A finite volume element method for simulating secondary settling tanks

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This contribution summarizes a new finite volume element (FVE) method as a unified scheme for coupled sedimentation-flow problems in secondary settling tanks in wastewater treatment. The model is two-dimensional in an axisymmetric setting. A new numerical example is presented.

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1 Model problem

Models for the simulation of secondary settling tanks in wastewater treatment are usually of macroscopic type, where the governing PDEs are used to describe the mass and linear momentum balance equations for the solid and liquid phases. A well-known challenge for 2D or 3D numerical simulation is the strong coupling between the field of solids concentration and the flow field of the mixture, which is defined by the velocity \(\mathbf{u}\) and the pressure \(p\). (Additional equations of motion for \(u\) and \(p\) do not arise in 1D models [11].) The governing equations can be written as follows, where \(\phi\), \(u\) and \(p\) are sought:

\[
\partial_t \phi + \nabla \cdot \mathbf{F}(\phi, \mathbf{u}) = \Delta A(\phi), \quad -\nabla \cdot (\mu(\phi)\varepsilon(\mathbf{u}) - \lambda p \mathbf{I}) = G(\phi), \quad \lambda \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \; t > 0. \tag{1}
\]

Here \(\mathbf{F}(\phi, \mathbf{u}) = \phi \mathbf{u} + f_{\text{fbk}}(\phi) \mathbf{k}\) is a flux vector, where \(f_{\text{fbk}}\) is the material specific Kynch batch flux density function and \(\mathbf{k}\) is the upward-pointing unit vector, the diffusion term \(\Delta A(\phi)\) models sediment compressibility (where \(A\) is assumed to be a strictly increasing function of \(\phi\)), and \(\mu(\phi)\varepsilon(\mathbf{u}) - \lambda p \mathbf{I}\) denotes the Cauchy stress tensor, where \(\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\) and \(\mu\) is concentration-dependent viscosity function. The initial and boundary conditions are as follows (see Figure 1). We consider \(t \in [0, T]\), and the spatial domain \((r, z) \in \Omega \subset \mathbb{R}^2\) corresponding to an axisymmetric setting. Initial data are specified for \(\phi\) and \(\mathbf{u}\). The unit is fed through the portion of the boundary \(\Gamma_{\mathrm{in}}\subseteq \partial \Omega\), where \(\mathbf{u} = \mathbf{u}_{\mathrm{in}}\) and \(\phi = \phi_{\mathrm{in}}\) are given. The boundary parts \(\Gamma_{\mathrm{out}}\) and \(\Gamma_{\mathrm{c}}\) correspond to discharge and overflow outlets, where \(\mathbf{u} = \mathbf{u}_{\mathrm{out}}\) and \(\mathbf{u} = \mathbf{u}_{\mathrm{c}}\) are specified. On the remainder of \(\partial \Omega\) we impose no-slip conditions (\(\mathbf{u} = 0\)) and zero-flux conditions for \(\phi\).

2 Finite Volume Element (FVE) method and numerical example

The FVE method can be summarized as follows [2]. We discretize (1) in time by a semi-implicit method. Starting from the appropriate weak formulation and employing standard stabilization techniques, we discretize (1) in space by a finite element (FE) scheme on a triangular “primal” mesh \(T_h\). Since \(\phi\) may be discontinuous (1) may be strongly degenerate), we choose piecewise linear, possibly discontinuous functions for \(\phi\), piecewise bilinear, continuous functions for \(\mathbf{u}\) and piecewise linear, continuous functions for \(p\). By standard discretization techniques we then obtain a Galerkin FE formulation for the coupled problem (1) including boundary conditions. Now, FVE methods are intermediate between FV and FE methods: FVE schemes generalize the FE scheme on a triangular “primal” mesh \(T_h\) and the diamond mesh \(T_h^\ast\) such that the FE method can then be expressed as a FV method where test functions are constant on the elements of \(T_h^\ast\) and \(T_h\). We define the following FE spaces for the approximation of \(\phi\), \(\mathbf{u}\) and \(p\), respectively:

\[
S_h := \{ s \in L^2_0(\Omega) : s|_K \in P_1(K) \forall K \in T \}, \quad V_h := \{ v \in H^1_0(\Omega) \cap C^0(\Omega) : v|_K \in P_1(K)^2 \forall K \in T \},
\]

\[
Q_h := \{ q \in L^2_0(\Omega) \cap C^0(\Omega) : q|_K \in P_1(K) \forall K \in T \}, \quad W_h := \{ w \in \mathbf{V} \cap C^0(\Omega) : w|_K \in P_1(K)^2 \forall K \},
\]

\[
V_h^\ast := \{ v \in L^2_0(\Omega) : v|_{K_j^\ast} \in P_0(K_j^\ast)^2 \forall K_j^\ast \in T^\ast, |v|_{K_j^\ast} = u_D \text{ if } K_j^\ast \text{ is a boundary volume} \},
\]

\[
S_h^\ast := \{ s \in L^2_0(\Omega) : s|_{K_F} \in P_0(K_F) \forall K_F \in T^\ast \}.
\]
Galerkin FE formulation. One eventually obtains the following space-time discrete scheme:

\[ \begin{align*}
\text{1. Given } & \phi_h^n, \text{ compute } u_h^n \text{ and } \rho^n_h \text{ by solving the Stokes problem} \\
& (\mu(\phi_h^n) \varepsilon_a(u_h^n), \varepsilon_a(v_h))_{1,\Omega} + \lambda(p_h^n, \nabla \cdot v_h)_{1,\Omega} - (G(\phi_h^n), P_h v_h)_{1,\Omega} \\
& + \sum_{F \in \mathcal{E}_h} \frac{1}{h_F^2} \left[ (\mu(\phi_h^n) \varepsilon_a(u_h^n) \cdot n_F)_{F}, [\mu(\phi_h^n) \varepsilon_a(v_h) \cdot n_F]_{F} \right]_{1,F} = 0 \\
& \forall v_h \in \mathcal{V}_h, \\
& \lambda(q_h, \nabla \cdot u_h^n)_{1,\Omega} + \sum_{K \in \mathcal{T}} h_K^2 \left( \lambda \nabla a_p h - G(\phi_h^n), \nabla a q_h \right)_{1,K} = 0 \\
& \forall q_h \in Q_h.
\end{align*} \]

\[ \begin{align*}
\text{2. Given } & u_h^n \text{ and } \rho_h^n \text{ computed in Step 1, obtain } \phi_h^{n+1} \text{ from} \\
& (\phi_h^{n+1} - \phi_h^n) / \Delta t^n, (R_h s_h)_{1,\Omega} - (F(\phi_h^{n+1}, u_h^n) - a(\phi_h^n, \nabla a \phi_h^{n+1}, \nabla a s_h))_{1,\Omega} \\
& + \sum_{F \in \mathcal{E}_h} \left( \frac{1}{h_F^2} (\phi_h^{n+1}, R_h s_h)_{F} + \frac{\mu}{h_F^2} (\phi_h^n, R_h s_h)_{F} \right)_{1,F} \\
& + \sum_{F \in \mathcal{E}_h} (\hat{F}(\phi_h^{n+1}, u_h^n) \cdot n_F - \left( a(\phi_h^n, \nabla a \phi_h^{n+1}, n_F \right))_{F} + \frac{\mu}{h_F^2} [R_h \phi_h^{n+1}]_{F} + [R_h \phi_h^{n+1}]_{F} \right)_{1,F} = 0 \\
& \forall s_h \in S_h.
\end{align*} \]

We choose the parameters \( \phi_0 = 0 \), \( \phi_{\text{max}} = 0.9 \), \( u_{\text{in}} = 2.2 \times 10^{-3} \text{ m/s} \), \( \phi_{\text{in}} = 0.1 \), \( g = 9.81 \text{ m/s}^2 \), \( u_{2,\text{out}} = 9 \nu u_{2,\text{in}} \), \( u_{3,\text{out}} = 0.019 \text{ m/s} \), \( \alpha = 5 \), \( \beta = 2.5 \), \( \lambda = 1 \), \( \sigma_0 = 0.05 \), \( b = 0.0961 \text{ m} \), \( \Delta t = 5 \text{ s} \), \( k_F^2 = 1/3 \), \( \kappa_F^2 = 100 \), \( k_F^2 = 500 \). The primal mesh \( \mathcal{T} \) is composed by 7410 elements and 4206 interior nodes. The boundary conditions for velocity at the suction lifts are given by \( u = (0, -u_{2,\text{out}}) / 4 \), where \( u_{2,\text{out}} = \nu u_{2,\text{in}}, \nu \) with \( \nu = 0.8 \) and \( \nu = 0.01 \) corresponding to operating conditions that differ from those studied in [2]. See Figure 3 for numerical results.

**Acknowledgements**

RB is supported by Fondecyt project 1090456 and BASAL project PFB 03 CMM (U. de Chile)/CT2MA. HT acknowledges support by Fondecyt project 11110264.

**References**