0}$
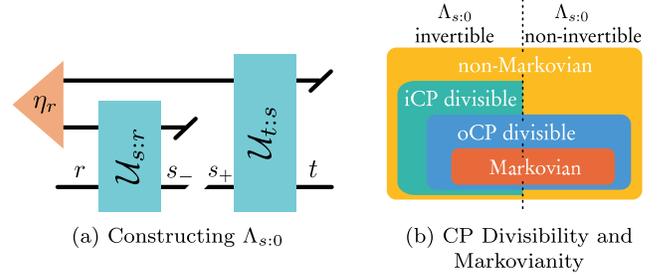
(b) CP Divisibility and Markovianity

FIG. 2. Divisibility and Markovianity. (a) The system interacts with one part of the correlated environment state leading to a non-Markovian OCP-divisible process. (b) The hierarchy of sets of processes with varying degrees of temporal correlations.

Operationally CP-divisible dynamics form a strict subset of ICP-divisible ones, see Fig. 2(b). While the operational requirement is harder to check experimentally, it has a threefold advantage: first the involved maps have a clear-cut operational meaning, and the property of OCP divisibility ties in effortlessly with frameworks tailored for the discussion of non-Markovian quantum processes. Second, the definition of OCP divisibility does not rely on the invertibility of $\Lambda_{s:0}$ and thus has wider applicability. Last, OCP divisibility breaks down for a larger class of memory effects than ICP divisibility, and consequently outperforms it as a witness of non-Markovianity.

CP divisibility $\neq$ Markovianity.—Even though OCP divisibility is a stricter requirement than ICP divisibility, it does not enforce Markovianity; we show this by means of a discrete time example. For an *ante litteram* continuous example of non-Markovian OCP-divisible dynamics, see [41,56]. We take inspiration from collision models [57–60] with correlated environment states [61,62]: Let the environment at $r = 0$ be in a correlated bipartite state that is uncorrelated with the system. The dynamics $\mathcal{U}_{y:x}$ between any two (of a set of three) times is such that the system only interacts with one part of the environment (denoted by x) that is discarded afterwards, see Fig. 2(a). This scenario satisfies the necessary nonsignaling condition. Now, if we choose the unitaries $U_{y:x}$ to be the swap operator $\mathbf{S}_{Sx}$ between system and part x of the environment, then we have $\Lambda_{t:r} = \Lambda_{t:s} \circ \Lambda_{s:r}$, and the dynamics is OCP divisible.

However, the process is non-Markovian; suppose the experimenter *stores* the system state at time s_-, and inserts a fresh state at s_+. The dynamics is allowed to continue to t and that state too is stored. The joint state ρ_{st} will be correlated even though the states inserted into the process, at times r and s_+, were independent. In particular, for the above case the resulting state ρ_{st} is exactly the correlated initial state of the environment. The experimenter could thus detect memory effects, although the dynamics is OCP divisible.

Nonetheless, an OCP-divisible process can be seen as one that is Markovian *on average*: Consider a multitime process where an experimenter measures the system at each

time, before independently preparing it in a new state; OCP divisibility implies that, if all past measurement outcomes are forgotten or averaged over then the future statistics only depend on the current preparation. A quantum Markov process, in contrast, requires that the future statistics only depend on the current preparation for *any* sequence of measurement outcomes [9,11,63–65]. We now fully characterize the temporal correlations that can persist in OCP-divisible dynamics, thus providing a quantifiable connection between Markovianity and the majority of witnesses of non-Markovianity employed in the literature.

Correlations in divisible processes.—The four classes of processes illustrated in Fig. 2(b) also have analogs in the classical domain. A classical stochastic process is described by a joint distribution

$$\mathbb{P}(x_n, t_n; \ldots; x_0, t_0), \qquad (6)$$

over the state of the system at different times, satisfying Kolmogorov conditions [66,67]. To check if a given process is Markovian necessitates checking all conditional probabilities given in Eq. (1), which requires the full distribution of Eq. (6). However, to infer the divisibility of a process, by inversion or operationally, requires only the bipartite marginal distributions of Eq. (6): $\{\mathbb{P}(x_s, t_s, x_0, t_0)\}_{s=1}^n$ and $\{\mathbb{P}(x_s, t_s, x_r, t_r)\}_{s>r\geq0}^n$, respectively. Thus, we have the same hierarchy as in Fig. 2(b) for temporal correlations in classical processes.

The quantum generalization of Eq. (6) is a multipartite positive operator $T_{n:\ldots:1:0}$ [10,11,68–71], called the *process tensor* in the field of open quantum system dynamics [10], which satisfies *generalized* Kolmogorov conditions [41,72]. Analogous to the classical case, the process tensor captures all temporal correlations in quantum processes, including across multiple time steps, in our case three. The probability of observing a sequence of events $\{x_r, x_s, x_t\}$, can be computed by contracting it with generalized measurement operators M_x:

$$\mathbb{P}(x_t, x_s, x_r | \mathcal{J}_t, \mathcal{J}_s, \mathcal{J}_r) = \text{tr}[(M_{x_t} \otimes M_{x_s} \otimes M_{x_r}) T_{t:s:r}], \quad (7)$$

which constitutes a generalization of the Born rule to processes in time [73,74]; $\mathcal{J}$ denotes an instrument [75], which is a collection of conditional transformations (CP maps) $\{M_{x_s}\}$ that update the system after a particular event is observed; these generalize the concept of positive operator valued measure (POVM). Without loss of generality, each element of Eq. (7) is expressed in terms of Choi states [71,76–78].

Mathematically, the process tensor $T := T_{t:s:r}$ is an operator on Hilbert spaces $\mathcal{H}_r \otimes \mathcal{H}_{s_-} \otimes \mathcal{H}_{s_+} \otimes \mathcal{H}_t$. For both panels in Fig. 1, the process tensor (the object within the dotted lines) is exactly the same. The difference between them lies entirely in the instrument at s. The instrument at r is a preparation with one element $M_{x_r} = \rho_r$,

and the instrument at t is a measurement $\{M_{x_t} = \Pi_{x_t}\}$, where the latter are POVM elements. The instrument at s for Fig. 1(a) implements the identity channel, which has Choi state $M_{x_s} = \varphi_{s_\pm}^+$, where $\varphi_{s_\pm}^+ := \sum_{jk}^d |jj\rangle\langle kk|$ and $s_\pm := s_- s_+$. For Fig. 1(b) the instrument at s also has a single element: $M_{x_s} = \mathbb{1} \otimes \rho_s$, which denotes the trace at s_- followed by a preparation of ρ_s. We review the details of the process tensor formalism in the Supplemental Material [54] and only include important details here.

Using Eq. (7) and the details of the instruments, we recover the maps in Eq. (5) from the process tensor. Let $L_{x:y}$ denote the Choi state of $\Lambda_{x:y}$. For an OCP-divisible process, we can show that $L_{t:r} = tr_{s_\pm}(\varphi_{s_\pm}^+ T)$, while $L_{s:r} = tr_{s_+t}[T]/d$ and $L_{t:s} = tr_{rs_-}[T]/d$. With this, we can rephrase OCP divisibility as

$$tr_s(\varphi_{s_\pm}^+ T) = \frac{1}{d^2} tr_{s_\pm}[tr_{rs_-}(T)\varphi_{s_\pm}^+ tr_{s_+t}(T)]. \qquad (8)$$

A detailed derivation of above statements is given in the Supplemental Material [54].

On the other hand, a quantum process is Markovian iff the Choi state of the corresponding process tensor has the form $T^{\text{Markov}} = L_{t:s} \otimes L_{s:r}$ [9,11,63,65]; any deviation from this product form implies detectable non-Markovian correlations. Since Eq. (8) does not force T to be of Markov form, OCP-divisible processes are not necessarily memoryless. Specifically, representing $T = L_{t:s} \otimes L_{s:r} + \chi_{tsr}$, where the matrix χ contains all tripartite non-Markovian correlations and satisfies $tr_{s_-r}[\chi_{tsr}] = tr_{ts_+}[\chi_{tsr}] = 0$, we see that Eq. (8) implies $tr_{s_\pm}(\varphi_{s_\pm}^+ \chi_{tsr}) = 0$, which provides a full classification of non-Markovian temporal correlations that can be present in OCP-divisible processes.

Conclusions.—In this Letter, we have provided an operationally motivated definition of CP divisibility that is stricter than the frequently used one relying on the invertibility of $\Lambda_{s:0}$. We showed that OCP divisibility is closely connected to nonsignaling conditions and implies the absence of information flow from the environment to the system. Additionally, we have demonstrated that OCP divisibility can be interpreted as Markovianity on average, yet OCP-divisible processes can still display nontrivial memory effects, which we have fully characterized. These results lay the foundation for a quantitative interpretation of all studies of memory effects that are based upon CP divisibility or witnesses derived thereof.

Near-term quantum technologies will require effective methods for detecting and addressing non-Markovian noise [79]. We have shed light on divisibility from an operational point of view, which helps us to identify the classes of temporal correlations that may evade regularly used checks for non-Markovianity. However, there are trade-offs between uncovering temporal correlations and the requisite number of experiments that must be performed. Our results enable experimentalists to make informed decisions about

investing resources in classifying the non-Markovian noise at hand.

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