Adv. Appl. Prob. **48,** 926–946 (2016)
*doi:*10.1017/apr.2016.34

RISK MINIMIZATION FOR GAME OPTIONS IN MARKETS IMPOSING MINIMAL TRANSACTION COSTS

YAN DOLINSKY * AND
YURI KIFER,** *The Hebrew University of Jerusalem*

Abstract

We study partial hedging for game options in markets with transaction costs bounded from below. More precisely, we assume that the investor's transaction costs for each trade are the maximum between proportional transaction costs and a fixed transaction cost. We prove that in the continuous-time Black–Scholes (BS) model, there exists a trading strategy which minimizes the shortfall risk. Furthermore, we use binomial models in order to provide numerical schemes for the calculation of the shortfall risk and the corresponding optimal portfolio in the BS model.

Keywords: Game option; transaction cost; hedging with friction; risk minimization

2010 Mathematics Subject Classification: Primary 91G10; 91G20
Secondary 60F15; 60G40; 60G44

1. Introduction

A game contingent claim (GCC) or game option, which was introduced in [11], is defined as a contract between the seller and the buyer of the option such that both have the right to exercise the option at any time up to a maturity date (horizon) T. If the buyer exercises the contract at time t then he receives the payment Y_t, but if the seller exercises (cancels) the contract before the buyer then the latter receives X_t. The difference $\Delta_t = X_t - Y_t$ is the penalty the seller pays to the buyer for the contract cancellation. In short, if the seller will exercise at a stopping time $\sigma \le T$ and the buyer at a stopping time $\tau \le T$ then the former pays to the latter the amount $H(\sigma, \tau)$, where $H(\sigma, \tau) = X_\sigma \mathbf{1}_{\{\sigma<\tau\}} + Y_\tau \mathbf{1}_{\{\tau\le\sigma\}}$. We set $\mathbf{1}_A = 1$ if an event A occurs and $\mathbf{1}_A = 0$ if not.

A hedge (for the seller) against a GCC is defined as a pair (π, σ) that consists of a self-financing strategy π and a stopping time σ which is the cancellation time for the seller, whom we also call an investor.

In this paper we study hedging with transaction costs of the following form. If the investor makes a small trade then he pays a fixed transaction cost, and if the investor makes a large trade he pays a proportional transaction cost. Formally, for buying (or selling) $\beta \ne 0$ stocks the transaction costs are given by $\max(\delta, \mu|\beta|S)$, where $\delta > 0$, $0 < \mu < 1$ are constants, and S is the stock price at the moment of the trade. The investor's total transaction cost should be finite;

Received 5 June 2015; revision received 31 August 2015.
* Postal address: Department of Statistics, The Hebrew University of Jerusalem, Mount Scopus, Jerusalem, 91905, Israel. Email address: yan.dolinsky@mail.huji.ac.il
** Postal address: Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, Jerusalem, 9190401, Israel. Email address: kifer@ma.huji.ac.il

hence, in our setup the investor can trade only a finite (random) number of times. Although this type of transaction cost is very natural and widespread it has received little attention in the literature.

In [3] it was proved that super-replication of game options under proportional transaction costs is expensive and leads to trivial (buy-and-hold) strategies. Since the friction in our setup is larger than the friction in the proportional costs setup, then similar results hold for our case as well. Therefore, with the presence of transaction costs, it is reasonable to assume that the seller's (investor's) initial capital is less than the superhedging price, and so hedging with risk comes into the picture. We deal with a certain type of risk called the shortfall risk (see [5] and [6]), which is the maximal expectation with respect to the buyer exercise times of the shortfall, but our results can be extended to other convex loss functions.

We refer the reader to, for example, [4], [7], [8], and [10] for background on shortfall risk minimization with friction. In all of these papers the authors considered the proportional transaction costs setup (for European and American options) for which they proved an existence of an optimal hedge.

In real market conditions transaction costs generally contain a fixed component, i.e. the transaction costs are bounded from below by a positive constant. Many authors considered utility maximization under transaction costs with a fixed component; see, for example, [1], [14], and [16]–[18]. However, for partial hedging of derivative securities this setup has not been studied before.

In this paper we consider a game option in the Black–Scholes (BS) model with continuous path-dependent payoffs. Our first result says that for our type of friction there is an optimal hedge. In general, the problem of the existence of an optimal hedge for the shortfall risk measure in a game options setup is much more complicated than for European and American options. The reason is that for game options the shortfall risk measure fails to be a convex functional of the portfolio strategy, and so the compactness principle which relies on the Komlós lemma can not be applied here. This is the principle that was applied for European and American options in all of the above mentioned papers. For the type of transaction costs considered here convexity arguments do not work for any type of options, and so our result is new even for American options (as a special case of game options) though in this case the proof is simpler. For game options without transaction costs or with only proportional transaction costs the existence of a shortfall minimization hedge remains an open question.

Our approach is to establish the continuity of the shortfall risk function and to reduce the optimization problem to a Dynkin game with continuous payoffs. Then we apply the theory of Dynkin games in a Brownian setup and use the fact that the set of all permissible trades is compact. Our results can be extended to more general continuous underlying processes than the geometric Brownian motion considered here.

Next, we deal with the computational aspect of shortfall risk minimization. We employ an appropriate sequence of binomial models in order to approximate the shortfall risk and to construct almost optimal portfolios in the BS model. For Lipschitz continuous (may be path dependent) payoffs, we obtain two-sided error estimates for the binomial approximations. So far, shortfall risk approximations for game options were considered only in the frictionless setup (see [6]), where the authors obtained only one-sided error estimates. In general, in the presence of transaction costs which larger than $\mu|\beta|S$ (β is the number of traded shares) for some constant $\mu > 0$, we find a uniform bound for the growth of admissible portfolios.

For game options in binomial models the shortfall risk and the corresponding optimal portfolio can be calculated by a dynamical programming algorithm. Thus, these approximation theorems provide an efficient tool for numerical calculation of the shortfall risk and the corresponding optimal hedge in the BS model.

The main results of this paper are formulated in the next section. In Section 3 we prove the existence of an optimal hedge (Theorem 2.1). In Section 4 we prove the approximation results (Theorem 2.2). The proof of both Theorems 2.1 and 2.2 rely on some regularity properties of the shortfall risk whose proofs we postpone till Section 5.

2. Preliminaries and main results

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a standard one-dimensional Brownian motion $\{W_t\}_{t=0}^{\infty}$, and the filtration $\mathcal{F}_t = \sigma\{W_s \mid s \le t\}$ completed by the null sets. Our BS financial market consists of a safe asset B used as a numeraire; hence, $B \equiv 1$, and of a risky asset S whose value at time t is given by

$$S_t^{(s)} = s\exp\biggl(\kappa W_t + \biggl(\vartheta - \frac{\kappa^2}{2}\biggr)t\biggr), \qquad s > 0,\ t \ge 0,$$

where $\kappa > 0$ is called volatility and $\vartheta \in \mathbb{R}$ is another constant. By Girsanov's theorem, the probability measure Q whose restrictions to each σ-algebra $\mathcal{F}_t$ satisfy

$$Z_t := \frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\bigg| \mathcal{F}_t = \exp\biggl(-\frac{\vartheta}{\kappa}W_t - \frac{1}{2}\biggl(\frac{\vartheta}{\kappa}\biggr)^2 t\biggr) \tag{2.1}$$

is the unique probability measure equivalent to $\mathbb{P}$ such that the stock price $S^{(s)}$ is a Q martingale.

Next, let $T < \infty$ and let $C[0, T]$ be the space of all continuous functions $f\colon [0, T] \to \mathbb{R}$ equipped with the uniform topology. Denote by $C_{++}[0, T] \subset C[0, T]$ the subset of all strictly positive functions. Let F and $G\colon C[0, T] \to C[0, T]$ be continuous progressively-measurable functions which means that for any $t \in [0, T]$ and $x, y \in C[0, T]$, $G(x)_{[0,t]} = G(y)_{[0,t]}$ and $F(x)_{[0,t]} = F(y)_{[0,t]}$ if $x_{[0,t]} = y_{[0,t]}$. We assume that $F \le G$ and there exist constants $C, p > 0$ for which

$$||F(x)|| + ||G(x)|| \le C(1 + ||x||^p), \tag{2.2}$$

where $||\cdot||$ denotes the standard norm on the space $C[0, T]$. Consider a game option with maturity date T and continuous payoffs given by

$$Y_t^{(s)} = [F(S^{(s)})](t) \le [G(S^{(s)})](t) = X_t^{(s)}, \qquad t \in [0, T].$$

In our model, purchase and sale of the risky asset are subject to transaction costs that are the maximum of a constant fee and a proportional transaction cost. Namely, if the investor buys (or sells) β stocks then his/her transaction cost is equal to

$$g(\beta, S) := \max(\delta, \mu|\beta|S)\,\mathbf{1}_{\{\beta \ne 0\}},$$

where $\delta > 0$, $0 < \mu < 1$ are constants and S is the stock price at the moment of trade. The presence of this minimal transaction cost yields that in order to avoid infinite transaction costs portfolios can only be rebalanced finitely many (but a random number of) times.

Next, we define hedging and shortfall risk in the above setup. A (self-financing) trading strategy with an initial position (z, y) is a triple $\pi = (z, y, \gamma)$, where z is the cash value of the portfolio at the initial time, y is the number of stocks at this moment, and $\gamma = \{\gamma_t\}_{t=0}^{T}$ is an adapted, left-continuous, pure jump process with finite (random) number of jumps and initial value $\gamma_0 = y$. The random variable γ_t denotes the number of shares in the portfolio π at time t before any change is made at this time (which is the reason why we assume that the process γ is left-continuous). Observe that at time 0 the investor has the value $z + g(y, s) - ys$ on his/her savings account. Thus, the portfolio (cash) value of a trading strategy π at time t is given by

$$V_t^{\pi} = z + \int_0^t \gamma_u \,\mathrm{d}S_u^{(s)} + g(y, s) - g(\gamma_t, S_t^{(s)}) - \sum_{u \in [0,t)} g(\gamma_{u+} - \gamma_u, S_u^{(s)}), \tag{2.3}$$

where in the last sum there is only finitely many terms which are not equal to 0. A portfolio π will be called *admissible* if $V_t^{\pi} \geq 0$ for any t. A hedge consists of a trading strategy and a cancellation time. Thus, formally, a hedge with initial position (z, y) is a pair (π, σ) such that π is an *admissible* portfolio and $\sigma \leq T$ is a stopping time (with respect to the Brownian filtration). From (2.3), it follows that for an *admissible* portfolio π the stochastic process V_t^{π}, $t \geq 0$, is a supermartingale with respect to the martingale measure Q. The set of all hedges with initial position $(z, y) \in \mathbb{R}_+ \times \mathbb{R}$ will be denoted by $\mathcal{A}(T, s, z, y)$. The set of all hedges will be denoted by $\mathcal{A}(T, s)$, where s is the initial stock price and T is the maturity date.

Next, we define the shortfall risk. Denote by $\mathcal{T}_T$ the set of all stopping times less than or equal to T. For a hedge (π, σ), the shortfall risk is defined by

$$\mathcal{R}(T, s, \pi, \sigma) = \sup_{\tau \in \mathcal{T}_T} \mathbb{E}_{\mathbb{P}}(X_\sigma^{(s)} \mathbf{1}_{\{\sigma < \tau\}} + Y_\tau^{(s)} \mathbf{1}_{\{\tau \leq \sigma\}} - V_{\sigma \wedge \tau}^{\pi})^+,$$

which is the maximal possible expectation with respect to the probability measure $\mathbb{P}$ of the shortfall. The shortfall risk for an initial position (z, y) is given by

$$R(T, s, z, y) = \inf_{(\pi,\sigma) \in \mathcal{A}(T,s,z,y)} \mathcal{R}(T, s, \pi, \sigma).$$

From the following theorem we see that for a given initial position (z, y) there exists a hedge which minimizes the shortfall risk.

Theorem 2.1. *Let $(z, y) \in \mathbb{R}_+ \times \mathbb{R}$ be an initial position. There exists a hedge (which may not be unique) $(\hat{\pi}, \hat{\sigma}) \in \mathcal{A}(T, s, z, y)$ such that $\mathcal{R}(T, s, \hat{\pi}, \hat{\sigma}) = R(T, s, z, y)$.*

Next, we approximate the shortfall risk in the BS model by a sequence of binomial models. In order to obtain error estimates we will assume that the functions F and G can be extended to the space $D[0, T]$ (of all càdlàg functions on the interval $[0, T]$) and satisfy the following Lipschitz condition. There exists a constant L such that, for any $0 \leq t_1 < t_2 \leq T$ and $x, y \in D[0, T]$,

$$||F(y) - F(x)|| + ||G(y) - G(x)|| \leq L||y - x||, \tag{2.4}$$

$$|F(x)(t_2) - F(x)(t_1)| + |G(x)(t_2) - G(x)(t_1)| \leq L\Big((t_2 - t_1)(1 + ||x||) + \sup_{t_1 \leq u \leq t_2} |x(u) - x(t)|\Big).$$

For any n consider a binomial model which consists of a savings account equivalent to 1, and of a piecewise constant risky asset $\{S_t^{n,s}\}_{t=0}^T$ given by

$$S_t^{n,s} = s \exp\bigg(\kappa \sqrt{\frac{T}{n}} \sum_{i=1}^{[nt/T]} \xi_i\bigg), \qquad t \in [0, T],$$

where $[\cdot]$ denotes the integer part and $\xi_1, \xi_2, \dots$ are independent and identically distributed random variables taking values 1 and -1 with probabilities

$$p^{(n)} = \left(\exp\left(\left(\kappa - \frac{2\vartheta}{\kappa}\right)\sqrt{\frac{T}{n}}\right) + 1\right)^{-1}, \qquad 1 - p^{(n)} = \left(\exp\left(\left(\frac{2\vartheta}{\kappa} - \kappa\right)\sqrt{\frac{T}{n}}\right) + 1\right)^{-1},$$

respectively. Let $\mathbb{P}_n$ be the corresponding probability measure and let $\mathcal{F}_t^{(n)} = \sigma\{S_u^{n,s} : u \le t\}$ be the filtration generated by $S^{n,s}$.

Denote by $\mathcal{A}^{(n)}(T, s, z, y)$ the set of all hedges with an initial position (z, y). The definition of a hedge is performed in an analogous way to the BS model, replacing the Brownian filtration by $\mathcal{F}^{(n)}$ and $S^{(s)}$ by $S^{n,s}$ in (2.3).

We introduce game options with the piecewise constant payoffs

$$Y_t^{n,s} = [F(S^{n,s})]\left(\frac{[nt/T]T}{n}\right) \le X_t^{n,s} = [G(S^{n,s})]\left(\frac{[nt/T]T}{n}\right), \qquad t \in [0, T].$$

Define the shortfall risk

$$\mathcal{R}_n(T, s, \pi, \sigma) = \sup_{\tau \in \mathcal{T}_T^{(n)}} \mathbb{E}_{\mathbb{P}_n}(X_\sigma^{n,s}\mathbf{1}_{\{\sigma<\tau\}} + Y_\tau^{n,s}\mathbf{1}_{\{\tau\le\sigma\}} - V_{\sigma\wedge\tau}^\pi)^+,$$

where $\mathcal{T}_T^{(n)}$ is the set of all stopping times less than T. The shortfall risk for an initial position (z, y) is given by $R_n(T, s, z, y) = \inf_{(\pi,\sigma)\in\mathcal{A}^{(n)}(T,s,z,y)} \mathcal{R}_n(T, s, \pi, \sigma)$.

Next, we introduce a simple form of Skorokhod embedding which allows us to consider the above binomial markets and the BS model on the same probability space. Set

$$W_t^* = \frac{\ln S_t^{(s)} - \ln s}{\kappa}, \qquad t \ge 0,$$

and, for any $n \in \mathbb{N}$, define recursively

$$\theta_0^{(n)} = 0, \qquad \theta_{k+1}^{(n)} = \inf\left\{t > \theta_k^{(n)} : |W_t^* - W_{\theta_k^{(n)}}^*| = \sqrt{\frac{T}{n}}\right\}.$$

Observe (see [6]) that, for any k, $W_{\theta_{k+1}^{(n)}}^* - W_{\theta_k^{(n)}}^*$ is independent of $\mathcal{F}_{\theta_k^{(n)}}$ and takes on the values $\sqrt{T/n}$ and $-\sqrt{T/n}$, with probabilities $p^{(n)}$ and $1 - p^{(n)}$, respectively. For any n, define the map $\Pi_n : L^\infty(\mathcal{F}_T^{(n)}, \mathbb{P}_n) \to L^\infty(\mathcal{F}_{\theta_n^{(n)}}, \mathbb{P})$ by $\Pi_n(U) = \widetilde{U}$ so that if

$$U = f\left(\sqrt{\frac{T}{n}}\xi_1, \dots, \sqrt{\frac{T}{n}}\xi_n\right)$$

for a function f on $\{\sqrt{T/n}, -\sqrt{T/n}\}^n$ then

$$\widetilde{U} = f(W_{\theta_1^{(n)}}^*, W_{\theta_2^{(n)}}^* - W_{\theta_1^{(n)}}^*, \dots, W_{\theta_n^{(n)}}^* - W_{\theta_{n-1}^{(n)}}^*).$$

The map Π_n allows us to lift hedges from the binomial models to the BS model. For an initial position (z, y) denote by $\mathcal{A}^{W,n}(s, z, y)$ the set of all *admissible* self-financing strategies which are managed on the set $\{0, \theta_1^{(n)}, \dots, \theta_n^{(n)}\}$ such that after the time $\theta_n^{(n)}$ the number of stocks in the portfolio is 0. Namely, $\pi = (z, y, \{\widetilde{\gamma}_t\}_{t=0}^\infty) \in \mathcal{A}^{W,n}(s, z, y)$ if $\widetilde{\gamma}_t$ is constant on the interval $(\theta_k^{(n)}, \theta_{k+1}^{(n)}]$, $k < n$, and $\widetilde{\gamma}_t \equiv 0$ on $(\theta_n^{(n)}, \infty)$. The portfolio value is given by (2.3). Define the lifting $\Psi_n \colon \mathcal{A}^{(n)}(T, s, z, y) \to \mathcal{A}^{W,n}(s, z, y) \times \mathcal{T}_T$, $\Psi_n(\pi, \sigma) = (\widetilde{\pi}, \widetilde{\sigma})$ as follows. Let $\pi = (z, y, \gamma)$. Then $\widetilde{\pi} = (z, y, \widetilde{\gamma})$, where

$$\widetilde{\gamma}_t = y\,\mathbf{1}_{\{t=0\}} + \sum_{i=0}^{n-1} \Pi_n(\gamma_{(i+1)T/n})\,\mathbf{1}_{\{\theta_i^{(n)} < t \le \theta_{i+1}^{(n)}\}}.$$

Similar arguments as in [4, Section 2] yield

$$V^{\widetilde{\pi}}_{\theta_k^{(n)}} = \Pi_n(V^{\pi}_{kT/n}) \quad \text{for } k = 0, 1, \dots, n,$$

and $V_t^{\widetilde{\pi}} \ge 0$, $t \ge 0$. Furthermore, the portfolio value $V_t^{\widetilde{\pi}}$ is constant after $\theta_n^{(n)}$. Observe that if we restrict the portfolio $\widetilde{\pi}$ to the interval $[0, T]$, we obtain an element in $\mathcal{A}(T, s, z, y)$. Next, we define $\widetilde{\sigma} \in \mathcal{T}_T$ by

$$\widetilde{\sigma} = \begin{cases} T \wedge \theta^{(n)}_{\Pi_n(\sigma)} & \text{if } \Pi_n(\sigma) < n, \\ \widetilde{\sigma} = T & \text{if } \Pi_n(\sigma) = n. \end{cases}$$

From the following theorem we see that the shortfall risk in the BS model can be approximated by the shortfall risks in the binomial models defined above. Furthermore, by lifting the optimal hedges in the binomial models we obtain 'almost' optimal hedges in the BS model.

Theorem 2.2. *Let $(z, y) \in \mathbb{R}_{++} \times \mathbb{R}$ be an initial position. There exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$,*

$$|R(T, s, z, y) - R_n(T, s, z, y)| \le C n^{-1/4} (\ln n)^{3/4}.$$

Furthermore, let $(\pi_n, \sigma_n) \in \mathcal{A}^{(n)}(T, s, z, y)$ be an optimal hedge, i.e. $\mathcal{R}_n(T, s, \pi_n, \sigma_n) = R_n(T, s, z, y)$. Then for the hedges $(\widetilde{\pi}_n, \widetilde{\sigma}_n) = \Psi_n(\pi_n, \sigma_n)$, $n \in \mathbb{N}$, we have

$$\mathcal{R}(T, s, \widetilde{\pi}_n, \widetilde{\sigma}_n) \le R(T, s, z, y) + C n^{-1/4} (\ln n)^{3/4},$$

where on the right-hand side we take the restriction of $\widetilde{\pi}_n$ to the interval $[0, T]$.

Remark 2.1. Theorems 2.1 and 2.2 can be extended to the case where the constant component δ becomes a Lipschitz continuous function of time $\delta(t)$. This makes sense since then we can consider the constant minimal transaction cost δ with respect to the original currency, which measured by a numeraire has the form $\delta(t) = \delta \mathrm{e}^{rt}$ with r being the interest rate. In this case the proof will become more technical and somewhat unwieldy. In order to simplify the exposition we will deal here only with the case where δ is constant with respect to the numeraire.

3. Proof of Theorem 2.1

We start with some preparations. For any $\boldsymbol{T} \le T$ and $v \in C_{++}[0, T]$, define the continuous stochastic process $\{S_t^{\boldsymbol{T}, v}\}_{t=0}^{T}$ by

$$S_t^{\boldsymbol{T}, v} = v_{T-\boldsymbol{T}}\,\mathbf{1}_{\{t \le T-\boldsymbol{T}\}} + S^{(v_{T-\boldsymbol{T}})}_{t+\boldsymbol{T}-T}\,\mathbf{1}_{\{t > T-\boldsymbol{T}\}}.$$

Namely, $S^{\boldsymbol{T}, v}$ coincides with v on the interval $[0, T - \boldsymbol{T}]$ and $S_t^{\boldsymbol{T}, v}$ is a geometric Brownian motion for $t > T - \boldsymbol{T}$. Consider a cash-settled game option with a maturity date $\boldsymbol{T} < \infty$ defined in a BS financial market which is described in Section 2. The payoffs are given by

$$Y_t^{\boldsymbol{T}, v} = [F(S^{\boldsymbol{T}, v})](t + T - \boldsymbol{T}) \quad \text{and} \quad X_t^{\boldsymbol{T}, v} = [G(S^{\boldsymbol{T}, v})](t + T - \boldsymbol{T}), \qquad t \in [0, \boldsymbol{T}].$$

Observe that the processes $X_t^{\boldsymbol{T},v} \ge Y_t^{\boldsymbol{T},v}$, $t \in [0, \boldsymbol{T}]$, are continuous and adapted. Furthermore, if $\boldsymbol{T} = T$ then $X_t^{\boldsymbol{T},v} = X_t^{(v_0)}$ and $Y_t^{\boldsymbol{T},v} = Y_t^{(v_0)}$ for $t \in [0, T]$.

Next, we define the shortfall risk for a maturity $\boldsymbol{T} \le T$. The sets $\mathcal{A}(\boldsymbol{T}, s, z, y)$ and $\mathcal{T}_{\boldsymbol{T}}$ are defined as in Section 2, replacing T by $\boldsymbol{T}$. For $v \in C_{++}[0, T]$ and a hedge $(\pi, \sigma) \in \mathcal{A}(\boldsymbol{T}, v_{T-\boldsymbol{T}}, z, y)$, the shortfall risk is defined by

$$\mathcal{R}(\boldsymbol{T}, v, \pi, \sigma) = \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}(X_\sigma^{\boldsymbol{T},v}\, \mathbf{1}_{\{\sigma<\tau\}} + Y_\tau^{\boldsymbol{T},v}\, \mathbf{1}_{\{\tau\le\sigma\}} - V_{\sigma\wedge\tau}^\pi)^+.$$

Similarly to the above set $R(\boldsymbol{T}, v, z, y) = \inf_{(\pi,\sigma)\in\mathcal{A}(\boldsymbol{T}, v_{T-\boldsymbol{T}}, z, y)} \mathcal{R}(\boldsymbol{T}, v, \pi, \sigma)$. Observe that $\mathcal{R}(\boldsymbol{T}, v, \cdot, \cdot\cdot)$ and $R(\boldsymbol{T}, v, \cdot, \cdot\cdot)$ depend only on $v_{[0,T-\boldsymbol{T}]}$. Furthermore, for $\boldsymbol{T} = T$, if $v_0 = s$ then $\mathcal{R}(T, v, \cdot, \cdot\cdot) = \mathcal{R}(T, s, \cdot, \cdot\cdot)$ and $R(T, v, \cdot, \cdot\cdot) = R(T, s, \cdot, \cdot\cdot)$.

Now, assume that at a given time a portfolio value is z, the number of stocks is y, and the stock price at this moment is S. If the investor buys $\beta \ne 0$ stocks then the new (cash-settled) portfolio value will be

$$h(S, z, y, \beta) := z + g(y, S) - g(y + \beta, S) - g(\beta, S). \tag{3.1}$$

For $\beta = 0$, we define the function h such that it will be continuous in $\beta = 0$. Thus, we set $h(S, z, y, 0) = z - \delta$. Let $\Gamma(S, z, y)$ be the set of all β which satisfy $h(S, z, y, \beta) \ge 0$. It is clear that $\Gamma(S, z, y)$ is a compact set. Observe also that a portfolio strategy is *admissible* if and only if it consists of permissible trades. For $y \ne 0$, we have $-y \in \Gamma(S, z, y)$, and so the set $\Gamma(S, z, y)$ is not empty. For $y = 0$, the set $\Gamma(S, z, y)$ is empty if and only if $z < \delta$. Define

$$\hat{R}(\boldsymbol{T}, v, z, y) = \min\Big((X_0^{\boldsymbol{T},v} - z)^+, \inf_{\beta\in\Gamma(v_{T-\boldsymbol{T}}, z, y)} R(\boldsymbol{T}, v, h(v_{T-\boldsymbol{T}}, z, y, \beta), \beta + y)\Big), \tag{3.2}$$

where the infimum over an empty set is ∞.

Next, let s be the initial stock price and (z, y) be the initial position of the investor. Set

$$\boldsymbol{V}_t^{s,z,y} = z + g(y, s) + y(S_t^{(s)} - s) - g(y, S_t^{(s)}), \qquad t \ge 0.$$

Observe that $\boldsymbol{V}_t^{s,z,y}$ is the portfolio value at time t of the investor who did not trade until this time. The following result is the first step in the proof of Theorem 2.1.

Lemma 3.1. *Let $(\boldsymbol{T}, v, z, y) \in [0, T] \times C_{++}[0, T] \times \mathbb{R}_+ \times \mathbb{R}$ and let $s = v_{T-\boldsymbol{T}}$ be the initial stock price. Define the stopping time $\Theta = \boldsymbol{T} \wedge \inf\{t : \boldsymbol{V}_t^{s,z,y} < 0\}$. Then*

$$R(\boldsymbol{T}, v, z, y) \ge \inf_{\substack{\sigma\in\mathcal{T}_{\boldsymbol{T}}\\ \sigma\le\Theta}} \sup_{\substack{\tau\in\mathcal{T}_{\boldsymbol{T}}\\ \tau\le\Theta}} \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\tau\le\sigma\}}(Y_\tau^{\boldsymbol{T},v} - \boldsymbol{V}_\tau^{s,z,y})^+ + \mathbf{1}_{\{\sigma<\tau\}}\, \hat{R}(\boldsymbol{T} - \sigma, S^{\boldsymbol{T},v}, \boldsymbol{V}_\sigma^{s,z,y}, y)).$$

Proof. Let $(\pi, \sigma) \in \mathcal{A}(\boldsymbol{T}, s, z, y)$. Set $\sigma_1 = \sigma \wedge \min\{t : \gamma_t \ne \gamma_{t+}\}$, where $\pi = (z, y, \gamma)$. Clearly, $\sigma_1 \in \mathcal{T}_{\boldsymbol{T}}$ is a stopping time. Introduce the stochastic process

$$U_t = \operatorname{ess}\sup_{\tau\ge t,\, \tau\in\mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}((\mathbf{1}_{\{\tau\le\sigma\}}\, Y_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\sigma<\tau\}}\, X_\sigma^{\boldsymbol{T},v} - V_{\sigma\wedge\tau}^\pi)^+ \mid \mathcal{F}_t), \qquad t \in [0, \boldsymbol{T}].$$

The stochastic process $\{\mathbf{1}_{\{t\le\sigma\}}\, Y_t^{\boldsymbol{T},v} + \mathbf{1}_{\{\sigma<t\}}\, X_\sigma^{\boldsymbol{T},v} - V_{\sigma\wedge t}^\pi\}_{t=0}^{\boldsymbol{T}}$ is left-continuous with right-hand limits. Furthermore, this process is lower semicontinuous from the right. Thus, from the general theory of optimal stopping (see [13] and the references therein), it follows that $\{U_t\}_{t=0}^{\boldsymbol{T}}$ is a càdlàg process and, for any stopping time $\rho \le \boldsymbol{T}$,

$$U_\rho = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{\boldsymbol{T}}^\rho} \mathbb{E}_{\mathbb{P}}((\mathbf{1}_{\{\tau\le\sigma\}}\, Y_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\sigma<\tau\}}\, X_\sigma^{\boldsymbol{T},v} - V_{\sigma\wedge\tau}^\pi)^+ \mid \mathcal{F}_\rho), \tag{3.3}$$

where $\mathcal{T}_{\boldsymbol{T}}^\rho$ is the set of all stopping times $\rho \le \tau \le \boldsymbol{T}$ which satisfy $\tau > \rho\, \mathbf{1}_{\rho<\boldsymbol{T}}$.

Clearly, there is no trade until the time σ_1. Thus, $V_t^{\pi} = \boldsymbol{V}_t^{s,z,y}$ for $t \le \sigma_1$. This together with the fact that $\sigma_1 \le \sigma$ and (3.3) (for $\rho = \sigma_1$) yields

$$\mathcal{R}(\boldsymbol{T}, v, \pi, \sigma) = \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\sigma_1 < \tau\}} U_{\sigma_1} + \mathbf{1}_{\{\tau \le \sigma_1\}} (Y_\tau^{\boldsymbol{T},v} - \boldsymbol{V}_\tau^{s,z,y})^+). \tag{3.4}$$

From the Markov property of the Brownian motion, it follows that

$$U_t \ge R(\boldsymbol{T} - t, S^{\boldsymbol{T},v}, V_t^{\pi}, \gamma_t).$$

Thus, by the continuity of R (see Proposition 5.1) it follows that on the event $\{\sigma_1 < \boldsymbol{T}\}$,

$$\begin{aligned} U_{\sigma_1} &= \mathbf{1}_{\{\sigma_1 = \sigma\}} (X_{\sigma_1}^{\boldsymbol{T},v} - V_{\sigma_1}^{\pi})^+ + \mathbf{1}_{\{\sigma_1 < \sigma\}} \lim_{t \downarrow \sigma_1} U_t \\ &\ge \min((X_{\sigma_1}^{\boldsymbol{T},v} - V_{\sigma_1}^{\pi})^+, R(\boldsymbol{T} - \sigma_1, S^{\boldsymbol{T},v}, V_{\sigma_1+}^{\pi}, \gamma_{\sigma_1+})) \\ &\ge \hat{R}(\boldsymbol{T} - \sigma_1, S^{\boldsymbol{T},v}, V_{\sigma_1}^{\pi}, \gamma_{\sigma_1}) \\ &= \hat{R}(\boldsymbol{T} - \sigma_1, S^{\boldsymbol{T},v}, \boldsymbol{V}_{\sigma_1}^{s,z,y}, y). \end{aligned} \tag{3.5}$$

From (3.4), (3.5), and the inequality $\sigma_1 \le \Theta$, we obtain

$$\begin{aligned} \mathcal{R}(\boldsymbol{T}, v, \pi, \sigma) \ge \inf_{\sigma \in \mathcal{T}_{\boldsymbol{T}},\, \sigma \le \Theta} \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}},\, \tau \le \Theta} \mathbb{E}_{\mathbb{P}}(&\mathbf{1}_{\{\tau \le \sigma\}} (Y_\tau^{\boldsymbol{T},v} - \boldsymbol{V}_\tau^{s,z,y})^+ \\ &+ \mathbf{1}_{\{\sigma < \tau\}} \hat{R}(\boldsymbol{T} - \sigma, S^{\boldsymbol{T},v}, \boldsymbol{V}_\sigma^{s,z,y}, y)), \end{aligned} \tag{3.6}$$

and since (π, σ) was arbitrary, the proof is completed. □

Next, we construct an optimal hedge and verify that it is indeed optimal. The verification will be done by showing that for the constructed hedge the left-hand side and the right-hand side of (3.6) are equal.

Let (z, y) be an initial position. Set $\hat{\sigma}_0 = 0$, $\hat{\gamma}_0 = y$, and $\hat{Z}_0 = z$. For $k \ge 1$, define the random time

$$\Theta_k = T \wedge \inf\Big\{t \ge \hat{\sigma}_{k-1} \colon \boldsymbol{V}_t^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}} < 0\Big\}$$

and the stochastic process $\{\boldsymbol{R}_t^{(k)}\}_{t=\hat{\sigma}_{k-1}}^{\Theta_k}$,

$$\begin{aligned} \boldsymbol{R}_t^{(k)} = \operatorname*{ess\,inf}_{\substack{\sigma \in \mathcal{T}_T \\ \hat{\sigma}_{k-1} \le \sigma \le \Theta_k}} \operatorname*{ess\,sup}_{\substack{\tau \in \mathcal{T}_T \\ \hat{\sigma}_{k-1} \le \tau \le \Theta_k}} \mathbb{E}_{\mathbb{P}}\Big(&\mathbf{1}_{\{\tau \le \sigma\}} \Big(Y_\tau^{(s)} - \boldsymbol{V}_\tau^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}\Big)^+ \\ &+ \mathbf{1}_{\{\sigma < \tau\}} \hat{R}\Big(T - \sigma, S^{(s)}, \boldsymbol{V}_\sigma^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}\Big) \;\Big|\; \mathcal{F}_t\Big). \end{aligned}$$

Next, introduce the random time

$$\hat{\sigma}_k = \Theta_k \wedge \inf\Big\{t \ge \hat{\sigma}_{k-1} \colon \boldsymbol{R}_t^{(k)} = \hat{R}\Big(T - t, S^{(s)}, \boldsymbol{V}_t^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}\Big)\Big\}$$

and the random variable

$$\hat{\beta}_k = \beta^*\Big(T - \hat{\sigma}_k, S^{(s)}, \boldsymbol{V}_{\hat{\sigma}_k}^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}\Big),$$

where the function β^* was introduced in Proposition 5.1. Finally, set $\hat{\gamma}_k = \hat{\gamma}_{k-1} + \hat{\beta}_k$ and

$$\hat{Z}_k = h\Big(S^{(s)}_{\hat{\sigma}_k}, \boldsymbol{V}_\tau^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}, \hat{\beta}_k\Big),$$

with the function h given by (3.1). The following lemma completes the proof of Theorem 2.1 and gives a characterization of the optimal hedge.

Lemma 3.2. *For any $k \geq 1$, the stochastic process $\{\boldsymbol{R}^{(k)}_t\}_{t=\hat{\sigma}_{k-1}}^{\Theta_k}$ is well defined, continuous, and $\hat{\sigma}_k, \Theta_k$ are stopping times. Furthermore, the random variables $\hat{\gamma}_k$ and $\hat{Z}_k$ are $\mathcal{F}_{\hat{\sigma}_k}$-measurable. Next, the structure of an optimal hedge is described in the following way. Set*

$$\hat{N} = \min\{k : \hat{\sigma}_k = T\} \wedge \min\Big\{k : \hat{R}\Big(T - \hat{\sigma}_k, S^{(s)}, \boldsymbol{V}_{\hat{\sigma}_k}^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}\Big) = \Big(X^{(s)}_{\hat{\sigma}_k} - \boldsymbol{V}_{\hat{\sigma}_k}^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}\Big)^+\Big\}.$$

Then $\hat{N} < \infty$ almost surely and the hedge $(\hat{\pi}, \hat{\sigma}) \in \mathcal{A}(T, s, z, y)$, which is given by $\hat{\pi} = (z, y, \hat{\gamma})$, where $\hat{\gamma}_t = y\,\mathbf{1}_{\{t=0\}} + \sum_{i=1}^{\hat{N}-1} \hat{\gamma}_i\,\mathbf{1}_{(\hat{\sigma}_i, \hat{\sigma}_{i+1}]}$ and $\hat{\sigma} = \hat{\sigma}_{\hat{N}}$, is an optimal hedge. Namely, $R(T, s, z, y) = \mathcal{R}(T, s, \hat{\pi}, \hat{\sigma})$.

Proof. First we establish a stronger version of (3.6). For any stopping time $\theta \in \mathcal{T}_T$ and random variables $Z \geq 0$, Y which are $\mathcal{F}_\theta$-measurable,

$$\begin{aligned} R(T-\theta, S^{(s)}, Z, Y) \\ \geq \inf_{\sigma \in \mathcal{T}_T,\, \theta \leq \sigma \leq \Theta}\ \sup_{\tau \in \mathcal{T}_T,\, \theta \leq \tau \leq \Theta} \mathbb{E}_{\mathbb{P}}\Big(\mathbf{1}_{\{\tau \leq \sigma\}}\Big(Y^{(s)}_\tau - \boldsymbol{V}_\tau^{S^{(s)}_\theta, Z, Y}\Big)^+ \\ + \mathbf{1}_{\{\sigma < \tau\}}\,\hat{R}\Big(T - \sigma, S^{(s)}, \boldsymbol{V}_\sigma^{S^{(s)}_\theta, Z, Y}, Y\Big)\Big), \end{aligned} \tag{3.7}$$

where $\Theta = T \wedge \inf\{t \geq \theta : \boldsymbol{V}_t^{S^{(s)}_\theta, Z, Y} < 0\}$. In order to derive (3.7) we introduce the stochastic processes

$$\boldsymbol{X}_t := \hat{R}\Big(T - t, S^{(s)}, \boldsymbol{V}_t^{S^{(s)}_\theta, Z, Y}, Y\Big) \quad \text{and} \quad \boldsymbol{Y}_t := \Big(Y^{(s)}_t - \boldsymbol{V}_t^{S^{(s)}_\theta, Z, Y}\Big)^+, \quad t \in [\theta, \Theta].$$

From Proposition 5.1, it follows that the above processes are continuous. Observe that (since the buyer can stop at 0) for any $(\hat{\boldsymbol{T}}, \hat{v}, \hat{z}, \hat{y}) \in [0, T] \times C_{++}[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, we have $\hat{R}(\hat{\boldsymbol{T}}, \hat{v}, \hat{z}, \hat{y}) \geq (Y_0^{\hat{\boldsymbol{T}}, \hat{v}} - \hat{z})^+$. We conclude that $\boldsymbol{X} \geq \boldsymbol{Y}$, and so by applying standard results on Dynkin games (see [15]), we derive (3.7) from (3.6).

Next for a given k, define on the random time interval $[\hat{\sigma}_{k-1}, \Theta_k]$ the processes

$$\hat{\boldsymbol{X}}_t := \hat{R}\Big(T - t, S^{(s)}, \boldsymbol{V}_t^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}, \hat{\gamma}_{k-1}\Big) \quad \text{and} \quad \hat{\boldsymbol{Y}}_t := \Big(Y^{(s)}_t - \boldsymbol{V}_t^{S^{(s)}_{\hat{\sigma}_{k-1}}, \hat{Z}_{k-1}, \hat{\gamma}_{k-1}}\Big)^+,$$

where $\hat{Z}_k, \hat{\gamma}_k, \hat{\sigma}_k, \Theta_k, k \in \mathbb{N}$ were defined above. Observe that $\hat{\boldsymbol{X}} \geq \hat{\boldsymbol{Y}}$. Thus, (induction on k) by applying [9, Proposition 3.9] (see also [2, Theorem 4.1]), we conclude that $\{\boldsymbol{R}^{(k)}_t\}_{t=\hat{\sigma}_{k-1}}^{\Theta_k}$ is a continuous stochastic process, $\hat{\sigma}_k$ is a stopping time, and $\hat{\gamma}_k$, $\hat{Z}_k$ are $\mathcal{F}_{\hat{\sigma}_k}$-measurable. From the definition of the function $\hat{R}$, it follows that $\hat{Z}_k \geq 0$ for $k \leq \hat{N}$. This together with the definition of the stopping times Θ_k, $k \in \mathbb{N}$ and the fact that the portfolio value is constant after $\hat\sigma_{\hat N}$ yields that $\hat\pi$ is an *admissible* portfolio. Observe that, for any $n \in \mathbb{N}$,

$$0 \le V^{\hat\pi}_{\hat\sigma_n} \le z + \max(\delta, \mu|y|s) + \sum_{i=1}^{n} \gamma^{\pi}_{\hat\sigma_{i-1}} (S^{(s)}_{\hat\sigma_i} - S^{(s)}_{\hat\sigma_{i-1}}) - n\delta\, \mathbf{1}_{\{\hat N > n\}}\,.$$

Taking the expectation with respect to the martingale measure Q, we obtain

$$Q\{\hat N > n\} \le \frac{z + \max(\delta, \mu|y|s)}{n\delta} \tag{3.8}$$

and so $\hat N < \infty$ almost surely. Thus, $(\hat\pi, \hat\sigma) \in \mathcal{A}(T, s, z, y)$.

Finally, we prove that $(\hat\pi, \hat\sigma)$ is an optimal hedge. Choose $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\mathbb{P}\{\hat N > N\} < \varepsilon. \tag{3.9}$$

Define the continuous martingale $M_t = \mathbb{E}_{\mathbb{P}}(\sup_{0\le t\le T}(X^{(s)}_t)^2\, \mathbf{1}_{\{\hat N > N\}} \mid \mathcal{F}_t)$, $t \in [0, T]$, and the stochastic process

$$\hat U_t = \operatorname*{ess\,sup}_{\tau \ge t,\, \tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{P}}((\mathbf{1}_{\{\tau\le\hat\sigma\}}\, Y^{(s)}_\tau + \mathbf{1}_{\{\hat\sigma<\tau\}}\, X^{(s)}_{\hat\sigma} - V^{\hat\pi}_{\hat\sigma\wedge\tau})^+ \mid \mathcal{F}_t), \qquad t \in [0, T].$$

We show that, for any $0 \le k \le N$,

$$\hat U_{\hat\sigma_{k\wedge\hat N}} \le \hat R(T - \hat\sigma_{k\wedge\hat N}, S^{(s)}, V^{\hat\pi}_{\hat\sigma_{k\wedge\hat N}}, \hat\gamma_{k\wedge\hat N-1}) + \sqrt{M_{\hat\sigma_{k\wedge\hat N}}}, \tag{3.10}$$

where, for $k = 0$, we set $\hat R(T, S^{(s)}, z, \hat\gamma_{-1}) := R(T, s, z, y)$.

We start with $k = N$. From the definition of $\hat N$ and the Jensen inequality, it follows that

$$\begin{aligned}\hat U_{\hat\sigma_{N\wedge\hat N}} &\le \mathbf{1}_{\{\hat\sigma_{N\wedge\hat N}=\hat\sigma\}}(X^{(s)}_{\hat\sigma_{N\wedge\hat N}} - V^{\hat\pi}_{\hat\sigma_{N\wedge\hat N}})^+ + \mathbf{1}_{\{N<\hat N\}}\, \mathbb{E}_{\mathbb{P}}\Big(\sup_{0\le t\le \mathbb{T}} (X^{(s)}_t) \Bigm| \mathcal{F}_{\hat\sigma_{N\wedge\hat N}}\Big)\\ &\le \hat R(T - \hat\sigma_{N\wedge\hat N}, S^{(s)}, V^{\hat\pi}_{\hat\sigma_{N\wedge\hat N}}, \hat\gamma_{N\wedge\hat N-1}) + \sqrt{M_{\hat\sigma_{N\wedge\hat N}}}.\end{aligned}$$

Next, we prove that if (3.10) holds for $k + 1$ then it holds for k. From the definition of $\hat N$, on the event $\{k \ge \hat N\}$ (which is $\mathcal{F}_{\hat\sigma_{k\wedge\hat N}}$-measurable), (3.10) trivially holds. Consider the event $\{k < \hat N\}$. On this event, similarly to (3.4), we have

$$\hat U_{\hat\sigma_k} = \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{P}}\Big(\mathbf{1}_{\{\hat\sigma_{k+1}<\tau\}}\, \hat U_{\hat\sigma_{k+1}} + \mathbf{1}_{\{\tau\le\hat\sigma_{k+1}\}}\Big(Y^{(s)}_\tau - \boldsymbol{V}_\tau^{S^{(s)}_{\hat\sigma_k}, \hat Z_k, \hat\gamma_k}\Big)^+ \Bigm| \mathcal{F}_{\hat\sigma_{k\wedge\hat N}}\Big).$$

This together with the induction assumption and the fact that $\sqrt{M}$ is a supermartingale yields

$$\begin{aligned}\hat U_{\hat\sigma_k} &\le \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{P}}\Big(\mathbf{1}_{\{\tau\le\hat\sigma_{k+1}\}}\Big(Y^{(s)}_\tau - \boldsymbol{V}_\tau^{S^{(s)}_{\hat\sigma_k}, \hat Z_k, \hat\gamma_k}\Big)^+\\ &\qquad + \mathbf{1}_{\{\hat\sigma_{k+1}<\tau\}}\, \hat R(T - \hat\sigma_{k+1}, S^{(s)}, V^{\hat\pi}_{\hat\sigma_{k+1}}, \hat\gamma_k) \Bigm| \mathcal{F}_{\hat\sigma_{k\wedge\hat N}}\Big) + \sqrt{M_{\hat\sigma_{k\wedge\hat N}}}\\ &= \sqrt{M_{\hat\sigma_{k\wedge\hat N}}}\\ &\qquad + \operatorname*{ess\,inf}_{\substack{\sigma\in\mathcal{T}_T\\ \hat\sigma_k\le\sigma\le\Theta_{k+1}}} \operatorname*{ess\,sup}_{\substack{\tau\in\mathcal{T}_T\\ \hat\sigma_k\le\tau\le\Theta_{k+1}}} \mathbb{E}_{\mathbb{P}}\Big(\mathbf{1}_{\{\tau\le\sigma\}}\Big(Y^{(s)}_\tau - \boldsymbol{V}_\tau^{S^{(s)}_{\hat\sigma_k}, \hat Z_k, \hat\gamma_k}\Big)^+\\ &\qquad\qquad + \mathbf{1}_{\{\sigma<\tau\}}\, \hat R\Big(T - \sigma, S^{(s)}, \boldsymbol{V}_\sigma^{S^{(s)}_{\hat\sigma_k}, \hat Z_k, \hat\gamma_k}, \hat\gamma_k\Big) \Bigm| \mathcal{F}_{\hat\sigma_{k\wedge\hat N}}\Big).\end{aligned}$$

Hence, by (3.7)

$$\hat{U}_{\hat{\sigma}_k} \le \sqrt{M_{\hat{\sigma}_{k\wedge\hat{N}}}} + R(T-\hat{\sigma}_k, S^{(s)}, V^{\hat{\pi}}_{\hat{\sigma}_k}, \hat{\gamma}_k) = \sqrt{M_{\hat{\sigma}_{k\wedge\hat{N}}}} + \hat{R}(T-\hat{\sigma}_{k\wedge\hat{N}}, S^{(s)}, V^{\hat{\pi}}_{\hat{\sigma}_k}, \hat{\gamma}_{k\wedge\hat{N}-1}),$$

where the last equality follows from the definition of $\hat{\gamma}_k$ and the fact that we are on the event $\{k < \hat{N}\}$. This completes the proof of (3.10).

From (2.2), the Cauchy–Schwarz inequality, and (3.9), it follows that $M_0 = O(\varepsilon^{1/2})$. Thus, by applying (3.10), for $k = 0$,

$$\mathcal{R}(T, s, \hat{\pi}, \hat{\sigma}) = \hat{U}_0 \le \sqrt{M_0} + R(T, s, z, y) = O(\varepsilon^{1/4}) + R(T, s, z, y),$$

and by taking $\varepsilon \downarrow 0$ we complete the proof. □

Remark 3.1. Observe that our proof relies on the positivity of both δ and μ. The fact that $\delta > 0$ is used in (3.8) implies that the number of transactions is finite almost surely which no longer holds for $\delta = 0$. The latter leads us to the setup of proportional transaction costs where portfolio strategies may no longer be piecewise constant (with a finite number of transactions), and so this case can not be treated by the dynamical programming approach of this section.

If $\delta > 0$ and $\mu = 0$ then we can still argue that the investor can trade only a finite number of times, but the set of all permissible trades $\Gamma(S, z, y)$ is no longer a compact set. The compactness of $\Gamma(S, z, y)$ is essential in Proposition 5.1 where we introduced the function β^*. From an intuitive point of view, for reasonable payoffs, even in the $\mu = 0$ case it does not make sense to make very large trades which can lead to a large positive or negative stock component in the portfolio and substantially increase the risk which we want to minimize. Hence, it may still be possible to construct the function β^* and to proceed with our method. However, this requires additional formal arguments which are beyond the scope of this paper.

4. Proof of Theorem 2.2

First, we follow [4, Lemma 4.1] and obtain a bound on the growth of *admissible* portfolios.

Lemma 4.1. *Let $(z, y) \in \mathbb{R}_{++} \times \mathbb{R}$ be an initial position. There exists a constant $\widetilde{C}$ such that, for any $\pi \in \mathcal{A}(T, s, z, y)$,*

$$\mathbb{E}_Q\left(\max_{0\le t\le T} |\gamma_t| S^{(s)}_t + \int_0^T S^{(s)}_u |\mathrm{d}\gamma_u|\right)^2 \le \widetilde{C}(1 + z^2 + y^2).$$

Proof. Let $\pi = (z, y, \gamma) \in \mathcal{A}(T, s, z, y)$. We will use the integration by parts formula

$$\int_0^T \gamma_u \,\mathrm{d}S^{(s)}_u = \gamma_t S^{(s)}_t - ys - \int_{[0,t]} S^{(s)}_u \,\mathrm{d}\gamma_u, \tag{4.1}$$

and the decomposition $\gamma_t = \gamma^+_t - \gamma^-_t$ into a positive variation γ^+ and a negative variation γ^-. From (2.3) and (4.1), it follows that, for any $t \in [0, T]$,

$$0 \le V^{\pi}_t \le z + g(y, s) + |y|s - (1+\mu)\int_{[0,t]} S^{(s)}_u \,\mathrm{d}\gamma_u - 2\mu \int_{[0,t]} S^{(s)}_u \,\mathrm{d}\gamma^-_u + (1+\mu)\gamma_t S^{(s)}_t,$$

and

$$0 \le V^{\pi}_t \le z + g(y, s) + |y|s - (1-\mu)\int_{[0,t]} S^{(s)}_u \,\mathrm{d}\gamma_u - 2\mu \int_{[0,t]} S^{(s)}_u \,\mathrm{d}\gamma^+_u + (1-\mu)\gamma_t S^{(s)}_t.$$

This together with (4.1) yields

$$\int_0^t S_u^{(s)}\,\mathrm{d}\gamma_u^- \le \frac{1}{2\mu}\bigg(z + g(y,s) + |y|s + (1+\mu)\bigg(|y|s + \int_0^t \gamma_u\,\mathrm{d}S_u^{(s)}\bigg)\bigg), \tag{4.2}$$
$$\int_0^t S_u^{(s)}\,\mathrm{d}\gamma^+ u \le \frac{1}{2\mu}\bigg(z + g(y,s) + |y|s + (1-\mu)\bigg(|y|s + \int_0^t \gamma_u\,\mathrm{d}S_u^{(s)}\bigg)\bigg),$$

and so

$$\begin{aligned}\gamma_t S_t^{(s)} &\ge \int_0^t \gamma_u\,\mathrm{d}S_u^{(s)} - \int_{[0,t]} S_u^{(s)}\,\mathrm{d}\gamma_u^- - |y|s\\ &\ge -\frac{1}{2\mu}(z + g(y,s) + (2+3\mu)|y|s) - \frac{1-\mu}{2\mu}\int_0^t \gamma_u \mathrm{d}S_u^{(s)},\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\gamma_t S_t^{(s)} &\le |y|s + \int_0^t \gamma_u\,\mathrm{d}S_u^{(s)} + \int_{[0,t]} S_u^{(s)}\,\mathrm{d}\gamma_u^+\\ &\le \frac{1}{2\mu}(z + g(y,s) + (2+\mu)|y|s) - \frac{1+\mu}{2\mu}\int_0^t \gamma_u \mathrm{d}S_u^{(s)}.\end{aligned} \tag{4.4}$$

From (4.3) and $(a+b)^2 \le 2(a^2+b^2)$, we obtain

$$|\gamma_t S_t^{(s)}|^2 \le \frac{(z + g(y,s) + (2+3\mu)|y|s)^2}{2\mu^2} + \frac{(1+\mu)^2}{2\mu^2}\bigg(\int_0^t \gamma(u)\,\mathrm{d}S(u)\bigg)^2. \tag{4.5}$$

Following the arguments of [4, Lemma 4.1], we see that there is a constant $\widetilde{c}$ such that

$$\mathbb{E}_Q\bigg(\sup_{0\le t\le T}\bigg(\int_0^t \gamma_u\,\mathrm{d}S_u^{(s)}\bigg)^2\bigg) \le \widetilde{c}(z + g(y,s) + (2+3\mu)|y|s)^2.$$

This together with (4.2) and (4.5) completes the proof. □

Next, for any $n \in \mathbb{N}$, introduce the piecewise constant stochastic process $\hat{S}_t^{n,s} = S^{(s)}_{\theta^{(n)}_{[nt/T]}}$, $t \in [0,T]$, and the payoffs

$$\hat{Y}_t^{n,s} = [F(\hat{S}^{n,s})]\bigg(\frac{[nt/T]T}{n}\bigg) \le \hat{X}_t^{n,s} = [G(\hat{S}^{n,s})]\bigg(\frac{[nt/T]T}{n}\bigg), \qquad t \in [0,T].$$

By applying the results of [12, Section 4], it follows that there exists a constant C_1 such that, for any n,

$$\mathbb{E}_{\mathbb{P}}\Big(\max_{1\le k\le n}\ \max_{\theta^{(n)}_{k-1}\le t\le\theta^{(n)}_k}(|X^{(s)}_{t\wedge T} - \hat{X}^{n,s}_{kT/n}| + |Y^{(s)}_{t\wedge T} - \hat{Y}^{n,s}_{kT/n}|)\Big) \le C_1 n^{-1/4}(\ln n)^{3/4}. \tag{4.6}$$

Let $\hat{\mathcal{T}}_n$ be the set of all stopping times with respect to the filtration $\{\mathcal{F}_{\theta_k^{(n)}}\}_{k=0}^n$ with values $k = 0, 1, \dots, n$. Set

$$\hat{R}_n(T,s,z,y) = \inf_{\pi\in\mathcal{A}^{W,n}(s,z,y)}\ \inf_{\zeta\in\hat{\mathcal{T}}_n}\ \sup_{\eta\in\hat{\mathcal{T}}_n} \mathbb{E}_{\mathbb{P}}\Big(\hat{X}^{n,s}_{\zeta T/n}\mathbf{1}_{\{\zeta<\eta\}} + \hat{Y}^{n,s}_{\eta T/n}\mathbf{1}_{\{\eta\le\zeta\}} - V^\pi_{\theta^{(n)}_{\zeta\wedge\eta}}\Big)^+.$$

Observe that $\hat{S}_t^{n,s} = \Pi_n(S_t^{n,s})$, $\hat{Y}_t^{n,s} = \Pi_n(Y_t^{n,s})$, and $\hat{X}_t^{n,s} = \Pi_n(X_t^{n,s})$, $t \in [0,T]$. Thus, by using similar arguments as in [4, Section 3] and [6, Section 3], we obtain

$$\hat{R}_n(T,s,z,y) = R_n(T,s,z,y). \tag{4.7}$$

Equation (4.7) plays a key role in the proof of the following result.

Lemma 4.2. *Let* $(z, y) \in \mathbb{R}_{++} \times \mathbb{R}$ *be an initial position. There exists a constant* $C_2 > 0$ *such that, for any* $n \in \mathbb{N}$,

$$R(T, s, z, y) \geq R_n(T, s, z, y) - C_2 n^{-1/4} (\ln n)^{3/4}.$$

Proof. Fix n and set $z_n = z - n^{-1/3}$. Assume that n is sufficiently large, so $z_n \geq 0$. Let $(\pi, \sigma) \in \mathcal{A}(T, s, z_n, y)$ be such that

$$R(T, s, z_n, y) > \mathcal{R}(T, s, \pi, \sigma) - \frac{1}{n}. \tag{4.8}$$

Introduce the random variable $\Gamma = \max_{0 \leq t \leq T} |\gamma_t| S_t^{(s)} + \int_0^T S_u^{(s)} |\mathrm{d}\gamma_u|$. Define the stopping time

$$\Upsilon = T \wedge \inf\left\{t \colon \max_{0 \leq u \leq t} |\gamma_u| S_u^{(s)} + \int_0^t S_u^{(s)} |\mathrm{d}\gamma_u| > n^{1/7}\right\},$$

and the portfolio $\dot{\pi} = (z, y, \dot{\gamma})$ by $\dot{\gamma}_t = \gamma_t \mathbf{1}_{\{t \leq \Upsilon\}}$. Namely, we liquidate the portfolio at the stopping time Υ. Observe that the initial capitals of the portfolios $\dot{\pi}$ and π are equal to z and z_n, respectively. From Lemma 4.1 and the Chebyshev inequality, it follows that $Q\{\Upsilon < T\} = Q\{\Gamma > n^{1/7}\} = O(n^{-2/7})$. Using this together with (2.1), (2.2), and the Hölder inequality (for $p = 8, q = \frac{8}{7}$), we obtain

$$\mathcal{R}(T, s, \dot{\pi}, \sigma) - \mathcal{R}(T, s, \pi, \sigma) \leq \mathbb{E}_Q\Big(Z_T^{-1} \sup_{0 \leq t \leq T} X_t^{(s)} \mathbf{1}_{\Upsilon < T}\Big) = O(n^{-1/4}). \tag{4.9}$$

Introduce the portfolio $\widetilde{\pi} = (z, y, \widetilde{\gamma}) \in \mathcal{A}^{W,n}(s, z, y)$ which is managed at the stopping times $0, \theta_1^{(n)}, \dots, \theta_n^{(n)}$ and is given by

$$\widetilde{\gamma}_t = y \mathbf{1}_{\{t=0\}} + \sum_{i=0}^{n-1} \dot{\gamma}_{\theta_i^{(n)}} \mathbf{1}_{(\theta_i^{(n)}, \theta_{i+1}^{(n)}]}.$$

Let $t \in [0, T]$ and consider the event $\{\theta_k^{(n)} < t \leq \theta_{k+1}^{(n)}\}$ for some $k < n$. From the integration by parts formula, we obtain

$$\begin{aligned} V_t^{\dot{\pi}} &= z_n + \dot{\gamma}_t S_t^{(s)} - ys - \bigg(\sum_{i=1}^{k} \int_{[\theta_{i-1}^{(n)}, \theta_i^{(n)})} S_u^{(s)} \, \mathrm{d}\dot{\gamma}_u - \int_{[\theta_k^{(n)}, t)} S_u^{(s)} \, \mathrm{d}\dot{\gamma}_u\bigg) \\ &\quad - \bigg(\sum_{i=1}^{k} \sum_{u \in [\theta_{i-1}^{(n)}, \theta_i^{(n)})} g(\dot{\gamma}_{u+} - \dot{\gamma}_u, S_u^{(s)}) + \sum_{u \in [\theta_k^{(n)}, t)} g(\dot{\gamma}_{u+} - \dot{\gamma}_u, S_u^{(s)})\bigg). \end{aligned}$$

Since $|S_u^{(s)} - S_{\theta_{k+1}^{(n)}}^{(s)}| \leq 4\kappa\sqrt{T/n} \min(S_u^{(s)}, S_{\theta_{k+1}^{(n)}}^{(s)})$ for $u \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ and large n. Then the previous inequality implies

$$\begin{aligned} V_t^{\dot{\pi}} &\leq z_n + \dot{\gamma}_t S_{\theta_{k+1}^{(n)}}^{(s)} + 4\kappa \dot{\gamma}_t \sqrt{\frac{T}{n}} S_t^{(s)} - ys \\ &\quad - \sum_{i=1}^{k+1} S_{\theta_i^{(n)}}^{(s)} \int_{[\theta_{i-1}^{(n)}, \theta_i^{(n)})} \mathrm{d}\dot{\gamma}_u + 4\kappa \sqrt{\frac{T}{n}} \int_0^T S_u^{(s)} |\mathrm{d}\dot{\gamma}_u| \\ &\quad - \sum_{i=1}^{k} g\Big(\dot{\gamma}_{\theta_i^{(n)}} - \dot{\gamma}_{\theta_{i-1}^{(n)}}, S_{\theta_i^{(n)}}^{(s)}\Big) + 4\kappa\mu \sqrt{\frac{T}{n}} \int_0^T S_u^{(s)} |\mathrm{d}\dot{\gamma}_u| - \int_{[\theta_k^{(n)}, t)} \mu S_t^{(s)} |\mathrm{d}\dot{\gamma}_t| \end{aligned}$$

since $\max_{0\le u\le T}|\dot\gamma_u|S^{(s)}_u+\int_0^T S^{(s)}_u|\mathrm{d}\dot\gamma_u|\le n^{1/7}$, then, for large n,

$$\begin{aligned}V^{\dot\pi}_t &\le z-ys+\dot\gamma_{\theta^{(n)}_k}S^{(s)}_{\theta^{(n)}_{k+1}}+S^{(s)}_{\theta^{(n)}_{k+1}}\int_{[\theta^{(n)}_k,t)}|\mathrm{d}\dot\gamma_u|-\sum_{i=1}^{k+1}S^{(s)}_{\theta^{(n)}_i}\int_{[\theta^{(n)}_{i-1},\theta^{(n)}_i)}\mathrm{d}\dot\gamma_u\\ &\quad-\sum_{i=1}^{k}g\Big(\dot\gamma_{\theta^{(n)}_i}-\dot\gamma_{\theta^{(n)}_{i-1}},S^{(s)}_{\theta^{(n)}_i}\Big)-\int_{[\theta^{(n)}_k,t)}\mu S^{(s)}_t|\mathrm{d}\dot\gamma_t|\\ &\le z-ys+\dot\gamma_{\theta^{(n)}_k}S^{(s)}_{\theta^{(n)}_{k+1}}-\sum_{i=1}^{k+1}S^{(s)}_{\theta^{(n)}_{i+1}}\int_{[\theta^{(n)}_{i-1},\theta^{(n)}_i)}\mathrm{d}\dot\gamma_u-\sum_{i=1}^{k}g\Big(\dot\gamma_{\theta^{(n)}_i}-\dot\gamma_{\theta^{(n)}_{i-1}},S^{(s)}_{\theta^{(n)}_i}\Big)\\ &=V^{\tilde\pi}_{\theta^{(n)}_{k+1}},\end{aligned}$$

where the last equality follows from the summation by parts. We conclude that (for sufficiently large n)

$$V^{\tilde\pi}_{\theta^{(n)}_{k+1}}\ge V^{\dot\pi}_t,\qquad t\in(\theta^{(n)}_k,\theta^{(n)}_{k+1}],\ k<n. \tag{4.10}$$

In particular, $\tilde\pi\in\mathcal{A}^{W,n}(s,z,y)$ is an *admissible* portfolio.

Let $\zeta\in\hat{\mathcal{T}}_n$ be given by

$$\zeta=\begin{cases}n\wedge\min\{k\colon\theta^{(n)}_k\ge\sigma\} & \text{if }\sigma<T,\\ n & \text{if }\sigma=T,\end{cases} \tag{4.11}$$

where (π,σ) satisfies (4.8). From (4.7)–(4.9) and Lemma 5.3, we obtain

$$\begin{aligned}&R_n(T,s,z,y)-R(T,s,z,y)\\ &\quad\le O(|z_n-z|^{3/4})+\frac{1}{n}+O(n^{-1/4})+\hat R_n(T,s,z,y)-R(T,s,\dot\pi,\sigma)\\ &\quad\le O(n^{-1/4})+\sup_{\eta\in\hat{\mathcal{T}}_n}\mathbb{E}_{\mathbb{P}}\Big(\hat X^{n,s}_{\zeta T/n}\mathbf{1}_{\{\zeta<\eta\}}+\hat Y^{n,s}_{\eta T/n}\mathbf{1}_{\{\eta\le\zeta\}}-V^{\tilde\pi}_{\theta^{(n)}_{\zeta\wedge\eta}}\Big)^+\\ &\qquad-\sup_{\tau\in\mathcal{T}_T}\mathbb{E}_{\mathbb{P}}\Big(X^{(s)}_\sigma\mathbf{1}_{\{\sigma<\tau\}}+Y^{(s)}_\tau\mathbf{1}_{\{\tau\le\sigma\}}-V^{\dot\pi}_{\sigma\wedge\tau}\Big)^+\end{aligned}$$

since $T\wedge\theta^{(n)}_\eta\in\mathcal{T}_T$ for any $\eta\in\mathcal{T}_n$, then

$$\begin{aligned}&R_n(T,s,z,y)-R(T,s,z,y)\\ &\quad\le O(n^{-1/4})+\sup_{\eta\in\hat{\mathcal{T}}_n}\mathbb{E}_{\mathbb{P}}\Big(\hat X^{n,s}_{\zeta T/n}\mathbf{1}_{\{\zeta<\eta\}}+\hat Y^{n,s}_{\eta T/n}\mathbf{1}_{\{\eta\le\zeta\}}-V^{\tilde\pi}_{\theta^{(n)}_{\zeta\wedge\eta}}\Big)^+\\ &\qquad-\sup_{\eta\in\hat{\mathcal{T}}_n}\mathbb{E}_{\mathbb{P}}\Big(X^{(s)}_\sigma\mathbf{1}_{\{\sigma<\theta^{(n)}_\eta\wedge T\}}+Y^{(s)}_{\theta^{(n)}_\eta\wedge T}\mathbf{1}_{\{\theta^{(n)}_\eta\wedge T\le\sigma\}}-V^{\dot\pi}_{\sigma\wedge\theta^{(n)}_\eta}\Big)^+\end{aligned}$$

since $\sigma<\theta^{(n)}_\eta\wedge T$ by (4.11) if $\zeta<\eta$, then

$$\begin{aligned}&R_n(T,s,z,y)-R(T,s,z,y)\\ &\quad\le O(n^{-1/4})+\mathbb{E}_{\mathbb{P}}\Big(|\hat X^{n,s}_{\zeta T/n}-X^{(s)}_\sigma|\mathbf{1}_{\{\sigma<\theta^{(n)}_n\}}\Big)+\sup_{\eta\in\hat{\mathcal{T}}_n}\mathbb{E}_{\mathbb{P}}|\hat Y^{n,s}_{\eta_n T/n}-Y^{(s)}_{\theta^{(n)}_\eta\wedge T}|\\ &\qquad+\sup_{\eta\in\hat{\mathcal{T}}_n}\mathbb{E}_{\mathbb{P}}\Big(V^{\dot\pi}_{\sigma\wedge\theta^{(n)}_\eta}-V^{\tilde\pi}_{\theta^{(n)}_{\zeta\wedge\eta}}\Big)^+\end{aligned}$$

since $V^{\dot{\pi}_n}_{\theta^{(n)}_{\zeta\wedge\eta}} \geq V^{\dot{\pi}}_{\sigma\wedge\theta^{(n)}_\eta}$ by (4.10), together with (4.6),

$$\begin{aligned} R_n(T,s,z,y) - R(T,s,z,y) &\leq O(n^{-1/4}) + \mathbb{E}_{\mathbb{P}}\Big(\max_{1\leq k\leq n}\max_{\theta^{(n)}_{k-1}\leq t\leq\theta^{(n)}_k}(|X^{(s)}_{t\wedge T} - \hat{X}^{n,s}_{kT/n}| + |Y^{(s)}_{t\wedge T} - \hat{Y}^{n,s}_{kT/n}|)\Big) \\ &= O(n^{-1/4}(\ln n)^{3/4}), \end{aligned}$$

and the result follows. □

In view of Lemma 4.2, in order to complete the proof of Theorem 2.2 it remains to establish the following result.

Lemma 4.3. *There exists a constant* C_3 *such that, for any* $n \in \mathbb{N}$ *and* $(\pi,\sigma) \in \mathcal{A}^{(n)}(T,s,z,y)$,

$$\mathcal{R}(T,s,\tilde{\pi},\tilde{\sigma}) \leq \mathcal{R}_n(T,s,\pi,\sigma) + C_3 n^{-1/4}(\ln n)^{3/4} \quad \textit{for } (\tilde{\pi},\tilde{\sigma}) = \Psi_n(\pi,\sigma).$$

Proof. The proof follows the proof of [6, Equation (2.26)], where we showed that if we lift a hedge to the BS model, the shortfall risk can increase only by the amount $O(n^{-1/4}(\ln n)^{3/4})$. Although in [6] there is no friction, the only property that we used there is that the portfolio value process is a supermartingale with respect to the martingale measure Q. In the current setup this fact still holds and so we just follow the proof from [6]. □

5. Regularity properties of shortfall risk

In this section we do not assume Lipschitz continuity of the functions F and G (that is (2.4)) but just continuity and (2.2).

Lemma 5.1. *Let* $v \in C_{++}[0,T]$. *There exists a continuous function* $m_v\colon \mathbb{R}_+ \to \mathbb{R}_+$ *(modulus of continuity) with* $m_v(0) = 0$ *such that, for any* $\boldsymbol{T}_1, \boldsymbol{T}_2 \in [0,T]$, $z \geq 0$, *and* $y \in \mathbb{R}$, *we have*

$$\left| R(\boldsymbol{T}_1, v, z, y) - R\left(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\right)\right| \leq m_v(|\boldsymbol{T}_1 - \boldsymbol{T}_2|).$$

Proof. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in [0,T]$, $z \geq 0$, and $y \in \mathbb{R}$. Without loss of generality, we assume that $\boldsymbol{T}_1 < \boldsymbol{T}_2$. Choose $\varepsilon > 0$. There exists a hedge $(\pi_1,\sigma_1) \in \mathcal{A}(\boldsymbol{T}_1, v(T-\boldsymbol{T}_1), z, y)$ such that

$$R(\boldsymbol{T}_1, v, z, y) > \mathcal{R}(\boldsymbol{T}_1, v, \pi_1, \sigma_1) - \varepsilon. \tag{5.1}$$

Set $\pi_1 = (z, y, \gamma^{(1)})$. Let

$$(\pi_2,\sigma_2) \in \mathcal{A}\left(\boldsymbol{T}_2, v(T-\boldsymbol{T}_2), z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\right)$$

be a hedge such that $\pi_2 = (z, y, \gamma^{(2)})$ is given by

$$\gamma^{(2)}_t = \mathbf{1}_{\{t\leq \boldsymbol{T}_1\}}\frac{\gamma^{(1)}_t v(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}, \qquad t \in [0,\boldsymbol{T}_2],$$

and $\sigma_2 = \sigma_1\mathbf{1}_{\{\sigma_1<\boldsymbol{T}_1\}} + \boldsymbol{T}_2\,\mathbf{1}_{\{\sigma_1=\boldsymbol{T}_1\}}$. Namely, the portfolio $\gamma^{(2)}$ is proportional $\gamma^{(1)}$ until the moment $\boldsymbol{T}_1$ and then we sell the stocks. The stopping time σ_2 is almost the same as σ_1 with a small modification such that if σ_1 is equal to $\boldsymbol{T}_1$ then $\sigma_2 = \boldsymbol{T}_2$. From (2.3), it follows that

$$(\pi_2,\sigma_2) \in \mathcal{A}\left(\boldsymbol{T}_2, v(T-\boldsymbol{T}_2), z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\right), \qquad V^{\pi_2}_{\sigma_2\wedge t} = V^{\pi_1}_{\sigma_1\wedge t}, \quad t\in[0,\boldsymbol{T}_2].$$

Let $\tau \in \mathcal{T}_{\boldsymbol{T}_2}$. Observe that if $\sigma_2 < \tau$ then $\sigma_1 < \tau \wedge \boldsymbol{T}_1$. Thus, from (5.1),

$$\begin{aligned} R\bigg(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\bigg) &\le \mathcal{R}(\boldsymbol{T}_2, v, \pi_2, \sigma_2) \\ &= \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}_2}} \mathbb{E}_{\mathbb{P}}(X^{\boldsymbol{T}_2,v}_{\sigma_2} \mathbf{1}_{\{\sigma_2<\tau\}} + Y^{\boldsymbol{T}_2,v}_{\tau} \mathbf{1}_{\{\tau\le\sigma_2\}} - V^{\pi_2}_{\sigma_2\wedge\tau})^+ \\ &\le \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}_2}} \mathbb{E}_{\mathbb{P}}(X^{\boldsymbol{T}_1,v}_{\sigma_1} \mathbf{1}_{\{\sigma_1<\tau\wedge\boldsymbol{T}_1\}} + Y^{\boldsymbol{T}_1,v}_{\tau\wedge\boldsymbol{T}_1} \mathbf{1}_{\{\tau\wedge\boldsymbol{T}_1\le\sigma_1\}} - V^{\pi_1}_{\sigma_1\wedge\tau})^+ \\ &\quad + \mathbb{E}_{\mathbb{P}}\Big(\sup_{0\le t\le \boldsymbol{T}_2} (|Y^{\boldsymbol{T}_1,v}_{t\wedge\boldsymbol{T}_1} - Y^{\boldsymbol{T}_2,v}_t| + |X^{\boldsymbol{T}_1,v}_{t\wedge\boldsymbol{T}_1} - X^{\boldsymbol{T}_2,v}_t|)\Big) \\ &\le \varepsilon + R(\boldsymbol{T}_1, v, z, y) + m_v(|\boldsymbol{T}_1 - \boldsymbol{T}_2|), \end{aligned}$$

where $m_v(\delta) = \sup_{|t_2-t_1|\le\delta} \mathbb{E}_{\mathbb{P}}(\sup_{0\le t\le T}(|Y^{t_1,v}_{t\wedge t_1} - Y^{t_2,v}_t| + |X^{t_1,v}_{t\wedge t_1} - X^{t_2,v}_t|))$.

From (2.2) and the fact that F and G are continuous, it follows that m_v is indeed a modulus of continuity. Since $\varepsilon > 0$ was arbitrary, we obtain

$$R\bigg(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\bigg) - R(\boldsymbol{T}_1, v, z, y) \le m_v(|\boldsymbol{T}_1 - \boldsymbol{T}_2|).$$

Next, we prove that

$$R(\boldsymbol{T}_1, v, z, y) - R\bigg(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\bigg) \le m_v(|\boldsymbol{T}_1 - \boldsymbol{T}_2|).$$

Choose $\varepsilon > 0$ and a hedge $(\tilde{\pi}_2, \tilde{\sigma}_2) \in \mathcal{A}(\boldsymbol{T}_2, v(T-\boldsymbol{T}_2), z, yv(T-\boldsymbol{T}_1)/v(T-\boldsymbol{T}_2))$ which satisfy

$$R\bigg(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\bigg) > \mathcal{R}(\boldsymbol{T}_2, v, \tilde{\pi}_2, \tilde{\sigma}_2) - \varepsilon.$$

Denote $\tilde{\pi}_2 = (z, yv(T-\boldsymbol{T}_1)/v(T-\boldsymbol{T}_2), \tilde{\gamma}^{(2)})$. Let $(\tilde{\pi}_1, \tilde{\sigma}_1) \in \mathcal{A}(\boldsymbol{T}_1, v(T-\boldsymbol{T}_1), z, y)$ be a hedge such that $\tilde{\pi}_1 = (z, y, \tilde{\gamma}^{(1)})$ is given by

$$\tilde{\gamma}^{(1)}_t = \frac{\gamma^{(2)}_t v(T-\boldsymbol{T}_2)}{v(T-\boldsymbol{T}_1)}, \qquad t \in [0, \boldsymbol{T}_1], \quad \text{and} \quad \tilde{\sigma}_1 = \tilde{\sigma}_2 \wedge \boldsymbol{T}_1.$$

Namely, we take a multiple of the hedge $(\tilde{\pi}_2, \tilde{\sigma}_2)$ and restrict it to the interval $[0, \boldsymbol{T}_1]$. Let $\tau \in \mathcal{T}_{\boldsymbol{T}_1}$ and observe that if $\sigma_2 < \tau$ then $\sigma_1 = \sigma_2 < \tau$. Thus, we obtain

$$\begin{aligned} R(\boldsymbol{T}_1, v, z, y) &\le \mathcal{R}(\boldsymbol{T}_1, v, \tilde{\pi}_1, \tilde{\sigma}_1) \\ &= \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}_1}} \mathbb{E}_{\mathbb{P}}(X^{\boldsymbol{T}_1,v}_{\tilde{\sigma}_1} \mathbf{1}_{\{\tilde{\sigma}_1<\tau\}} + Y^{\boldsymbol{T}_1,v}_{\tau} \mathbf{1}_{\{\tau\le\tilde{\sigma}_1\}} - V^{\tilde{\pi}_1}_{\tilde{\sigma}_1\wedge\tau})^+ \\ &\le \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}_1}} \mathbb{E}_{\mathbb{P}}(X^{\boldsymbol{T}_2,v}_{\tilde{\sigma}_2} \mathbf{1}_{\{\tilde{\sigma}_2<\tau\}} + Y^{\boldsymbol{T}_2,v}_{\tau} \mathbf{1}_{\{\tau\le\tilde{\sigma}_2\}} - V^{\tilde{\pi}_2}_{\tilde{\sigma}_2\wedge\tau})^+ \\ &\quad + \mathbb{E}_{\mathbb{P}}\Big(\sup_{0\le t\le \boldsymbol{T}_2} (|Y^{\boldsymbol{T}_1,v}_{t\wedge\boldsymbol{T}_1} - Y^{\boldsymbol{T}_2,v}_t| + |X^{\boldsymbol{T}_1,v}_{t\wedge\boldsymbol{T}_1} - X^{\boldsymbol{T}_2,v}_t|)\Big) \\ &\le \varepsilon + R\bigg(\boldsymbol{T}_2, v, z, \frac{yv(T-\boldsymbol{T}_1)}{v(T-\boldsymbol{T}_2)}\bigg) + m_v(|\boldsymbol{T}_1 - \boldsymbol{T}_2|), \end{aligned}$$

and by taking $\varepsilon \downarrow 0$ we complete the proof. □

Next, we obtain the following simple result.

Lemma 5.2. *Let* $v \in C_{++}[0,T]$. *There exists a continuous function* $\widetilde{m}_v : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\widetilde{m}_v(0) = 0$ *such that, for any* $\widetilde{v} \in C_{++}[0,T]$, $\boldsymbol{T} \in [0,T]$, $z \geq 0$, *and* $y \in \mathbb{R}$, *we have*

$$\left| R(\boldsymbol{T}, v, z, y) - R\left(\boldsymbol{T}, \widetilde{v}, z, \frac{yv(T-\boldsymbol{T})}{\widetilde{v}(T-\boldsymbol{T})}\right)\right| \leq \widetilde{m}_v(||v-\widetilde{v}||).$$

Proof. Fix $\boldsymbol{T} \in [0,T]$, $\widetilde{v} \in C_{++}[0,T]$, $z \geq 0$, and $y \in \mathbb{R}$. Let $\pi_1 = (z, y, \gamma)$ be an *admissible* portfolio in the market with maturity date $\boldsymbol{T}$ and an initial stock price $v(T-\boldsymbol{T})$. Consider the portfolio

$$\pi_2 = \left(z, \frac{yv(T-\boldsymbol{T})}{\widetilde{v}(T-\boldsymbol{T})}, \frac{\gamma v(T-\boldsymbol{T})}{\widetilde{v}(T-\boldsymbol{T})}\right)$$

as an *admissible* portfolio in the market with maturity date $\boldsymbol{T}$ and an initial stock price $\widetilde{v}(T-\boldsymbol{T})$. The map $\pi_1 \to \pi_2$ is a bijection between the corresponding sets of portfolios. From (2.3), it follows that $V^{\pi_1} = V^{\pi_2}$, and so

$$\begin{aligned}\left| R(\boldsymbol{T}, v, z, y) - R\left(\boldsymbol{T}, \widetilde{v}, z, \frac{yv(T-\boldsymbol{T})}{\widetilde{v}(T-\boldsymbol{T})}\right)\right| &\leq \mathbb{E}_{\mathbb{P}}\Big(\sup_{0\leq t\leq \boldsymbol{T}} (|Y_t^{\boldsymbol{T},v} - Y_t^{\boldsymbol{T},\widetilde{v}}| + |X_t^{\boldsymbol{T},v} - X_t^{\boldsymbol{T},\widetilde{v}}|)\Big) \\ &\leq \widetilde{m}_v(||v-\widetilde{v}||),\end{aligned}$$

where $\widetilde{m}_v(\delta) = \sup_{\boldsymbol{T}\in[0,T]} \sup_{||\widetilde{v}-v||\leq\delta} \mathbb{E}_{\mathbb{P}}(||X^{\boldsymbol{T},v} - X^{\boldsymbol{T},\widetilde{v}}|| + ||Y^{\boldsymbol{T},v} - Y^{\boldsymbol{T},\widetilde{v}}||)$. From (2.2) and the fact that F and G are continuous, it follows that $\widetilde{m}_v$ is indeed a modulus of continuity. □

Next, we establish continuity properties of the shortfall risk as a function of the initial position.

Lemma 5.3. *Let* $K > 0$. *There exists a constant* $\hat{C} = \hat{C}(K)$ *such that the following holds. For any* $v \in C_{++}[0,T]$, $\boldsymbol{T} \in [0,T]$, $(z_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$, $i = 1, 2$, *such that* $||v|| \leq K$ *and* $y_1 = 0$ *if and only if* $y_2 = 0$, *we have*

$$|R(\boldsymbol{T}, v, z_1, y_1) - R(\boldsymbol{T}, v, z_2, y_2)| \leq \hat{C}(|z_1 - z_2| + |y_1 - y_2|)^{3/4}.$$

Proof. Fix $v \in C_{++}[0,T]$, $\boldsymbol{T} \in [0,T]$, and $(z_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$, $i = 1, 2$. Assume that $||v|| \leq K$ and $y_1 = 0$ if and only if $y_2 = 0$. Denote $s = v(T-\boldsymbol{T})$ and $A = |z_1 - z_2| + |y_1 - y_2|$. Without loss of generality, we assume that $0 < A < 1$. Let $(\pi_1, \sigma_1) \in \mathcal{A}(\boldsymbol{T}, s, z_1, y_1)$ be such that

$$\mathcal{R}(\boldsymbol{T}, v, \pi_1, \sigma_1) < R(\boldsymbol{T}, v, z_1, y_1) + A. \tag{5.2}$$

Denote $\pi_1 = (z, y, \gamma^{(1)})$. Define the stopping times $\varrho = \inf\{t : \gamma_{t+}^{(1)} = 0\} \wedge \boldsymbol{T}$ and

$$\varsigma = \inf\{t : V_t^{\pi_1} \leq |z_2 - z_1| + |y_2 - y_1|(1+\mu)(s + S_{t\wedge\varrho}^{(s)})\} \wedge \boldsymbol{T}.$$

Introduce the hedge $(\pi_2, \sigma_2) \in \mathcal{A}(\boldsymbol{T}, s, z_2, y_2)$ by $\sigma_2 = \sigma_1$ and

$$\gamma_t^{(2)} = \mathbf{1}_{\{t\leq\varsigma\}}((\gamma_t^{(1)} + y_2 - y_1)\,\mathbf{1}_{\{t\leq\varrho\}} + \gamma_t^{(1)}\,\mathbf{1}_{\{t>\varrho\}}).$$

Namely, at the time ς the investor liquidates the portfolio. Until the time ς, the portfolio strategy $\gamma^{(2)}$ is a shift of $\gamma^{(1)}$ until the first stock liquidation time ϱ of $\gamma^{(1)}$ and after this time the portfolios are the same. From (2.3), it follows that

$$V_t^{\pi_2} = V_{t\wedge\varsigma}^{\pi_2} \geq V_{t\wedge\varsigma}^{\pi_1} - (|z_2 - z_1| + |y_2 - y_1|(1+\mu)(s + S_{t\wedge\varsigma\wedge\varrho}^{(s)})) \geq 0,$$

where the last inequality follows from the definition of ς. Thus, $(\pi_2, \sigma_2) \in \mathcal{A}(\boldsymbol{T}, s, z_2, y_2)$. Furthermore, for any random variable Φ, we have

$$(\Phi - V_t^{\pi_1})^+ \geq (\Phi - V_t^{\pi_2})^+ - \mathbf{1}_{\{t \leq \varsigma\}}(|z_2 - z_1| + |y_2 - y_1|(1+\mu)(s + S_{t\wedge\varsigma\wedge\varrho}^{(s)})) \\ - \mathbf{1}_{\{t>\varsigma\}}(\mathbf{1}_{\{V_t^{\pi}>1\}}\,\Phi + \mathbf{1}_{\{V_t^{\pi}\leq 1\}}\,V_t^{\pi_1}). \tag{5.3}$$

From (5.2) and (5.3), we obtain

$$\begin{aligned}
&R(\boldsymbol{T}, v, z_2, y_2) - R(\boldsymbol{T}, v, z_1, y_1) \\
&\quad\leq A + \mathcal{R}(\boldsymbol{T}, v, \pi_2, \sigma_2) - \mathcal{R}(\boldsymbol{T}, v, \pi_1, \sigma_1) \\
&\quad\leq A + |z_1 - z_2| + \sup_{\tau\in\mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}(|y_2 - y_1|(1+\mu)(s + S_{\tau\wedge\varsigma\wedge\varrho}^{(s)}) + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} \\
&\qquad\qquad + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) \\
&\quad\leq A + |z_1 - z_2| + 2|y_2 - y_1|(1+\mu)\mathbb{E}_{\mathbb{P}}\Big(\max_{0\leq t\leq T} S_t^{(s)}\Big) \\
&\qquad + \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}).
\end{aligned}$$

Thus, in order to complete the proof it remains to show that

$$\sup_{\tau\in\mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) = O(A^{3/4}).$$

Let $\tau \in \mathcal{T}_T$. From (2.1), (2.2), and the Hölder inequality (for $p = 4$, $q = \frac{4}{3}$), it follows that there exists a constant $c = c(K)$ such that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) \\
&\quad= \mathbb{E}_Q(Z_T^{-1}\,\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + Z_T^{-1}\,\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) \\
&\quad\leq c((\mathbb{E}_Q[\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}])^{3/4} + (\mathbb{E}_Q[\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}<1\}}(V_\tau^{\pi_1})^{4/3}])^{3/4}) \\
&\quad\leq c((\mathbb{E}_Q[\mathbf{1}_{\{\varsigma<\tau\}}\,V_\tau^{\pi_1}])^{3/4} + (\mathbb{E}_Q[\mathbf{1}_{\{\varsigma<\tau\}}\,V_\tau^{\pi_1}])^{3/4})
\end{aligned}$$

since$\{V_t^{\pi_1}\}_{t=0}^{\boldsymbol{T}}$ is a supermartingale with respect to Q, then from the previous inequality

$$\mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) \leq 2c(\mathbb{E}_Q[\mathbf{1}_{\{\varsigma<\tau\}}\,V_{\varsigma+}^{\pi_1}])^{3/4}$$

since $V_{\varsigma+}^{\pi_1} \leq |z_2 - z_1| + |y_2 - y_1|(1+\mu)(s + S_{\{\varsigma\wedge\varrho\}}^{(s)})$ on the event $\{\varsigma < \tau\}$, we conclude

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}>1\}}\,X_\tau^{\boldsymbol{T},v} + \mathbf{1}_{\{\varsigma<\tau\}}\,\mathbf{1}_{\{V_\tau^{\pi_1}\leq 1\}}\,V_\tau^{\pi_1}) &\leq 2c(|z_2 - z_1| + 2s|y_2 - y_1|(1+\mu))^{3/4} \\
&= O(A^{3/4}),
\end{aligned}$$

completing the proof. □

Lemma 5.4. *Let $K > 0$, $v \in C_{++}[0, T]$, and $y \in \mathbb{R}$. Assume that $||v|| \leq K$ and $|y| \leq \delta/\mu \min_{t\in[0,T]} 1/v(t)$. Then, for any $z \geq 0$ and $\boldsymbol{T} \in [0, T]$,*

$$R(\boldsymbol{T}, v, z + \delta, 0) - R(\boldsymbol{T}, v, z, y) \leq \hat{C}(K)|y|^{3/4}.$$

Proof. Denote $s = v(T - \boldsymbol{T})$. Without loss of generality, we assume that $y \neq 0$. Let $(\pi_1, \sigma_1) \in \mathcal{A}(\boldsymbol{T}, s, z, y)$ such that $R(\boldsymbol{T}, v, z, y) > \mathcal{R}(\boldsymbol{T}, v, \pi_1, \sigma_1) - |y|$. Set $\pi_1 = (z, y, \gamma^{(1)})$. Define the stopping times

$$\varrho = \inf\{t : \gamma^{(1)}_{t+} = 0\} \wedge \boldsymbol{T} \quad \text{and} \quad \varsigma = \inf\{t : V^{\pi_1}_t \leq |y|(1+\mu)S^{(s)}_{t\wedge\varrho}\} \wedge \boldsymbol{T}.$$

Introduce the hedge $(\pi_2, \sigma_2) \in \mathcal{A}(\boldsymbol{T}, s, z+\delta, 0)$ by $\sigma_2 = \sigma_1$ and $\pi_2 = (z+\delta, 0, \gamma^{(2)})$, where $\gamma^{(2)}_t = \mathbf{1}_{\{t\leq\varsigma\}}((\gamma^{(1)}_t - y)\,\mathbf{1}_{\{t\leq\varrho\}} + \gamma^{(1)}_t\,\mathbf{1}_{\{t>\varrho\}})$. Our assumptions imply that $g(y, s) = \delta$, and so from (2.3), we obtain $V^{\pi_2}_t = V^{\pi_2}_{t\wedge\varsigma} \geq V^{\pi_1}_{t\wedge\varsigma} - |y|(1+\mu)S^{(s)}_{t\wedge\varsigma\wedge\varrho} \geq 0$. Thus, similarly to Lemma 5.3, we have $R(\boldsymbol{T}, v, z+\delta, 0) - R(\boldsymbol{T}, v, z, y) \leq \hat{C}(K)|y|^{3/4}$, as required. □

The following proposition is the main result of this section.

Proposition 5.1. (i) *The function* $R\colon [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ *is upper semicontinuous (and, hence, measurable). If we restrict the function* R *to the domain* $[0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\setminus\{0\}$ *then the function* $R\colon [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *is a continuous function.*

(ii) *There exists a measurable function* $\beta^*\colon [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ *such that* $\beta^*(\boldsymbol{T}, v, \cdot, \cdot\cdot)$ *depends only on* $v_{[0,T-\boldsymbol{T}]}$ *and the infimum in* β *in (3.2) is attained at* $\beta^* = \beta^*(\boldsymbol{T}, v, z, y)$.

(iii) *For any* $y\in\mathbb{R}$, *the function* $\hat{R}_y\colon [0,T]\times C_{++}[0,T]\times[\delta\,\mathbf{1}_{\{y=0\}},\infty)\to\mathbb{R}$ *defined by* $\hat{R}_y(\boldsymbol{T}, v, z) = \hat{R}(\boldsymbol{T}, v, z, y)$ *is a continuous function.*

Proof. (i) Let $(\boldsymbol{T}, v, z, y)\in[0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}$. Clearly, for any $(\widetilde{\boldsymbol{T}}, \widetilde{v}, \widetilde{z}, \widetilde{y}) \in [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}$, we have

$$\begin{aligned}|R(\boldsymbol{T}, v, z, y) - R(\widetilde{\boldsymbol{T}}, \widetilde{v}, \widetilde{z}, \widetilde{y})| &\leq \left|R(\boldsymbol{T}, v, z, y) - R\left(\widetilde{\boldsymbol{T}}, v, z, \frac{yv(T-\boldsymbol{T})}{v(T-\widetilde{\boldsymbol{T}})}\right)\right| \\ &\quad + \left|R\left(\widetilde{\boldsymbol{T}}, v, z, \frac{yv(T-\boldsymbol{T})}{v(T-\widetilde{\boldsymbol{T}})}\right) - R\left(\widetilde{\boldsymbol{T}}, \widetilde{v}, z, \frac{yv(T-\boldsymbol{T})}{\widetilde{v}(T-\widetilde{\boldsymbol{T}})}\right)\right| \\ &\quad + \left|R\left(\widetilde{\boldsymbol{T}}, \widetilde{v}, z, \frac{yv(T-\boldsymbol{T})}{\widetilde{v}(T-\widetilde{\boldsymbol{T}})}\right) - R(\widetilde{\boldsymbol{T}}, \widetilde{v}, \widetilde{z}, \widetilde{y})\right|.\end{aligned}$$

This together with Lemmas 5.1, 5.2, and 5.3 yields that $R\colon [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is a continuous function. Next, we prove that $R\colon [0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is upper semicontinuous. Set $s = v(T-\boldsymbol{T})$. For any *admissible* portfolio $\pi_1 = (z, 0, \gamma^{(1)})$, introduce the portfolio $\pi_2 = (z, y, \gamma^{(2)})$ by $\gamma^{(2)}_0 = y$ and $\gamma^{(2)}_t = \gamma^{(1)}_t$ for $t > 0$. Observe that, for any $t > 0$,

$$V^{\pi_2}_t - V^{\pi_1}_t = V^{\pi_2}_{0+} - V^{\pi_1}_{0+} = g(y, s) - g(y - \gamma^{(1)}_{0+}, s) + g(\gamma^{(1)}_{0+}, s) \geq 0.$$

Thus, $R(\boldsymbol{T}, v, z, y) \leq R(\boldsymbol{T}, v, z, 0)$. This together with (5.4) and Lemmas 5.1, 5.2, and 5.3 completes the proof of (i).

(ii) Fix $(\boldsymbol{T}, v, z, y)\in[0,T]\times C_{++}[0,T]\times\mathbb{R}_+\times\mathbb{R}$. Set $s = v(T-\boldsymbol{T})$. Assume that $\Gamma(s, z, y) \neq \varnothing$. We want to show that the minimum in (3.2) is attained. Thus, let $\{\beta_n\}_{n=1}^\infty \subset \Gamma(s, z, y)$ for which

$$\lim_{n\to\infty} R(\boldsymbol{T}, v, h(s, z, y, \beta_n), y+\beta_n) = \inf_{\beta\in\Gamma(s,z,y)} R(\boldsymbol{T}, v, h(s, z, y, \beta), y+\beta). \tag{5.4}$$

The set $\Gamma(s, z, y)$ is compact, and so without loss of generality, we assume (by passing to a subsequence) that the sequence $\{\beta_n\}_{n=1}^{\infty} \subset \Gamma(s, z, y)$ converges; thus, let $\lim_{n\to\infty} \beta_n = \hat{\beta} \in \Gamma(s, z, y)$. First assume that $\hat{\beta} \neq -y$, then we have $h(s, z, y, \hat{\beta}) = \lim_{n\to\infty} h(s, z, y, \beta_n)$. Thus, from (5.4) and the fact that $R\colon [0, T] \times C_{++}[0, T] \times \mathbb{R}_+ \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous, we conclude that

$$R(\boldsymbol{T}, v, h(s, z, y, \hat{\beta}), y + \hat{\beta}) = \inf_{\beta\in\Gamma(s,z,y)} R(\boldsymbol{T}, v, h(s, z, y, \beta), y + \beta). \tag{5.5}$$

Next, we deal with the $\hat{\beta} = -y$ case. In this case $h(s, z, y, \hat{\beta}) = \delta + \lim_{n\to\infty} h(s, z, y, \beta_n)$, and so from (5.4) and Lemma 5.4, we obtain

$$R(\boldsymbol{T}, v, h(s, z, y, \hat{\beta}), 0) = \inf_{\beta\in\Gamma(s,z,y)} R(\boldsymbol{T}, v, h(s, z, y, \beta), y + \beta). \tag{5.6}$$

We conclude that the infimum in (3.2) is attained at $\hat{\beta} \in \Gamma(s, z, y)$. It follows that there exists a measurable map $\beta^*\colon [0, T]\times C_{++}[0, T]\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}$ such that $\beta^* = \beta^*(\boldsymbol{T}, v, z, y)$ depends only on $v_{[0,T-\boldsymbol{T}]}$ and the infimum in (3.2) is attained at β^* provided that $\Gamma(v(T-\boldsymbol{T}), z, y) \neq \varnothing$. For instance, $\beta^*(\boldsymbol{T}, v, z, y) = 0$ if $y = 0$ and $z < \delta$ (i.e. $\Gamma(v(T - \boldsymbol{T}), z, y) = \varnothing$), and if $\Gamma(v(T - \boldsymbol{T}), z, y) \neq \varnothing$ then

$$\begin{aligned}\beta^*(\boldsymbol{T}, v, z, y) &= \min_{\widetilde{\beta}\in\Gamma(v(T-\boldsymbol{T}),z,y)} R(\boldsymbol{T}, v, h(s, z, y, \widetilde{\beta}), y + \widetilde{\beta}) \\ &= \inf_{\beta\in\Gamma(s,z,y)} R(\boldsymbol{T}, v, h(s, z, y, \beta), y + \beta).\end{aligned}$$

(iii) Fix $y \in \mathbb{R}$. Choose a sequence $\{\boldsymbol{T}_n, v_n, z_n\}_{n=1}^{\infty} \subset [0, T] \times C[0, T] \times [\delta\, \mathbf{1}_{\{y=0\}}, \infty)$ which converges to $(\boldsymbol{T}, v, z)$. From the continuity of G, it follows that

$$X_0^{\boldsymbol{T},v} = \lim_{n\to\infty} X_0^{\boldsymbol{T}_n,v_n}. \tag{5.7}$$

Set $s = v(T - \boldsymbol{T})$ and $s_n = v_n(T - \boldsymbol{T}_n)$, $n \in \mathbb{N}$. Let $\beta_n = \beta^*(\boldsymbol{T}_n, v_n, z_n, y)$, $n \in \mathbb{N}$. The sequence β_n, $n \in \mathbb{N}$, is bounded, and so without loss of generality by taking a subsequence we can assume that it converges. Denote $\hat{\beta} = \lim_{n\to\infty} \beta_n$. It is straightforward to check that $\hat{\beta} \in \Gamma(s, z, y)$. By using (5.7) and applying similar arguments as in (5.5) and (5.6), we obtain

$$\begin{aligned}\hat{R}(\boldsymbol{T}, v, z, y) &\leq \min((X_0^{\boldsymbol{T},v} - z)^+, R(\boldsymbol{T}, v, h(s, z, y, \hat{\beta}), \hat{\beta} + y)) \\ &\leq \liminf_{n\to\infty} \hat{R}(\boldsymbol{T}_n, v_n, z_n, y).\end{aligned}$$

This yields the lower semicontinuity of $\hat{R}_y$. Thus, it remains to establish upper semicontinuity. Let $\widetilde{\beta} = \beta^*(\boldsymbol{T}, s, z, y)$. First assume that $h(s, z, y, \widetilde{\beta}) > 0$. Then for sufficiently large n, we have $\widetilde{\beta} \in \Gamma(s_n, z_n, y)$ and $h(s, z, y, \widetilde{\beta}) = \lim_{n\to\infty} h(s_n, z_n, y, \widetilde{\beta})$. Thus, from (5.7) and the fact that R is upper semicontinuous, we obtain

$$\begin{aligned}\hat{R}(\boldsymbol{T}, v, z, y) &= \min((X_0^{\boldsymbol{T},v} - z)^+, R(\boldsymbol{T}, v, h(s, z, y, \widetilde{\beta}), \widetilde{\beta} + y)) \\ &\geq \limsup_{n\to\infty} \hat{R}(\boldsymbol{T}_n, v_n, z_n, y).\end{aligned} \tag{5.8}$$

Finally, assume that $h(s, z, y, \widetilde{\beta}) = 0$. Let $\pi = (0, \widetilde{y}, \gamma)$ be an *admissible* portfolio for some $\widetilde{y}$. From the fact that the geometric Brownian motion can increase or decrease for any amount (with positive probability) on any time interval, it follows that $\gamma_t = 0$ for $t > 0$. Indeed, otherwise, the portfolio value can become negative with positive probability. Thus,

$$R(\boldsymbol{T}, v, 0, \widetilde{y}) = \inf_{\sigma \in \mathcal{T}_{\boldsymbol{T}}} \sup_{\tau \in \mathcal{T}_{\boldsymbol{T}}} \mathbb{E}_{\mathbb{P}}(X_\sigma^{\boldsymbol{T},v} \mathbf{1}_{\{\sigma<\tau\}} + Y_\tau^{\boldsymbol{T},v} \mathbf{1}_{\{\tau \le \sigma\}}).$$

In particular, $R(\boldsymbol{T}, v, 0, \widetilde{y})$ does not depend on $\widetilde{y}$. From (2.2) and the fact that F and G are continuous, it follows that $R(\cdot, \cdot\cdot, 0, \widetilde{y})$ is continuous. For any $n \in \mathbb{N}$, $-y \in \Gamma(s_n, z_n, y)$, and so from the upper semicontinuity of R, it follows that

$$\begin{aligned}\hat{R}(\boldsymbol{T}, v, z, y) &= \min((X_0^{\boldsymbol{T},v} - z)^+, R(\boldsymbol{T}, v, 0, \widetilde{\beta} + y)) \\ &\ge \limsup_{n\to\infty} \min((X_0^{\boldsymbol{T}_n,v_n} - z_n)^+, R(\boldsymbol{T}_n, v_n, z_n - \delta\, \mathbf{1}_{\{y=0\}}, 0)) \\ &\ge \limsup_{n\to\infty} \hat{R}(\boldsymbol{T}_n, v_n, z_n, y). \end{aligned} \tag{5.9}$$

From (5.8) and (5.9) we derive the upper semicontinuity of $\hat{R}_y$, completing the proof. □

Acknowledgements

The authors were partially supported by the Einstein Foundation (grant number A 2012 137). The first author also acknowledges support of the Marie Curie Actions Fellowships (grant number 618235), and the second author of the Israeli Science Foundation (grant number 82/10).

References

[1] Altarovici, A., Muhle-Karbe, J. and Soner, H. M. (2015). Asymptotics for fixed transaction costs. *Finance Stoch.* **19,** 363–414.
[2] Cvitanić, J. and Karatzas, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Prob.* **24,** 2024–2056.
[3] Dolinsky, Y. (2013). Hedging of game options with the presence of transaction costs. *Ann. Appl. Prob.* **23,** 2212–2237.
[4] Dolinsky, Y. (2014). Limit theorems for partial hedging under transaction costs. *Math. Finance* **24,** 567–597.
[5] Dolinsky, Y. and Kifer, Y. (2007). Hedging with risk for game options in discrete time. *Stochastics* **79,** 169–195.
[6] Dolinsky, Y. and Kifer, Y. (2008). Binomial approximations of shortfall risk for game options. *Ann. Appl. Prob.* **18,** 1737–1770.
[7] Guasoni, P. (2002). Optimal investment with transaction costs and without semimartingales. *Ann. Appl. Prob.* **12,** 1227–1246.
[8] Guasoni, P. (2002). Risk minimization under transaction costs. *Finance Stoch.* **6,** 91–113.
[9] Iron, Y. and Kifer, Y. (2011). Hedging of swing game options in continuous time. *Stochastics* **83,** 365–404.
[10] Kamizono, K. (2003). Partial hedging under transaction costs. *SIAM J. Control Optimization* **42,** 1545–1558.
[11] Kifer, Y. (2000). Game options. *Finance Stoch.* **4,** 443–463.
[12] Kifer, Y. (2006). Error estimates for binomial approximations of game options. *Ann. Appl. Prob.* **16,** 984–1033, 2273–2275. (Correction: **18** (2008), 1271–1277.)
[13] Kobylanski, M. and Quenez, M.-C. (2012). Optimal stopping time problem in a general framework. *Electron. J. Prob.* **17,** 28pp.
[14] Korn, R. (1998). Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance Stoch.* **2,** 85–114.
[15] Lepeltier, J.-P. and Maingueneau, M. A. (1984). Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* **13,** 25–44.
[16] Lo, A. W., Mamaysky, H. and Wang, J. (2004). Asset prices and trading volume under fixed transaction costs. *J. Political Econom.* **112,** 1054–1090.
[17] Morton, A. J. and Pliska, S. R. (1995). Optimal portfolio management with fixed transaction costs. *Math. Finance* **5,** 337–356.
[18] Øksendal, B. and Sulem, A. (2002). Optimal consumption and portfolio with both fixed and proportional transaction costs. *SIAM J. Control Optimization* **40,** 1765–1790.