Grade 3/4 students' understanding of geometrical objects: Australian case studies on (mis)conceptions of cubes

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Our paper presents early findings from a study that investigated Australian Grade 3 and 4 students' geometrical conceptual knowledge. This study is part of an international collaboration between Australian and German researchers. It draws on earlier research of Year 3 German students' cube constructions, errors and misconceptions and the categorization of these. The student data reported in this paper was analyzed using a framework for guiding and describing sources of errors and/or misconceptions related to cube constructions. These findings highlight the importance for students to construct three-dimensional objects using a variety of materials.

Keywords: Geometrical concept knowledge, cube constructions, primary students, errors.

Introduction

Research relating to children's geometric reasoning has focused on the classification of geometric objects and application of the Van Hiele levels of geometric thinking (Bleeker, Stols & Putten, 2013). Within the early years of primary school, less has been reported on children's knowledge and visualization of three-dimensional objects, in detail. There is also limited Australian research of young children's knowledge of geometry (MacDonald, Goff, Docket & Perry, 2016).

Our research is an international collaboration between German and Australian colleagues, and in this paper we report Australian data collected from Grades 3 and 4 students. Our objective was to investigate whether students' constructions of three-dimensional objects (e.g. prisms) elicits insights into their understanding of the properties of prisms that would not otherwise be revealed. In the German iteration students responded to a one-on-one task based interview using Froebel's Gifts, small wooden blocks (cubes and prisms). These blocks are not commonly used in Australian schools and we were interested in how Australian students might demonstrate their geometrical conceptual knowledge when using Froebel's Gifts to construct larger prisms.

Geometrical conceptual knowledge refers to students' spatial concepts, including visualization, verbal, and construction skills, their understanding of the relationships between two-dimensional shapes (2-D) and three-dimensional (3-D) objects, and their reasoning processes (e.g., Dindyal, 2015). In this study, geometrical conceptual knowledge referred to students' perceptions, visualization skills and identification of distinct properties of 3-D objects, in particular prisms, prior to and when constructing different sized cubes.

Prior to this collaboration, the third author used Froebel's Gifts and construction tasks to investigate Grade 3 German students' knowledge of geometrical solids (Reinhold & Wöller, 2016). Students' construction strategies and products were interpreted according to the Van Hiele framework. The results indicated a wide variety of students' geometric conceptual knowledge of solids. Further studies of Grade 3 and 4 students' understanding of geometrical objects, including Australian students, would provide further insights of students' conceptual knowledge of 3-D objects, and into common errors or (mis)conceptions that might occur. Ulusoy (2015) suggested a model that identified a wide variety of selection types and errors relating to the identification of 2-D shapes (trapezoids). Our pilot study builds on other studies such as Fujita, Kondo, Kumakura and Kumakura (2017) who described older students' errors in relation to cube representations, and Finesilver (2016), who examined the spatial structuring, enumeration and errors of students working with 3-D arrays. Informed by these previous studies, these research questions guided our study:

- What errors are identified when Australian Grade 3/4 students use small cubes and rectangular prisms to construct cubes?
- What codes and categories can be used to classify students' responses and sources of errors and/or misconceptions when constructing cubes?

Theoretical framework

The initial framework that guided our analysis was developed in a previous study (Reinhold & Wöller, 2016). However, an in-depth analysis of the Australian data necessitated a widening of the initial framework. This also included drawing on the works of scholars who investigated children's errors of cube constructions, providing an opportunity to investigate further insights into children's geometrical conceptual knowledge through the lens of the students' errors.

Cubes and cube constructions: revisiting the Piagetian framework

The Swiss psychologist Piaget is well known for his contributions to children's cognitive development, including a framework of their intellectual development. According to Piaget and his colleagues, young children begin to understand space by exploring simple relationships such as order and enclosure, separation and proximity. This encompasses an understanding of topological relationships – which is ensued by the development of understanding projective and Euclidian concepts of space (Piaget & Inhelder, 1967). Euclidean concept of space, objects (or parts of them) are located not only relative to each other, but also according to co-ordinate axes in 3-D space. These three dimensions and relationships determine an understanding of a cube as one of the five Platonic solids. A cube is classified as a 3-D object comprising 12 edges of equal length, 6 square faces with 3 faces meeting at each of the 8 vertices. In terms of class inclusion, the cube is the only regular hexahedron, and differs from other rectangular prisms because it can be classified as a regular square prism (all rectangular faces are squares).

Children's cube constructions, in particular their errors and strategies when copying (cube) buildings have been investigated from the 1920s onwards (Reinhold, 2007). For example, Piaget and Inhelder (1971) investigated how children copied cube constructions (consisting of seven or more single cubes, displayed on a picture or as a solid model) when using wooden cubes. The analysis of these studies by Piaget and Inhelder included a classification of children's errors. For example, young children may identify units of a construction, but assemble them incorrectly (wrong orientation of segments in the construction). Furthermore, a global similarity of the cube construction may stand in contrast to incorrect details (e.g., wrong number of cubes). These observations are in line with Piaget's overall notion that young children focus on single aspects of a phenomenon and prevents them from simultaneously taking various viewpoints into account and relates to their struggle when coordination different perspectives of 3-D objects.

Visualization, coordination and integration of views in cube construction

Recent studies have also explored the notion of children's inability to coordinate and integrate their ideas simultaneously in visualization tasks. For example, Reinhold (2007) found that children tended to focus on isolated features, as they visualize "in bits" rather than sequentially when doing mental rotation tasks that included cube constructions. Instead of visualizing the entire structure of a construction, students were more likely to take into account only small portions of a 3-D array in an additive manner. Gorgorió (1998) found that focusing on single elements of a construction appears to happen more frequently when students construct by themselves, in contrast to drawing their responses.

Battista and Clements (1996) reported students' inability to coordinate two different orthogonal views for the construction of a rectangular prism when counting the number of single cubes. They found a lack of coordination whenever the students were unable "to recognize how they [the orthogonal views of a prism] should be placed in proper position relative to each other" (p. 267). This prevented some students from forming "one integrated mental image of the objects" (p. 272). Students' initial conceptions of rectangular prisms were characterized as uncoordinated sets of faces, and in transition to a more elaborated understanding, students may then reconstruct in layers, yet focus locally, piece by piece. Battista & Clements also argued that in order to enumerate the number of cubes students must have a mental model of the prism, and that spatial structuring provides the input and organization for enumeration. They defined spatial structuring as:

> the mental act of structuring an organization or form for an object or set of objects. The process … includes establishing units, establishing relationships between units … and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit) (p. 282).

Understanding of class inclusion in the Van Hiele framework

The *development of geometrical conceptual knowledge* from primary to secondary school has been defined as five levels of development (Van Hiele, 1986). According to Van Hiele (1986), younger children identify shapes by recognizing resemblances to everyday objects or by identifying prototypes in the stage of *Visualization.* In the ensuing stage of *Analysis* they are more likely to take a shape's properties into account when they decide upon categorization. Battista (2007) suggested renaming the *Analysis* stage as 'Analytical/Componential' which encompass three stages. The first is the "visual-informal componential reasoning" stage in which students focus on parts of shapes and then on the spatial relationships between the parts using visually based descriptions, that utilize informal language. In the next stage, "informal and insufficient-formal componential reasoning" their descriptions are a combination of informal and formal language, whereas in the final stage, "sufficient formal property-based reasoning" they use formal geometric language when describing and conceptualizing shapes. Students who have not reached the stage of *Analysis*, as defined by the Van Hiele, are unable to give a concise definition of a geometrical object whilst taking account of mathematical properties (Reinhold & Wöller, 2016). When students begin debating about the impact of various properties of relationships between shapes, they achieve the stage of *Abstraction*.

Method

We report three case studies that were representative of the Australian cohort in relation to the errors identified from the analysis that investigated Grade 3 and 4 students' (N=24) understanding of geometric objects whilst constructing a "cube". Students responded to a one-to-one interview (designed by German colleagues) using Froebel's cubes and cuboids (rectangular prisms) (see Figures 1-3), identifying insights into their understanding of the properties of prisms. Each interview was conducted by the Australian authors and video-recorded for later analysis. Students were asked to construct different cubes using smaller cubes (Froebel's Gift # 3) and then using rectangular prisms (Froebel's Gift # 4) (Reinhold, Downton & Livy, 2017). Examples of questions asked during interview included:

"Close your eyes and imagine a cube. Describe what you see."
"I want you to build a cube using these blocks (2cm cubes). How do you know this is a cube?"
"Can you build a different cube? How do you know this is a cube?"
"I want you to build a cube using these blocks (small rectangular prisms) …"
"Compare this cube (one 2 cm cube) with this rectangular prism (small rectangular prism).
"How are they the same and different? (blocks shown to student)"

Following an interpretative paradigm in qualitative data analysis, we used student responses and performance data for the analysis, guided by *Grounded Theory* methods (Corbin & Strauss, 2015). The first two authors independently used open coding, while viewing the videos, identifying key themes related to their responses and to their construction process. We highlighted evidence of students' difficulties with their visualization skills as well as when students mentally structured solids or assembled parts of their constructions. Three case studies are presented to show examples of the following coding and categories:

COORDINATION & INTEGRATION is evident when students notice the properties of the solid they are constructing including an ability to coordinate distinct units of (visual or cognitive) information, and simultaneously integrate all spatial information (spatial relationships and properties). Whenever some of these properties are not yet fully developed, the students focus on isolated units of visual or cognitive information. Elements of this category and visualisation encompass spatial structuring (Battista, 2007).

VISUALIZATION includes cognitive processes related to visual perception, mental images and mental transformation of the entire cube or of smaller units. This may challenge students' ability to see and imagine spatial relationships and the representations of a cube or parts of the object (e.g., mentally rotate or (dis)assemble parts of the cube that is under construction).

CLASS INCLUSION refers to knowledge of the relationships between two solids (e.g., cube as a special rectangular prism) and students' understanding the structure of the hierarchical classification. In single cases, this may include the use of terms indicating class inclusion (e.g., prism, rectangular prism, rectangular square prism). Similarly, when describing a cube, the three students related their description to everyday objects, such as, *it's like a box*, *a yellow block*, which is evidence of Level 1, Visualisation (Van Hiele, 1986).

FLEXIBILITY consists of students' ability to visualize and construct more than a single appropriate representation of a cube (2x2x2 as most common prototype). This may also address the idea to mentally (dis)assemble their own construction in various ways in order to flexibly find more than one solution (e.g., to enlarge a cube in length, width and height by adding one block). Whereas *STABILITY* refers to having a strong (mental) image of an object.

These categories prompted a critical debate allowing us to consider further interpretations for classifying and discussing students' sources of errors.

Results

The results focus on selected examples of three students' errors and (mis)conceptions showing screenshots to support our findings. We grouped and categorized their errors and (mis)conceptions, providing examples of each category (described above and in Figure 4).

Edward (pseudonyms used throughout) (aged 10 in Grade 4)

When asked to describe a cube Edward said, "A cube has 8 corners and 12 edges and is 3-D." He was able to construct a 2x2x2 cube (length x width x height) using eight smaller cubes and said, "It is equal on all sides and is 3-D". For his second cube construction, he built a 3x3x3 cube using 27 cubes stating, "All faces are equal." However, when asked to construct a cube using small rectangular prisms Edward did not draw on this insight. After an attempt to construct a cube using six rectangular prisms (see Figure 1 left), he removed two blocks (top layer) leaving a 2x2x2 construction (Figure 1 middle). He said, "This is now a cube." Edward was focusing incorrectly on the dimensions of the object (2x2x2) rather than the properties of a cube that all faces are equal. Next, he incorrectly constructed another cube (Figure 1 right) he responded that it was a cube because, "All the faces were even," revealing another misconception. Edward's construction of a cube using cubes was the 'prototypical' cube and he recognised that all the faces were equal. This response also aligns with his initial answer to the first interview question, in which he said, "a cube has six faces that are all equal and it looks like a dice". When challenged to construct different cubes using rectangular prisms he struggled. When asked why the third construction was a cube he indicated that all faces are even. It is possible that he was attending to the top face (a square) and did not notice that the other faces were rectangles.

Figure 1: Edwards' response when making cubes with smaller rectangular prisms.

Emma (aged 8 in Grade 3)

When asked to describe a cube Emma said, "It is a yellow block and it has 8 points and 6 sides and […] is a 3-D square." Emma referred to a cube as a 3-D square and attended to a holistic appearance (Van Hiele, 1986). With smaller cubes, she constructed a 2x2x2 cube by using a strategy of trial and error. Again, she said, “It is a cube because it is a 3-D square (counting the edges twice) with 8 points (counting the faces) and 6 sides.” When using the smaller rectangular prisms (Figure 2 left) she struggled and said, “I know it’s not a cube, it’s a rectangle but I don’t know how to make it a cube from here.” After several attempts (Figure 2 right) Emma concluded that she could not construct a cube. Emma had difficulty visualizing and constructing a cube when using rectangular prisms, most likely attributed by her lack of experiences with these blocks.

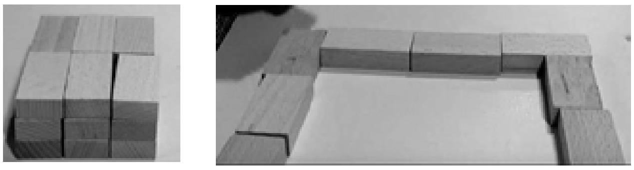

Figure 2: Emma’s response when making cubes with smaller rectangular prisms.

Chloe (aged 8 in Grade 3)

When asked to describe a cube, Chloe said, “A cube is like a box … it has 12 faces … is a square shape.” Chloe had difficulty constructing cubes (using cubes) and attempted to build an outline of the base of a cube (5+5+5+4) (Figure 3 left). Next, she built “towers” four cubes high on each corner (Figure 3 second) and filled in the ‘walls’ (Figure 3 third). Chloe said, “This is not a real cube because it needs to be filled up,” nor did she recognize the faces were not equivalent. Her final incorrect cube construction (Figure 3 fourth) had an uneven frame (5+5+6+4). After several attempts of counting the cubes Chloe still could not identify her error or make each length of the ‘faces’ equal. Her error related to spatial structuring when counting single elements (“Four, four, four and four.”) in a cube construction (Figure 3 right, the arrow visualizes her starting point for counting).

Figure 3: Chloe’s single steps when making a cube with smaller cubes and her counting of each cube.

The results of these case studies highlight the following (mis)conceptions or errors:

- Attention to the dimensions rather than properties of a cube (when using rectangular prisms)
- A focus on the top face being square (rather than all faces being square)
- An inability to visualize and construct a cube (when using cubes and/or rectangular prisms)

Our results confirm those of Reinhold (2007), that students tend to visualize “in bits” rather than visualizing the entire constructions, and their lack of spatial structuring (Battista, 2007), when constructing cubes. Many of these misconceptions or limited conceptual knowledge were evidence of: *co-ordination* and *integration, visualization, class inclusion* and/or *flexibility* and *stability* (Figure 4).

Discussion

The following discussion includes the results in relation to the four categories shown in Figure 4. Doing so highlights the cognitive demands and developmental aspects that we identified as possibly influencing the variety of students' correct solutions (not reported within this paper) and errors. As indicated in the literature, these categories were not entirely new to this field of research. Each of the four cases revealed difficulties in one or more of these categories.

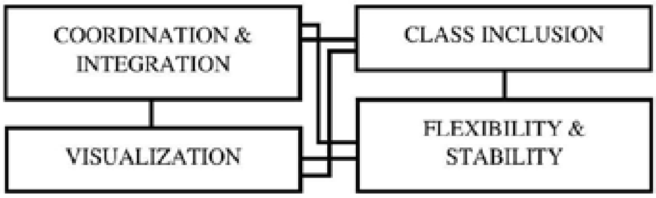

Figure 4: Sources of errors in Grade 3/4 students' concepts of cubes.

COORDINATION & INTEGRATION: Edward focused on the dimensions of the constructions rather than the properties of a cube when constructing a cube using rectangular prisms. He was focusing on isolated bits of information and units when visualising (Figure 1). *VISUALIZATION*: Emma and Chloe had difficulties with their spatial structuring when visualizing and making a cube using smaller rectangular prisms (Figures 2 & 3). They were unable to see and imagine spatial relationships and the representations of a cube or parts of the object (e.g., mentally rotate or (dis)assemble parts of the cube that is under construction). *CLASS INCLUSION*: Students such as Edward had not yet developed class inclusion as they could not conceptualize that a cube could also be classified as a rectangular prism. *FLEXIBILITY*: Most students in our study require further development of their understanding of the properties of cubes and rectangular prisms in order to develop *flexibility and stability*.

Conclusion and Implications

A commonality across the results was students' inability to integrate all spatial information simultaneously (e.g., Battista & Clements, 1996; Reinhold, 2007) and their lack of spatial structuring. Many students within this study could state properties of a cube correctly ("6 faces, 8 vertices and 12 edges") yet further exploration of their ability to construct and classify geometric objects revealed lack of spatial structuring. For instance, some students struggled with the idea that two adjacent cubes formed a rectangular prism (mental transformation referring to visualization). Hence, we argue that a deep understanding of the concept of a "cube" is not entirely developed until a student has the capability to *coordinate* and *integrate*; consider *class inclusion* correctly; *visualize representations* of a cube or parts of an object; and to find an appropriate balance of *flexibility* and *stability*.

These insights into student errors and/or misconceptions has highlighted the need to provide students with many opportunities to construct cubes (and cuboids) beyond the prototypical using a range of materials. An implication of our study is to highlight the importance for (Australian) teachers to use construction tasks to develop students' geometric thinking related to three-dimensional objects. Ongoing in-depth analysis of individual students' geometrical concepts of rectangular prisms aims to strengthen these findings and provide opportunities for further research.

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