Dynamics of PDE, Vol.15, No.4, 265-282, 2018

On the energy-critical fractional Schrödinger equation in the radial case

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Communicated by Dong Li, received November 27, 2017.

Abstract. We consider the Cauchy problem for the energy-critical nonlinear Schrödinger equation with fractional Laplacian (fNLS) in the radial case. We obtain global well-posedness and scattering in the energy space in the defocusing case, and in the focusing case with energy below the ground state. The main feature of the present work is the nonlocality of the operator. This does not allow us to use standard computations for the rigidity part of the theorem. Instead we develop a commutator argument which has its own interest for problems with nonlocal operators.

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1. Introduction

In this paper, we study the Cauchy problem for the nonlinear Schrödinger equation with fractional Laplacian for $N \geq 2$:

$$\begin{cases} i\partial_t u + D^{2\alpha} u + \mu |u|^{\frac{4\alpha}{N-2\alpha}} u = 0 \quad (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^\alpha(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{N}{2N-1}, 1)$, $D^{2\alpha}$ is the Fourier multiplier of symbol $|\xi|^{2\alpha}$ with $D = \sqrt{-\Delta}$, $\mu \in \{-1, 1\}$. Here $\mu = 1$ corresponds to the defocusing case, and $\mu = -1$

1991 *Mathematics Subject Classification.* 35L70, 35Q55.
Key words and phrases. Nonlinear wave equation, nonlinear Schrödinger equation.

corresponds to the focusing case. When $\alpha = 1$, (1.1) is the well-known energy-critical nonlinear Schrödinger equation which has been extensively studied, and we refer the readers to [**27**] for a survey of the study. When $0 < \alpha < 1$, (1.1) is a nonlocal model known as nonlinear fractional Schrödinger equation which has also attracted much attentions recently (see [**12, 15, 14, 4, 5, 13, 6, 7, 17, 1, 25**]). The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics, which was derived by Laskin [**22, 23**] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The purpose of this paper is to prove some analogue global well-posedness and scattering for (1.1) in the radial case.

Under the flow of the equation (1.1), the following quantities (mass and energy) are conserved:

$$M(u) = \int_{\mathbb{R}^N} |u(x,t)|^2 dx,$$
$$E_\mu(u) = \int_{\mathbb{R}^N} \frac{1}{2}|D^\alpha u|^2 + \frac{\mu}{p+2}|u|^{p+2} dx.$$

We write $E_\pm(u) = E_{\pm 1}(u)$. Moreover, the equation (1.1) preserves the radial symmetry, and also has the following scaling invariance: for $\lambda > 0$

$$u(x,t) \to \lambda^{\frac{N-2\alpha}{2}} u(\lambda x, \lambda^{2\alpha} t), \quad u_0(x) \to \lambda^{\frac{N-2\alpha}{2}} u_0(\lambda x).$$

Thus, (1.1) is $\dot{H}^\alpha$-critical, since the scaling transform leaves $\dot{H}^\alpha$-norm invariant.

There are remarkable differences between the defocusing and focusing cases. In the focusing case, the flow has more kinds of dynamical behavior. An important role is played by the ground state W_α, namely the unique non-negative radial solution to the fractional elliptic equation

$$D^{2\alpha} W - |W|^{\frac{4\alpha}{N-2\alpha}} W = 0. \tag{1.2}$$

We have $W_\alpha \in \dot{H}^\alpha$, and so W_α is a stationary solution to (1.1) when $\mu = -1$. See section 3 for more properties of W_α. The main result of this paper is

THEOREM 1.1. *Assume* $\alpha \in (\frac{N}{2N-1}, 1)$ *and*

$$2\alpha < N < 4\alpha.$$

Let W_α *be as above. Assume* $u_0 \in \dot{H}^\alpha$, u_0 *radial. Then*

(1) *Defocusing case* ($\mu = 1$)*:* (1.1) *is globally well-posed, and scattering holds.*
(2) *Focusing case* ($\mu = -1$)*: if* $E_-(u_0) < E_-(W_\alpha)$ *and* $\|D^\alpha u_0\|_2 < \|D^\alpha W_\alpha\|_2$, *then* (1.1) *is globally well-posed, and scattering holds.*

REMARK 1. Notice that the cases $N = 2$ and $N = 3$ which are relevant for the Schrödinger equation are covered depending on the values of α. Indeed, one can take $N = 2$ with $\alpha \in (\frac{2}{3}, 1)$ and $N = 3$ with $\alpha \in (\frac{3}{4}, 1)$. These restrictions on the exponents for the powers of the laplacian come from the nonlocality of the operator and the arguments to handle the concentration-compactness argument. Indeed, the nonlocal character of the operator does not allow to have "exact" expressions but only estimates. At the moment, we do not know how to remove these assumptions.

Now we discuss the ideas of proof. We follow closely the Kenig-Merle's concentration compactness/rigidity method [**19**]. There are several different ingredients:

(1) Radial Strichartz estimates. When $\alpha < 1$, we know that the classical Strichartz estimates in non-radial case has loss of regularity. However, in the radial case, it was known that when $\alpha \in (\frac{N}{2N-1}, 1)$ one has generalized estimates which has no loss of derivatives, see [**16**]. In contrast to [**19**], radial symmetry for (1.1) plays crucial role in many aspects.
(2) The results from the study of the fractional elliptic equation. The fractional elliptic equation has been extensively studied recently. In the focusing case, we will apply the results for (1.2) which was obtained in [**24**], [**3**].
(3) Localization of virial identity. In the rigidity argument, we use the localization of virial identity. Due to the nonlocal nature of $|D|^{2\alpha}$, we need to deal with some commutator estimates.

The main difference between (1.1) and Schrödinger equation is the nonlocal property of the fractional Laplacian. In our proof, this nonlocal property makes only slight difference from the Kenig-Merle's argument in the concentration-compactness part (Thus we omit most of the details). However, it makes big difference in the space-time a-priori estimates, e.g. localization of virial estimates in the rigidity part. We do not know any other monotonicity, such as Morawetz estimates. To our best knowledge, this paper is the first one which generalizes Kenig-Merle's argument to the nonlocal setting.

2. The Cauchy problem and the variational estimates

2.1. The Cauchy problem. In this section, we review the local theory and small data global theory for the Cauchy problem (1.1) with radial symmetry. It has no difference between defocusing and focusing cases. The key ingredient is the radial Strichartz estimates obtained in [**16**].

LEMMA 2.1 (Proposition 3.9 [**16**]). *Suppose $N \geq 2$, $\alpha > 1/2$ and u, u_0, F are spherically symmetric in space and satisfy*

$$\begin{cases} i\partial_t u + D^{2\alpha} u = F & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0. \end{cases}$$

Then for $\gamma \in \mathbb{R}$ it holds

$$\|u\|_{L_t^q L_x^r} + \|u\|_{C(\mathbb{R}:\dot{H}^\gamma)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{2.1}$$

if the following conditions hold:

(1) (q,r) and $(\tilde{q}, \tilde{r})$ both satisfy the following conditions:

$$2 \leq q, r \leq \infty, \frac{1}{q} < (N - \frac{1}{2})(\frac{1}{2} - \frac{1}{r}); \tag{2.2}$$

(2) $\tilde{q}' < q$ and the "gap" condition:

$$\frac{2\alpha}{q} + \frac{N}{r} = \frac{N}{2} - \gamma, \ \frac{2\alpha}{\tilde{q}} + \frac{N}{\tilde{r}} = \frac{N}{2} + \gamma.$$

REMARK 2. The conditions in (1) can be relaxed to the following

$$2 \leq q, r \leq \infty, \frac{1}{q} \leq (N - \frac{1}{2})(\frac{1}{2} - \frac{1}{r}), \quad (q,r) \neq (2, \frac{4N-2}{2N-3}). \tag{2.3}$$

On the boundary line $\frac{1}{q} = (N - \frac{1}{2})(\frac{1}{2} - \frac{1}{r})$, [**16**] first proved it for $q \geq r$, and was later improved to other pairs independently by [**18**] and [**8**].

Definition 2.1. For $N \geq 2$, we say that a pair of exponents (q, r) is α-admissible if (q, r) verifies

$$\frac{2\alpha}{q} + \frac{N}{r} = \frac{N}{2}, \quad 2 \leq q, r \leq \infty. \tag{2.4}$$

By Lemma 2.1, we see that if $\alpha \in (\frac{N}{2N-1}, 1)$, then we have a full set of α-admissible Strichartz estimates which has no loss of derivatives. With these Strichartz estimates, we can proceed as the classical theory of Schrödinger equation. Let $I \subset \mathbb{R}$ be an interval, and we define $S_\alpha(I), W_\alpha(I)$ norm by

$$\|v\|_{S_\alpha(I)} = \|v\|_{L_t^{\frac{2(N+2\alpha)}{N-2\alpha}} L_x^{\frac{2(N+2\alpha)}{N-2\alpha}}} \quad \text{and} \quad \|v\|_{W_\alpha(I)} = \|v\|_{L_t^{\frac{2(N+2\alpha)}{N-2\alpha}} L_x^{\frac{2N(N+2\alpha)}{N^2+4\alpha^2}}}.$$

Note that $(\frac{2(N+2\alpha)}{N-2\alpha}, \frac{2N(N+2\alpha)}{N^2+4\alpha^2})$ is α-admissible pairs. By Sobolev embedding, we have that if $N > 2\alpha$,

$$\|v\|_{S_\alpha(I)} \leq C\|D^\alpha v\|_{W_\alpha(I)}.$$

Definition 2.2. Let $t_0 \in I$. We say that $u \in C(I; \dot{H}^\alpha(\mathbb{R}^N)) \cap \{D^\alpha u \in W_\alpha(I)\}$ is a solution of the (1.1) if

$$u|_{t_0} = u_0, \quad \text{and} \quad u(t) = e^{i(t-t_0)D^{2\alpha}} u_0 + \int_{t_0}^t e^{i(t-t')D^{2\alpha}} |u|^{\frac{4\alpha}{N-2\alpha}} u \, dt'.$$

Definition 2.3. Let $v_0 \in \dot{H}^\alpha$, $v(x, t) = e^{itD^{2\alpha}} v_0$ and let $\{t_n\}$ be a sequence, with $\lim_{n\to\infty} t_n = \overline{t} \in [-\infty, +\infty]$. We say that $u(x, t)$ is a non-linear profile associated with $(v_0, \{t_n\})$ if there exists an interval I, with $\overline{t} \in I$ (if $\overline{t} = \pm\infty$, $I = [a, +\infty)$ or $(-\infty, a]$) such that u is a solution of (CP) in I and

$$\lim_{n\to\infty} \|u(-, t_n) - v(-, t_n)\|_{\dot{H}^\alpha} = 0.$$

With the Strichartz estimates, we can obtain the following results for (1.1) by standard arguments (for example, see [**9**]).

Theorem 2.2. *(1) Assume $N \geq 2$, $\alpha \in (\frac{N}{2N-1}, 1)$, $2\alpha < N < 6\alpha$ and $u_0 \in \dot{H}^\alpha(\mathbb{R}^N)$, u_0 radial, $\|u_0\|_{\dot{H}^\alpha} \leq A$. Then $\exists \delta = \delta(A)$ s.t. if $\|e^{itD^{2\alpha}} u_0\|_{S_\alpha(I)} \leq \delta$, $0 \in \dot{I}$, there exists a unique solution to (1.1) on I such that $u \in C(I; \dot{H}^\alpha)$, $\sup_{t\in I} \|u(t)\|_{\dot{H}^\alpha} + \|D^\alpha u\|_{W_\alpha(I)} \leq C(A)$ and $\|u\|_{S_\alpha(I)} \leq 2\delta$. Moreover, we have*

- *Local existence: there exists a maximal open interval $I = (-T_-(u_0), T_+(u_0))$ where the solution u is defined.*
- *Small data global existence: if $A \ll 1$, then $I = (-\infty, +\infty)$.*
- *$D^\alpha u \in L_t^q L_x^r(I' \times \mathbb{R}^N)$ for any α-admissible pair (q, r), where $I' \subset I$ is a closed interval with finite length.*
- *Blowup criterion: If $T_+(u_0) < +\infty$, then $\|u\|_{S_\alpha([0,T_+(u_0)))} = +\infty$. A similar statement holds in the negative time direction.*
- *Scattering: If $T_+(u_0) = +\infty$ and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in \dot{H}^\alpha$ such that*

$$\lim_{t\to+\infty} \|u(t) - e^{itD^{2\alpha}} u_+\|_{H^\alpha(\mathbb{R}^N)} = 0. \tag{2.5}$$

A similar statement holds in the negative time direction.

(2) For any $u_+ \in \dot{H}^\alpha$, there exists a solution u to (1.1) *such that* (2.5) *holds. As a consequence, for any $(v_0, \{t_n\})$, there always exists a non-linear profile associated to $(v_0, \{t_n\})$ with a maximal interval of existence.*

Next, we need a perturbation theorem. It follows in a very similar way as Theorem 2.14 in [**19**] (see [**20**] for a correct proof), see also [**26**]. Since for $\alpha \in (\frac{N}{2N-1}, 1)$, we have generalized inhomogeneous Strichartz estimate given by Lemma 2.1, the proof is with slight change and we omit the details,

Theorem 2.3 (Stability). *Assume $N \geq 2$, $\alpha \in (\frac{N}{2N-1}, 1)$, $2\alpha < N < 6\alpha$. Let $I = [0, L)$, $L \leq +\infty$, and let $\tilde{u}$ be defined on $I \times \mathbb{R}^N$ such that*

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^\alpha(I\times\mathbb{R}^N)} \leq A, \quad \|\tilde{u}\|_{S_\alpha(I)} \leq M, \quad \|D^\alpha \tilde{u}\|_{W_\alpha(I)} < \infty \tag{2.6}$$

for some constants A and M, and $\tilde{u}$ verifies in the sense of integral equation

$$i\tilde{u}_t + D^{2\alpha}\tilde{u} + \mu|\tilde{u}|^{\frac{4\alpha}{N-2\alpha}}\tilde{u} = e$$

for some function e. Let $u_0 \in \dot{H}^\alpha$ be such that $\|u(0) - \tilde{u}(0)\|_{\dot{H}^\alpha} \leq A'$. Then $\exists \varepsilon_0 = \varepsilon_0(M, A, A')$ s.t. if $0 < \varepsilon < \varepsilon_0$ and

$$\|e^{itD^{2\alpha}}(u(0) - \tilde{u}(0))\|_{S_\alpha(I)} \leq \varepsilon, \quad \|D^\alpha e\|_{L_I^2 L_x^{\frac{2N}{N+2\alpha}}} \leq \varepsilon, \tag{2.7}$$

then, there exists a unique solution u on $I \times \mathbb{R}^N$ to (1.1) *with initial data u_0 satisfying*

$$\|u\|_{S_\alpha(I)} \leq C(A, A', M), \quad \sup_{t\in I}\|u(t) - \tilde{u}(t)\|_{\dot{H}^\alpha} \leq C(A, A', M).$$

2.2. Some variational estimates in the focusing case. In the focusing case, the ground state plays an important role. Consider the fractional elliptic equation

$$D^{2\alpha}W - |W|^{\frac{4\alpha}{N-2\alpha}}W = 0. \tag{2.8}$$

By the work of Lieb [**24**], it was known that: if $0 < \alpha < N/2$, then (2.8) has a solution in $\dot{H}^\alpha$

$$W(x) = C_1(N, \alpha)\left(\frac{1}{1 + C_2(N, \alpha)|x|^2}\right)^{\frac{N-2\alpha}{2}}$$

for some $C_1, C_2 > 0$. It arises in the study of the best constant for Hardy-Littlewood-Sobolev inequalities. The classification of positive regular solutions for (2.8) was studied in [**3**]. We also have the following characterization of W (see [**24**], [**11**]): W attains the best constant C_N in the Sobolev embedding inequality:

$$\|u\|_{L^{\frac{2N}{N-2\alpha}}} \leq C_N\|D^\alpha u\|_{L^2}. \tag{2.9}$$

Moreover, if $0 \neq u \in \dot{H}^\alpha$ verifies $\|u\|_{L^{\frac{2N}{N-2\alpha}}} = C_N\|D^\alpha u\|_{L^2}$, then $u = W_{\theta_0,x_0,\lambda_0} := e^{i\theta_0}\lambda_0^{(N-2\alpha)/2}W(\lambda_0(x - x_0))$ for some $\theta_0 \in [-\pi, \pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^N$.

W is a stationary solution to (1.1) when $\mu = -1$. By the equation (2.8), we have $\int |D^\alpha W|^2 = \int |W|^{2^*}$. Also, (2.9) yields $C_N^2 \int |D^\alpha W|^2 = \left(\int |W|^{2^*}\right)^{(N-2\alpha)/N}$, so that $C_N^2 \int |D^\alpha W|^2 = \left(\int |D^\alpha W|^2\right)^{\frac{N-2\alpha}{N}}$. Hence,

$$\int |D^\alpha W|^2 = \frac{1}{C_N^{N/\alpha}} \quad \text{and} \quad E_\mu(W) = \frac{\alpha}{N}\frac{1}{C_N^{N/\alpha}}\int |D^\alpha W|^2.$$

With the variational properties, we can follow Kenig-Merle's argument with slight change to prove the following lemma. We omit the proof.

LEMMA 2.10. *(1) Assume* $\alpha \in (\frac{N}{2N-1}, 1)$, $\|D^\alpha u\|_{L^2} < \|D^\alpha W\|_{L^2}$, *and* $E_-(u) \leq (1-\delta_0)E_-(W)$ *for some* $\delta_0 > 0$. *Then, there exists* $\overline{\delta} = \overline{\delta}(\delta_0, N) > 0$ *such that*

$$\int |D^\alpha u|^2 \leq (1-\overline{\delta}) \int |D^\alpha W|^2 \tag{2.11}$$

and

$$\int |D^\alpha u|^2 - |u|^{2^*} \geq \overline{\delta} \int |D^\alpha u|^2 . \tag{2.12}$$

(2) Assume $\alpha \in (\frac{N}{2N-1}, 1)$. *Let* u *be a solution of* (1.1) *with maximal interval* I, $\|D^\alpha u_0\|_{L^2} < \|D^\alpha W\|_{L^2}$, *and* $E_-(u_0) \leq (1-\delta_0)E_-(W)$ *for some* $\delta_0 > 0$. *Then, there exists* $\overline{\delta} = \overline{\delta}(\delta_0, N) > 0$ *such that for* $t \in I$

$$\int |D^\alpha u(t)|^2 \leq (1-\overline{\delta}) \int |D^\alpha W|^2 \tag{2.13}$$

$$\int |D^\alpha u(t)|^2 - |u(t)|^{2^*} \geq \overline{\delta} \int |D^\alpha u(t)|^2 \tag{2.14}$$

$$E_-(u(t)) \simeq \int |D^\alpha u(t)|^2 \simeq \int |D^\alpha u_0|^2 \tag{2.15}$$

with comparability constants which depend only on δ_0.

3. Profile decomposition and concentration-compactness alternative

A profile decomposition has been developped in [**4**] for the L^2-critical nonlocal Schrödinger equation. However, in our case, this is not enough. As in [**21**] and [**19**], one has the following theorem.

THEOREM 3.1. *Let* $\{v_{0,n}\} \in \dot{H}^\alpha$ *such that* $\int |D^\alpha v_{0,n}|^2 \leq A$ *with* $\alpha \in (\frac{N}{2N-1}, 1)$. *Assume furthermore that*

$$\|e^{-itD^{2\alpha}} v_{0,n}\|_{L^{2(N+2\alpha)/N-2\alpha}} \geq \delta(N)$$

Then there exists a sequence $\{V_{0,j}\} \in \dot{H}^\alpha$, *a subsequence of* $\{v_{0,n}\} \in \dot{H}^\alpha$ *and a triple* $(\lambda_{j,n}, x_{j,n}, t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$ *with the orthogonality condition as* $n \to \infty$ *for* $j \neq j'$

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to \infty$$

such that

$$\|V_{0,1}\|_{\dot{H}^\alpha} \geq \alpha_0(A) > 0.$$

If $V_j^\ell(x,t) = e^{-itD^{2\alpha}} V_{0,j}$ *then, given* $\epsilon_0 > 0$, *there exists* $J = J(\epsilon_0)$ *and a sequence* $\{w_n\} \in \dot{H}^\alpha$ *so that*

$$V_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{(N-2\alpha)/2}} V_j^\ell\Big(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}^2}\Big) + w_n$$

and for large enough n

$$\|e^{-itD^{2\alpha}} w_n\|_{S_\alpha(\mathbb{R})} \leq \epsilon_0$$

$$\int |D^\alpha V_{0,n}|^2 = \sum_{j=0}^{J} \int |D^\alpha V_{0,j}|^2 + \int |D^\alpha w_n|^2 + o(1), \quad \textit{as } n \to \infty \tag{3.1}$$

$$E_\pm(V_{0,n}) = \sum_{j=0}^{J} E_\pm(V_j^\ell(-t_{j,n}/\lambda_{j,n}^2)) + E_\pm(w_n) + o(1), \quad \textit{as } n \to \infty \tag{3.2}$$

PROOF. The original proof can be found in the work of Keraani [**21**]. Here we follow the approach of Visan [**27**]. For $\alpha \in (\frac{1}{2}, 1)$ and $N \geq 2$, we have that

$$\|e^{itD^{2\alpha}} f\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}(\mathbb{R}\times\mathbb{R}^N)} \lesssim \|f\|_{\dot{H}_x^\alpha(\mathbb{R}^N)} \tag{3.3}$$

holds for all radial $f \in \dot{H}^\alpha$.

LEMMA 3.4 (Refined Strichartz estimate). *Let $\alpha \in (\frac{N}{2N-1}, 1)$, $N \geq 2$, and $f \in \dot{H}^\alpha(\mathbb{R}^N)$, we have*

$$\|e^{itD^{2\alpha}} f\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}(\mathbb{R}\times\mathbb{R}^N)} \lesssim \sup_{j\in Z} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}(\mathbb{R}\times\mathbb{R}^N)}^{\frac{4\alpha}{N+2\alpha}} \|f\|_{\dot{H}_x^\alpha(\mathbb{R}^N)}^{\frac{N-2\alpha}{N+2\alpha}}, \tag{3.5}$$

where $f_j = P_j f$ and P_j is a Littlewood-Paley projection around the frequency annulus $\{\xi : |\xi| \sim 2^j\}$.

PROOF. First we consider the case when $N \geq 6\alpha$. We have $\frac{N+2\alpha}{2(N-2\alpha)} \leq 1$. We have

$$\begin{aligned}
&\|e^{itD^{2\alpha}} f\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}(\mathbb{R}\times\mathbb{R}^N)}^{\frac{2(N+2\alpha)}{N-2\alpha}} \\
&\lesssim \int\int_{\mathbb{R}\times\mathbb{R}^N} \left(\sum_{j\in\mathbb{Z}} |e^{itD^{2\alpha}} f_j|^2\right)^{\frac{N+2\alpha}{N-2\alpha}} dxdt \\
&\lesssim \int\int_{\mathbb{R}\times\mathbb{R}^N} \left(\sum_{j\in\mathbb{Z}} |e^{itD^{2\alpha}} f_j|^2\right)^{\frac{N+2\alpha}{2(N-2\alpha)}} \left(\sum_{k\in\mathbb{Z}} |e^{itD^{2\alpha}} f_k|^2\right)^{\frac{N+2\alpha}{2(N-2\alpha)}} dxdt \\
&\lesssim \sum_{j\leq k} \int\int_{\mathbb{R}\times\mathbb{R}^N} |e^{itD^{2\alpha}} f_j|^{\frac{N+2\alpha}{N-2\alpha}} |e^{itD^{2\alpha}} f_k|^{\frac{N+2\alpha}{N-2\alpha}} dxdt,
\end{aligned} \tag{3.6}$$

where we used the fact that $\frac{N+2\alpha}{2(N-2\alpha)} \leq 1$ in the last step. By applying Hölder inequality, Bernstein inequality, and Strichartz estimate (3.3), we obtain

$$
\begin{aligned}
(3.6) \lesssim & \sum_{j\leq k} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-4\alpha}}} \|e^{itD^{2\alpha}} f_j\|^{\frac{4\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \\
& \|e^{itD^{2\alpha}} f_k\|^{\frac{4\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \|e^{itD^{2\alpha}} f_k\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N}}} \\
& \lesssim \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|^{\frac{8\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \sum_{j\leq k} \\
& \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-4\alpha}}} \|e^{itD^{2\alpha}} f_k\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N}}} \\
& \lesssim \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|^{\frac{8\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \sum_{j\leq k} 2^{2\alpha j} \\
& \|e^{itD^{2\alpha}} f_j\|_{L_t^{\frac{2(N+2\alpha)}{N-4\alpha}} L_x^{\frac{2N(N+2\alpha)}{N^2+8\alpha^2}}} \|e^{itD^{2\alpha}} f_k\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N}}} \\
& \lesssim \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|^{\frac{8\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \sum_{j\leq k} 2^{2\alpha j} \|f_j\|_{L_x^2} \|f_k\|_{L_x^2} \\
& \lesssim \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|^{\frac{8\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \sum_{j\leq k} 2^{\alpha(j-k)} \|f_j\|_{\dot{H}_x^\alpha} \|f_k\|_{\dot{H}_x^\alpha} \\
(3.7) \qquad & \lesssim \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|^{\frac{8\alpha}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \|f\|^2_{\dot{H}_x^\alpha},
\end{aligned}
$$

which complete the proof for the case $N \geq 6\alpha$.

Now we turn to the case $2 \leq N < 6\alpha$. We have $\frac{N+2\alpha}{2(N-2\alpha)} > 1$ in this case. Set

$$\left[\frac{N+2\alpha}{N-2\alpha}\right] = m - 1,$$

for some integer $m \geq 3$. Proceeding as before, we have

$$
\begin{aligned}
& \|e^{itD^{2\alpha}} f\|^{\frac{2(N+2\alpha)}{N-2\alpha}}_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}(\mathbb{R}\times\mathbb{R}^N)} \\
& \lesssim \int\int_{\mathbb{R}\times\mathbb{R}^N} \Big(\sum_{j\in\mathbb{Z}} |e^{itD^{2\alpha}} f_j|^2 \Big)^{\frac{N+2\alpha}{N-2\alpha}} dxdt \\
& \lesssim \int\int_{\mathbb{R}\times\mathbb{R}^N} \prod_{i=1}^m \Big(\sum_{j_i\in\mathbb{Z}} |e^{itD^{2\alpha}} f_{j_i}|^2 \Big)^{\frac{N+2\alpha}{m(N-2\alpha)}} dxdt \\
& \lesssim \sum_{j_1\leq j_2\leq\cdots\leq j_m} \int\int_{\mathbb{R}\times\mathbb{R}^N} \prod_{i=1}^m |e^{itD^{2\alpha}} f_{j_i}|^{\frac{2(N+2\alpha)}{m(N-2\alpha)}} dxdt \\
(3.8) \qquad & \lesssim \sum_{j_1\leq j_2\leq\cdots\leq j_m} \int\int_{\mathbb{R}\times\mathbb{R}^N} \prod_{i=1}^m |e^{itD^{2\alpha}} f_{j_i}| |e^{itD^{2\alpha}} f_{j_i}|^{\frac{2(N+2\alpha)}{m(N-2\alpha)}-1} dxdt,
\end{aligned}
$$

where we use the fact $\frac{N+2\alpha}{m(N-2\alpha)} \leq 1$. By applying Hölder inequality, Bernstein inequality, and Strichartz estimate (3.3), we obtain

$$
\begin{aligned}
(3.8) \lesssim & \sum_{j_1\le\cdots\le j_m} \big\| |e^{itD^{2\alpha}} f_{j_1}||e^{itD^{2\alpha}} f_{j_m}| \big\|_{L_{t,x}^{\frac{N+2\alpha}{N-2\alpha}}} \\
& \prod_{i=2}^{m-1} \|e^{itD^{2\alpha}} f_{j_i}\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}} \prod_{i=1}^{m} \|e^{itD^{2\alpha}} f_{j_i}\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{2(N+2\alpha)}{m(N-2\alpha)}-1} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{m-2+m(\frac{2(N+2\alpha)}{m(N-2\alpha)}-1)} \sum_{j_1\le j_m} \Big(j_m - j_1\Big)^{m-2} \\
& \big\| |e^{itD^{2\alpha}} f_{j_1}||e^{itD^{2\alpha}} f_{j_m}| \big\|_{L_{t,x}^{\frac{N+2\alpha}{N-2\alpha}}} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{8\alpha}{N-2\alpha}} \sum_{j_1\le j_m} \Big(j_m - j_1\Big)^{m-2} \\
& \|e^{itD^{2\alpha}} f_{j_1}\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-4\alpha}}} \|e^{itD^{2\alpha}} f_{j_m}\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N}}} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{8\alpha}{N-2\alpha}} \sum_{j_1\le j_m} \Big(j_m - j_1\Big)^{m-2} 2^{2j_1\alpha} \\
& \|e^{itD^{2\alpha}} f_{j_1}\|_{L_t^{\frac{2(N+2\alpha)}{N-4\alpha}} L_x^{\frac{2N(N+2\alpha)}{N^2+8\alpha^2}}} \|e^{itD^{2\alpha}} f_{j_m}\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N}}} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{8\alpha}{N-2\alpha}} \sum_{j_1\le j_m} \Big(j_m - j_1\Big)^{m-2} 2^{2j_1\alpha} \|f_{j_1}\|_{L_x^2} \|f_{j_m}\|_{L_x^2} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{8\alpha}{N-2\alpha}} \sum_{j_1\le j_m} \Big(j_m - j_1\Big)^{m-2} 2^{(j_1-j_m)\alpha} \|f_{j_1}\|_{\dot H_x^\alpha} \|f_{j_m}\|_{\dot H_x^\alpha} \\
\lesssim & \sup_{j\in\mathbb{Z}} \|e^{itD^{2\alpha}} f_j\|_{L_{t,x}^{\frac{2(N+2\alpha)}{N-2\alpha}}}^{\frac{8\alpha}{N-2\alpha}} \|f\|_{\dot H_x^\alpha}^2, \qquad (3.9)
\end{aligned}
$$

which complete the proof for the case $2 \le N < 6\alpha$. Thus we complete the proof of the lemma. □

Once the refined Strichartz is established, the profile decomposition follows from standard techniques. See for instance [**27**].

□

4. Minimal energy non-scattering solution

We now assume $2\alpha < N < 6\alpha$. Denote $A_+ = \infty, A_- = E_-(W)$. For each $0 \le a \le A_\pm$, let

$$
\begin{aligned}
\mathcal{K}^-(a) :=& \{f \in \dot H^\alpha_{rad} : E_-(f) < a,\ \|D^\alpha f\|_2 < \|D^\alpha W\|_2\} \\
\mathcal{K}^+(a) :=& \{f \in \dot H^\alpha_{rad} : E_+(f) < a\}, \\
\mathcal{S}^\pm(a) :=& \sup\{\|u\|_{S_\alpha(I)} \mid u(0) \in \mathcal{K}^\pm(a),\ u \text{ sol. to (1.1) with } \pm\},
\end{aligned}
$$

Let

$$E^*_\pm := \sup\{a > 0 \mid \mathcal{S}^\pm(a) < \infty\}. \tag{4.1}$$

The small data scattering implies that $E^*_\pm > 0$. We will prove $E^*_\pm = A_\pm$ by contradiction, and thus finish the proof of Theorem 1.1.

Assume $E^*_\pm < A_\pm$, then we show the existence of a critical element which is compact modulo invariant groups. We have

LEMMA 4.1 (Existence of critical element). *Suppose $E^*_\pm < A_\pm$, then there is a radial solution $u_\pm$ to* (1.1)$\pm$ *with maximal interval $I_\pm$ satisfying*

$$E(u_\pm) = E^*_\pm, \quad \|D^\alpha u_-\|_2 < \|D^\alpha W\|_2, \quad \|u_\pm\|_{S_\alpha(I_\pm)} = \infty.$$

LEMMA 4.2. *Assume $u_\pm$ is as in Lemma 4.1 and say that $\|u_\pm\|_{S(I_\pm \cap (0,\infty))} = \infty$. Then there exists $\lambda(t) \in \mathbb{R}^+$, for $t \in I_\pm \cap (0,\infty)$, such that*

$$K = \{v(x,t) : v(x,t) = \frac{1}{\lambda(t)^{\frac{N-2\alpha}{2}}} u_\pm(\frac{x}{\lambda(t)}, t)\}$$

has the property that $\overline{K}$ is compact in $\dot{H}^\alpha$. A corresponding conclusion is reached if $\|u_\pm\|_{S_\alpha(I_\pm \cap (-\infty,0))} = \infty$.

The two lemmas above follow in the same way as in Kenig-Merle [**19**], by using stability Theorem and the profile decomposition given in Theorem 3.1.

5. Rigidity Theorem

The main purpose of this section is to disprove the existence of critical element that was constructed in the previous section under the assumption $E^*_\pm < A_\pm$ by using the structure of the equation (1.1). We will rely on the virial identity.

LEMMA 5.1 (virial identity). *Assume u is a smooth solution to* (1.1). *Then*

$$\frac{d}{dt}\Re \int iux \cdot \nabla \bar{u} dx = 2\alpha \int |D^\alpha u|^2 dx + \frac{N\mu p}{p+2} \int |u|^{p+2} dx.$$

The previous lemma is just a formal computation based on properties of the Fourier transform (recall that the symbol of $D^{2\alpha}$ is $|\xi|^{2\alpha}$).

Since the virial does not make sense in the energy space, we will use the localization of virial estimates. In this sequel, we fix $\psi \in C_0^\infty(\mathbb{R}^N)$, ψ radial, $\psi \equiv 1$ for $|x| < 1$, $\psi \equiv 0$ for $|x| \geq 2$. For $R \gtrsim 1$, let $\psi_R(x) = \psi(x/R)$, $\tilde{\psi}_R(x) = \frac{x}{R} \cdot \nabla \psi(\frac{x}{R})$. We have

LEMMA 5.2. *Assume u is solution to* (1.1). *Then*

$$\begin{aligned}\frac{d}{dt}\Re \int iux\psi_R \cdot \nabla \bar{u} dx =& 2\alpha \int |D^\alpha u|^2 \psi_R dx + \frac{p\mu N}{p+2} \int |u|^{p+2} \psi_R dx \\ &+ \Re 2 \int D^\alpha u [D^\alpha, \psi_R](x \cdot \nabla \bar{u}) dx + \frac{p\mu}{p+2} \int |u|^{p+2} \tilde{\psi}_R dx \\ &+ \Re N \int D^\alpha u [D^\alpha, \psi_R] \bar{u} dx + \Re \int D^\alpha u [D^\alpha, \tilde{\psi}_R] \bar{u} dx,\end{aligned}$$

where $[D^\alpha, f]g = D^\alpha(fg) - fD^\alpha g$.

PROOF. Using the equation (1.1), we get from direct computation that

$$\begin{aligned}&\frac{d}{dt}\Re\int iux\psi_R\cdot\nabla\bar{u}dx\\
=&\Re\int(D^{2\alpha}u+\mu|u|^pu)(2x\psi_R\cdot\nabla\bar{u}+N\psi_R\bar{u}+\tilde{\psi}_R\bar{u})dx\\
=&\Re\int D^{2\alpha}u2x\psi_R\cdot\nabla\bar{u}dx+\Re\int\mu|u|^pu2x\psi_R\cdot\nabla\bar{u}dx\\
&+\Re\int D^{2\alpha}u(d\psi_R\bar{u}+\tilde{\psi}_R\bar{u})dx+\Re\int\mu|u|^pu(N\psi_R\bar{u}+\tilde{\psi}_R\bar{u})dx\\
:=&I+II+III+IV.\end{aligned}$$

Obviously,

$$IV=N\mu\int|u|^{p+2}\psi_Rdx+\mu\int\tilde{\psi}_R|u|^{p+2}dx.$$

Using integration by part, we get

$$\begin{aligned}III=&\Re\int D^{2\alpha}u(d\psi_R\bar{u}+\tilde{\psi}_R\bar{u})dx\\
=&N\int|D^\alpha u|^2\psi_Rdx+\Re d\int D^\alpha u[D^\alpha,\psi_R]\bar{u}dx\\
&+\int|D^\alpha u|^2\tilde{\psi}_Rdx+\Re\int D^\alpha u[D^\alpha,\tilde{\psi}_R]\bar{u}dx.\end{aligned}$$

Similarly,

$$\begin{aligned}II=&\Re\int\mu|u|^pu2x\psi_R\cdot\nabla\bar{u}dx=\int\mu|u|^px\psi_R\cdot\nabla(|u|^2)dx\\
=&-\frac{2\mu N}{p+2}\int|u|^{p+2}\psi_Rdx-\frac{2\mu}{p+2}\int|u|^{p+2}\tilde{\psi}_Rdx\end{aligned}$$

Now we compute I. By Fourier transform, it is easy to check $[D^\alpha,x\cdot\nabla]=\alpha D^\alpha$. Then we have

$$\begin{aligned}I=&\Re2\int D^\alpha u\psi_R(x\cdot\nabla D^\alpha\bar{u}+\alpha D^\alpha\bar{u})dx+\Re2\int D^\alpha u[D^\alpha,\psi_R](x\cdot\nabla\bar{u})dx\\
=&2\alpha\int|D^\alpha u|^2\psi_Rdx-N\int|D^\alpha u|^2\psi_Rdx\\
&-\int|D^\alpha u|^2\tilde{\psi}_Rdx+\Re2\int D^\alpha u[D^\alpha,\psi_R](x\cdot\nabla\bar{u})dx.\end{aligned}$$

Summing over the four terms, we complete the proof. □

Due to the nonlocal properties of the fractional Schrödinger equation, the localization of virial estimates is not very clean. There are many remainder terms. However, all of them can be handled in the energy space. We have

LEMMA 5.3. *Assume* $0<\alpha\le1$, $0<\varepsilon<\alpha$ *and* $R\gtrsim1$. *Then*

$$\|[D^\alpha,\psi_R]f\|_{L^2}\lesssim\|g\|_{L^{\frac{2N}{N-2\alpha}}(|x|\gtrsim R^{1-\varepsilon})}+R^{-\varepsilon\alpha}\|D^\alpha f\|_{L^2},\tag{5.1}$$

$$\|[D^\alpha,\psi_R]x\cdot\nabla f\|_{L^2}\lesssim\|D^\alpha f\|_{L^2},\tag{5.2}$$

$$\|[D^\alpha,\psi_R]x\cdot\nabla f\|_{L^2(|x|\lesssim R^{1-\varepsilon})}\lesssim R^{-\varepsilon\alpha/2}\|D^\alpha f\|_{L^2}+\|g\|_{L^{\frac{2n}{n-2\alpha}}(|x|\gtrsim R^{1-\varepsilon})},\tag{5.3}$$

where $g = \mathcal{F}^{-1}(|\hat{f}|)$.

The proof of the lemma will be given in the end of this section. Now we use it to prove the main result of this section:

THEOREM 5.4. *Assume that* $u_0^\pm \in \dot{H}^\alpha$ *is such that*

$$E_\pm(u_0^\pm) < A_\pm, \quad \|D^\alpha u_0^-\|_2 < \|D^\alpha W\|_2.$$

Let $u_\pm$ *be the solution of* $(1.1)\pm$ *with* $u_\pm(0) = u_0^\pm$, *with maximal interval of existence* $I_\pm$. *Assume that there exists* $\lambda(t) > 0$, *for* $t \in I_\pm \cap [0,\infty)$, *with the property that*

$$K = \{v(x,t) : v(x,t) = \frac{1}{\lambda(t)^{\frac{N-2\alpha}{2}}} u_\pm(\frac{x}{\lambda(t)}, t)\}$$

is precompact in $\dot{H}^\alpha$. *Then we must have* $T_+(u_0) = \infty$, $u_0 \equiv 0$.

PROOF OF THEOREM 5.4. We only prove the focusing case, since the defocusing case follows in a similar way. Assume $I_- = (-T_-, T_+)$. It suffices to prove this theorem under the assumption that $\lambda(t) \geq A_0$ for some $A_0 > 0$ for all t, since the general case follows similarly as in [**19**]. The proof splits in two cases.

Case 1. $T_+(u_0) < +\infty$

With the same proof as in [**19**], we have $\lambda(t) \to \infty$ as $t \uparrow T_+(u_0)$. We define

$$y_R(t) = \int |u(x,t)|^2 \psi_R(x) dx, \quad t \in [0, T_+).$$

Then we have

$$\begin{aligned} y_R'(t) =& -2\,\mathrm{Im} \int D^{2\alpha}(u) \cdot \bar{u} \psi_R(x) dx \\ =& -2\,\mathrm{Im} \int D^\alpha u \cdot [D^\alpha(\bar{u}\psi_R) - \bar{u} \cdot D^\alpha \psi_R] dx - 2\,\mathrm{Im} \int D^\alpha u \cdot \bar{u} \cdot D^\alpha \psi_R dx. \end{aligned}$$

By the commutator estimates $\|D^\alpha(fg) - fD^\alpha g\|_2 \lesssim \|D^\alpha g\|_2 \|g\|_\infty$, $\alpha \in (0,1)$, we get

$$\begin{aligned} |y_R'(t)| \lesssim& \|D^\alpha u\|_2 \|D^\alpha(\bar{u}\psi_R) - \bar{u} \cdot D^\alpha \psi_R\|_2 + \|D^\alpha u\|_2 \|u\|_{\frac{2n}{n-2\alpha}} \|D^\alpha \psi_R\|_{\frac{n}{\alpha}} \\ \lesssim& \|D^\alpha u\|_2^2. \end{aligned}$$

Next, we show: for all $R > 0$,

$$\int_{|x|<R} |u(x,t)|^2 dx \to 0, \quad \text{as } t \to T_+(u_0). \tag{5.4}$$

In fact, $u(y,t) = \lambda(t)^{\frac{N-2\alpha}{2}} v(\lambda(t)y, t)$ so that

$$\begin{aligned} &\int_{|x|<R} |u(x,t)|^2 dx \\ =&\lambda(t)^{-2\alpha} \int_{|y|<R\lambda(t)} |v(y,t)|^2 dy \\ =&\lambda(t)^{-2\alpha} \int_{|y|<\varepsilon R\lambda(t)} |v(y,t)|^2 dy + \lambda(t)^{-2\alpha} \int_{\varepsilon R\lambda(t)<|y|<R\lambda(t)} |v(y,t)|^2 dy \\ :=&I + II. \end{aligned}$$

By Hölder and Sobolev, we have

$$I \lesssim \lambda(t)^{-2\alpha} \|v\|^2_{L^{\frac{2N}{N-2\alpha}}} (\varepsilon R\lambda(t))^{2\alpha} \lesssim (\varepsilon R)^{2\alpha} \|D^\alpha W\|_2^2,$$

while

$$II \lesssim \lambda(t)^{-2\alpha}(R\lambda(t))^{2\alpha}\|v\|^2_{L^{\frac{2N}{N-2\alpha}}(|x|\geq \varepsilon R\lambda(t))} \to 0, \text{ as } t \to T_+(u_0).$$

Thus (5.4) follows.

Therefore, we have

$$|y_R(0) - y_R(T_+(u_0))| \lesssim T_+(u_0)\|D^\alpha W\|_2^2,$$

which implies

$$y_R(0) \lesssim T_+(u_0)\|D^\alpha W\|_2^2.$$

Then letting $R \to \infty$, we obtain that $u_0 \in L^2(\mathbb{R}^N)$. Arguing as before,

$$|y_R(t) - y_R(T_+(u_0))| \lesssim (T_+(u_0) - t)\|D^\alpha W\|_2^2.$$

So

$$|y_R(t)| \lesssim (T_+(u_0) - t)\|D^\alpha W\|_2^2.$$

Letting $R \to \infty$, we see that

$$\|u(t)\|_2^2 \lesssim (T_+(u_0) - t)\|D^\alpha W\|_2^2$$

and so by the conservation of the L^2 norm $\|u_0\|_2 = \|u(t)\|_2 \to 0, t \to T_+(u_0)$. But that $u \equiv 0$ contradicting $T_+(u_0) < +\infty$.

Case 2. $T_+(u_0) = +\infty$

In this case we use the localized virial identity. Let $u(y,t) = \lambda(t)^{\frac{N-2\alpha}{2}} v(\lambda(t)y, t)$, then

$$\begin{aligned}
\int_{|y|>R(\varepsilon)} |D^\alpha u(y,t)|^2 dy &= \int_{|y|>R(\varepsilon)} \lambda(t)^N |D^\alpha v(\lambda(t)y,t)|^2 dy \\
&= \int_{|z|>\lambda(t)R(\varepsilon)} |D^\alpha v(z,t)|^2 dx \\
&\leq \int_{|z|\geq A_0 R(\varepsilon)} |D^\alpha v(z,t)|^2 dz \\
&\lesssim \varepsilon. \text{ (by the precompactness of K)}
\end{aligned}$$

By similar arguments, we have for any $\varepsilon > 0$, there exists $R(\varepsilon)$ such that

$$\int_{|x|>R(\varepsilon)} \Big(|D^\alpha u(x,t)|^2 + |u(x,t)|^{\frac{2N}{N-2\alpha}} + \frac{|u(x,t)|^2}{|x|^{2\alpha}}\Big) dx < \varepsilon \tag{5.5}$$

Let $\tilde{u} = \mathcal{F}_x^{-1}|\mathcal{F}u(\xi,t)|$. By Plancherel theorem we know $\tilde{u}$ has the same compactness as u. Thus we have: for each $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that, for all $t \in [0, \infty)$, we have

$$\int_{|x|>R(\varepsilon)} \Big(|D^\alpha \tilde{u}(x,t)|^2 + |\tilde{u}(x,t)|^{\frac{2N}{N-2\alpha}} + \frac{|\tilde{u}(x,t)|^2}{|x|^{2\alpha}}\Big) dx < \varepsilon. \tag{5.6}$$

Next, we consider

$$I_R(t) = \operatorname{Re}\int iux\psi_R \cdot \nabla\bar{u}dx.$$

By Cauchy-Schwarz inequality and integration by parts for fractional derivatives, we have

$$\begin{aligned}
|I_R(t)| \lesssim &\|D^{1-\alpha}(ux\psi_R)\|_2 \cdot \|D^{\alpha-1}\nabla\bar{u}\|_2 \\
\lesssim &\|D^\alpha u\|_2 \cdot \|D^{1+\frac{N}{2}-2\alpha}(x\psi_R)\|_2 \cdot \|D^\alpha u\|_2 \lesssim R^{2\alpha} \cdot \|D^\alpha u_0\|_2^2.
\end{aligned}$$

On the other hand, by Lemma 5.2 and Lemma 5.3, we have

$$I'_R(t) = 2\alpha \int |D^\alpha u|^2 dx - 2\alpha \int |u|^{\frac{2N}{N-2\alpha}} dx \tag{5.7}$$

$$+ 2\alpha \int (|D^\alpha u|^2 - |u|^{\frac{2N}{N-2\alpha}})(\psi_R - 1) dx - \frac{2\alpha}{N} \int |u|^{\frac{2N}{N-2\alpha}} \tilde{\psi}_R dx \tag{5.8}$$

$$+ 2 \operatorname{Re} \int D^\alpha u [D^\alpha, \psi_R](x \cdot \nabla \bar{u}) dx \tag{5.9}$$

$$+ d \operatorname{Re} \int D^\alpha u [D^\alpha, \psi_R] \bar{u} dx + \operatorname{Re} \int D^\alpha u [D^\alpha, \tilde{\psi}_R] \bar{u} dx. \tag{5.10}$$

By the variational estimates, we have

$$(5.7) \geq C_\delta \|D^\alpha u_0\|_2^2.$$

If $u_0 \neq 0$, then fix $0 < \varepsilon \ll \|D^\alpha u_0\|_2^2$. For (5.8), by (5.5) we get that

$$(5.8) \lesssim \varepsilon$$

for R sufficiently large. The term of (5.9) can be estimated as follows

$$\begin{aligned}
&|(5.9)| \\
\lesssim & \left| \int_{|x| \lesssim R^{1-\varepsilon}} D^\alpha u [D^\alpha, \psi_R](x \cdot \nabla \bar{u}) dx \right| + \left| \int_{|x| \gtrsim R^{1-\varepsilon}} D^\alpha u [D^\alpha, \psi_R](x \cdot \nabla \bar{u}) dx \right| \\
\lesssim & \|D^\alpha u\|_2 \|[D^\alpha, \psi_R](x \cdot \nabla \bar{u})\|_{L^2(|x| \lesssim R^{1-\varepsilon})} + \|D^\alpha u\|_{L^2(|x| \gtrsim R^{1-\varepsilon})} \|[D^\alpha, \psi_R](x \cdot \nabla \bar{u})\|_2 \\
\lesssim & R^{-\frac{\varepsilon\alpha}{2}} \|D^\alpha u\|_2 + \|\tilde{u}\|_{L^{\frac{2N}{N-2\alpha}}(|x| \gtrsim R^{1-\varepsilon})} + \|D^\alpha u\|_{L^2(|x| \gtrsim R^{1-\varepsilon})} \|D^\alpha u\|_{L^2}
\end{aligned}$$

where the last inequality follows from Lemma 3.5. Therefore, (5.9) $\lesssim \varepsilon$ if R is sufficiently large. The smallness of (5.10) can be obtained similarly. Thus

$$|I'_R(t)| \gtrsim \int |D^\alpha u_0|^2.$$

Integrating in t, we get $I_R(t) - I_R(0) \gtrsim t \int |D^\alpha u_0|^2$, but we also have $|I_R(t) - I_R(0)| \lesssim R^2 \int |D^\alpha u_0|^2$, which is a contradiction for t large. Thus $u_0 \equiv 0$ and the theorem is proved. $\square$

In the end, we give the proof of Lemma 5.3.

Proof of Lemma 5.3. First we show (5.1). Using Fourier transform, we have

$$\begin{aligned}
|\mathcal{F}([D^\alpha, \psi_R] f)(\xi)| \lesssim & \left| \int_{\xi = \xi_1 + \xi_2} (|\xi_1 + \xi_2|^\alpha - |\xi_2|^\alpha) \widehat{\psi_R}(\xi_1) \widehat{f}(\xi_2) \right| \\
\lesssim & \int_{\xi = \xi_1 + \xi_2} |\xi_1|^\alpha |\widehat{\psi_R}(\xi_1)| \cdot |\widehat{f}(\xi_2)|.
\end{aligned}$$

Then we get

$$\begin{aligned}
\|[D^\alpha, \psi_R] f\|_2 \lesssim & \|\mathcal{F}^{-1}(|\xi_1|^\alpha |\widehat{\psi_R}(\xi_1)|) \cdot g\|_2 \\
\lesssim & \|\mathcal{F}^{-1}(|\xi_1|^\alpha |\widehat{\psi_R}(\xi_1)|)\|_{\frac{N}{\alpha}} \cdot \|g\|_{L^{\frac{2N}{N-2\alpha}}(|x| \gtrsim R^{1-\varepsilon})} \\
& + \|\mathcal{F}^{-1}(|\xi_1|^\alpha |\widehat{\psi_R}(\xi_1)|)\|_{L^{\frac{N}{\alpha}}(|x| \lesssim R^{1-\varepsilon})} \cdot \|g\|_{L^{\frac{2N}{N-2\alpha}}} \\
\lesssim & \|g\|_{L^{\frac{2N}{N-2\alpha}}(|x| \gtrsim R^{1-\varepsilon})} + R^{-\varepsilon\alpha} \|D^\alpha f\|_{L^2}
\end{aligned}$$

where in the last inequality we used the fact that $\|\mathcal{F}^{-1}(|\xi_1|^\alpha|\widehat{\psi_R}(\xi_1)|)\|_{\frac{n}{\alpha}} \leq C$, $|\mathcal{F}^{-1}(|\xi_1|^\alpha|\widehat{\psi_R}(\xi_1)|)| \lesssim R^{-\alpha}$ and the Sobolev embedding.

Next, we prove (5.2). Direct computations show that

$$\begin{aligned}&\mathcal{F}([D^\alpha,\psi_R]x\cdot\nabla f)(\xi)\\ =&-\int(|\xi|^\alpha-|\xi_2|^\alpha)\widehat{\psi_R}(\xi-\xi_2)\nabla_{\xi_2}\cdot(\xi_2\widehat{f}(\xi_2))d\xi_2\\ =&\int_{\xi=\xi_1+\xi_2}-\alpha|\xi_2|^\alpha\widehat{\psi_R}(\xi_1)\widehat{f}(\xi_2)+i(|\xi_1+\xi_2|^\alpha-|\xi_2|^\alpha)\widehat{x\psi_R}(\xi_1)\cdot\xi_2\widehat{f}(\xi_2).\end{aligned}$$

Thus we get

$$|\mathcal{F}([D^\alpha,\psi_R]x\cdot\nabla f)(\xi)|\lesssim\int_{\xi=\xi_1+\xi_2}|\xi_2|^\alpha(|\widehat{\psi_R}(\xi_1)|+|\widehat{x\psi_R}(\xi_1)|\cdot|\xi_1|)\cdot|\widehat{f}(\xi_2)|$$

and then by Plancherel's equality

$$\|[D^\alpha,\psi_R]x\cdot\nabla f\|_{L^2}\lesssim\|D^\alpha f\|_{L^2}.$$

Finally, we prove (5.3). We have

$$\begin{aligned}&\mathcal{F}([D^\alpha,\psi_R]x\cdot\nabla f)(\xi)\\ =&\int_{\xi=\xi_1+\xi_2,|\xi_1|\ll|\xi_2|}-\alpha|\xi_2|^\alpha\widehat{\psi_R}(\xi_1)\widehat{f}(\xi_2)+i(|\xi_1+\xi_2|^\alpha-|\xi_2|^\alpha)\widehat{x\psi_R}(\xi_1)\cdot\xi_2\widehat{f}(\xi_2)\\ &+\int_{\xi=\xi_1+\xi_2,|\xi_1|\gtrsim|\xi_2|}-\alpha|\xi_2|^\alpha\widehat{\psi_R}(\xi_1)\widehat{f}(\xi_2)+i(|\xi_1+\xi_2|^\alpha-|\xi_2|^\alpha)\widehat{x\psi_R}(\xi_1)\cdot\xi_2\widehat{f}(\xi_2)\\ :=&\mathcal{F}[M(f)]+\mathcal{F}[R(f)].\end{aligned}$$

As before, we have

$$|\mathcal{F}[R(f)](\xi)|\lesssim\int_{\xi=\xi_1+\xi_2}|\xi_1|^\alpha(|\widehat{\psi_R}(\xi_1)|+|\widehat{x\psi_R}(\xi_1)|\cdot|\xi_1|)\cdot|\widehat{f}(\xi_2)|$$

and then as (5.1) we get

$$\|Rf\|_2\lesssim\|g\|_{L^{\frac{2n}{n-2\alpha}}(|x|\gtrsim R^{1-\varepsilon})}+R^{-\varepsilon\alpha}\|D^\alpha f\|_{L^2}.$$

To estimate $M(f)$, we need to exploit a cancelation. Since $|\xi_1|\ll|\xi_2|$, by fundamental theorem of calculus we have

$$|\xi_1+\xi_2|^\alpha-|\xi_2|^\alpha=\int_0^1\frac{d}{dt}|t\xi_1+\xi_2|^\alpha dt=\int_0^1\alpha|t\xi_1+\xi_2|^{\alpha-1}\frac{t\xi_1+\xi_2}{|t\xi_1+\xi_2|}dt\cdot\xi_1. \tag{5.11}$$

Recall that

$$\mathcal{F}[M(f)]=\int_{|\xi_1|\ll|\xi_2|}-\alpha|\xi_2|^\alpha\widehat{\psi_R}(\xi_1)\widehat{f}(\xi_2)+\int_{|\xi_1|\ll|\xi_2|}i(|\xi_1+\xi_2|^\alpha-|\xi_2|^\alpha)\widehat{x\psi_R}(\xi_1)\cdot\xi_2\widehat{f}(\xi_2)$$

Thus we get, using (5.11) for the second integral

$$\begin{aligned}\mathcal{F}[M(f)]=&\int_{|\xi_1|\ll|\xi_2|}-\alpha|\xi_2|^\alpha\widehat{\psi_R}(\xi_1)\widehat{f}(\xi_2)\\ &+\int_{|\xi_1|\ll|\xi_2|}\int_0^1\alpha|t\xi_1+\xi_2|^{\alpha-1}\frac{t\xi_1+\xi_2}{|t\xi_1+\xi_2|}dt\cdot i\xi_1\widehat{x\psi_R}(\xi_1)\cdot\xi_2\widehat{f}(\xi_2)\end{aligned} \tag{5.12}$$

Denote $\xi_s = (\xi_{s,1}, \cdots, \xi_{s,N}), s = 1, 2$, then the second term equals to

$$\int_{|\xi_1|\ll|\xi_2|} \sum_{j,k=1}^{N} \int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_{1,k} + \xi_{2,k}}{|t\xi_1 + \xi_2|} dt \cdot i\xi_{1,k} \widehat{x_j\psi_R}(\xi_1) \cdot \xi_{2,j} \widehat{f}(\xi_2)$$

$$= \int_{|\xi_1|\ll|\xi_2|} \int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_1 + \xi_2}{|t\xi_1 + \xi_2|} dt \cdot \xi_2 \widehat{\psi_R}(\xi_1) \widehat{f}(\xi_2)$$

$$+ \int_{|\xi_1|\ll|\xi_2|} \int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_1 + \xi_2}{|t\xi_1 + \xi_2|} dt \cdot \widehat{x \otimes \nabla\psi_R}(\xi_1) \cdot \xi_2 \widehat{f}(\xi_2).$$

Thus, we get

$$\mathcal{F}[M(f)] = \int_{|\xi_1|\ll|\xi_2|} (\int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_1 + \xi_2}{|t\xi_1 + \xi_2|} dt \cdot \xi_2 - \alpha|\xi_2|^\alpha) \widehat{\psi_R}(\xi_1) \widehat{f}(\xi_2)$$

$$+ \int_{|\xi_1|\ll|\xi_2|} i \int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_1 + \xi_2}{|t\xi_1 + \xi_2|} dt \cdot |\xi_2|^{-\alpha} \xi_2 \widehat{\tilde{\psi}_R}(\xi_1) \widehat{D^\alpha f}(\xi_2)$$

$$= \mathcal{F}[I] + \mathcal{F}[II].$$

For I, by mean value formula, we have

$$|I| \lesssim \int_{|\xi_1|\ll|\xi_2|} \alpha^2 |\xi_2|^{\alpha-1} |\xi_1| \cdot |\widehat{\psi_R}(\xi_1)| \cdot |\widehat{f}(\xi_2)|$$

and then

$$\|I\|_2 \lesssim R^{-\alpha} \|f\|_2.$$

For II, we see

$$II = \int K(x - y_1, x - y_2) \tilde{\psi}_R(y_1) D^\alpha f(y_2) dy_1 dy_2$$

where K is the kernel for the bilinear multiplier

$$K(x, y) = \int e^{i(x\xi_1 + y\xi_2)} m(\xi_1, \xi_2) d\xi_1 d\xi_2$$

with the symbol

$$m(\xi_1, \xi_2) = \int_0^1 \alpha|t\xi_1 + \xi_2|^{\alpha-1} \frac{t\xi_1 + \xi_2}{|t\xi_1 + \xi_2|} dt \cdot |\xi_2|^{-\alpha} \xi_2 \cdot 1_{|\xi_1|\ll|\xi_2|}.$$

It is easy to see from direct computations that m satisfy the Coifman-Meyer condition (see [**10**]), and then the kernel K is Calderón-Zygmund (see [**10**]) and

$$|K(x - y_1, x - y_2)| \lesssim (|x - y_1| + |x - y_2|)^{-2N}.$$

If $|y_1| \sim R, |x| \lesssim R^{1-\varepsilon}$, then

$$|K(x - y_1, x - y_2)| \lesssim R^{-2N}.$$

Hence we estimate by Cauchy-Schwarz inequality,

$$|\, II\,|^2 \le \|D^\alpha f\|_2^2 \frac{1}{R^{3N}} \int |\tilde{\psi}_1(y_1)|^2\, dy_1 \le \frac{C}{R^{3N}} \|D^\alpha f\|_2^2.$$

Hence

$$\|II\|^2_{L^2(|x|\lesssim R^{1-\varepsilon})} \le \frac{CR^{N(1-\varepsilon)}}{R^{3N}}.$$

Thus we get since R is large

$$\|II\|_{L^2(|x|\lesssim R^{1-\varepsilon})} \lesssim R^{-\varepsilon/2} \|D^\alpha f\|_2.$$

Therefore, the lemma is proved. □

Acknowledgement. This work was supported by Australian Research Council Discovery Project (Grant No. DP170101060). L. Zhao is supported by grant NSFC of China no. 11771415. Y. Wang is supported by NSFC of China, no. 11771140. Y. Sire is supported by a Simons fellowship.

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