APPROXIMATING STOCHASTIC VOLATILITY BY RECOMBINANT TREES[1]

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A general method to construct recombinant tree approximations for stochastic volatility models is developed and applied to the Heston model for stock price dynamics. In this application, the resulting approximation is a four tuple Markov process. The first two components are related to the stock and volatility processes and take values in a two-dimensional binomial tree. The other two components of the Markov process are the increments of random walks with simple values in $\{-1, +1\}$. The resulting efficient option pricing equations are numerically implemented for general American and European options including the standard put and calls, barrier, lookback and Asian-type pay-offs. The weak and extended weak convergences are also proved.

1. Introduction. Contrary to many mathematical models, the discrete counterpart of the celebrated Black–Scholes model [4] came after its continuous version, and it is generally accepted that this simple binomial approximation by Cox et al. [8] has been instrumental in the better understanding and the applicability of the model. Rubinstein [28] states that "the Black and Scholes model is widely viewed as one of the most successful in the social sciences and perhaps, including its binomial extension, the most widely used formula, with embedded probabilities, in human history."

This widespread use and practicality is extended by further research. In particular, stochastic volatility models have been introduced to address the volatility smiles observed in option markets and the heavy tails and high peaks of the underlying asset distributions. Hull and White [19], Chesney and Scott [5], Stein and Stein [29], Heston [17] and Hagan et al. [16] among many others, assume a bivariate diffusion framework in which a separate stochastic process represents the dynamics of asset price volatility. In all these models, the asset price process S_t

and its volatility factor process Y_t satisfy the following stochastic differential equations:

$$dS_t = S_t[\mu \, dt + f(Y_t) \, dW_t],$$
$$dY_t = \mu^Y(Y_t) \, dt + \sigma^Y(Y_t) \, dZ_t,$$

with correlated Brownian motions W, Z and different choices for the functions $\mu^Y(y)$, $\sigma^Y(y)$ and $f(y)$.

In this paper, we construct an approach that provides a *recombining* tree approximation for all stochastic volatility models of the above type. This approximation as the Cox–Ross–Rubinstein (CRR) model easily constructs a discrete time financial market that itself is arbitrage free and as such allows for simple analysis of related complex instruments.

For specificity, we implement our methodology on the Heston model. Well known among stochastic volatility models, it deserves special attention because of its ability to provide closed-form solutions for European options through Fourier transform. This unique feature allows for an efficient and quick calibration of the model to European options. However, for derivative products with early exercise features such as American options, closed-form solutions do not exist even under the Heston model. Hence, numerical methods such as binomial tree, finite difference schemes or Monte Carlo simulation have to be used to evaluate American and other exotic options under stochastic volatility models.

In any market with a nonconstant volatility, the CRR methodology encounters a basic difficulty. Indeed, since the volatility changes at each time, the nodes do not recombine on the lattice, and this fact results in an exponential and thus a computationally explosive tree that cannot be used in many realistic situations. Nelson and Ramaswamy [24] were the first to construct a computationally simple binomial process which approximates a diffusion process given in the form

$$dY_t = \mu(Y, t) \, dt + \sigma(Y, t) \, dZ_t.$$

They solve the node recombination problem by transforming the process given in the above equation into a process $X(Y, t)$ such that the instantaneous volatility of the transformed process is constant. Hilliard and Schwartz [18] follow this method to develop binomial trees for continuous-time risk-neutralized diffusion processes of a special form.

Our main tool is to apply correlated random walks in order to approximate diffusion processes. A correlated random walk is a generalized random walk in the sense that the increments are not identically and independently distributed, but they only satisfy some Markov-type of conditions. The exact definition will be given in Section 3. These processes naturally lie on a grid, and their Markov structure allows for an efficient computation of option prices.

The idea to use correlated random walks for approximating diffusion processes goes back to Gruber and Schweizer [15] and to Kusuoka [22]. In [15], the authors

prove a convergence result for one-dimensional diffusion processes that satisfies stronger regularity conditions than those that appear in stochastic volatility models. In [22], Kusuoka uses (also in one dimension) an original technique to modify random walks in order to get a diffusion in the limit. Again the regularity conditions that he assumes are stronger than those that are required in stochastic volatility models.

Our approach is also similar to that of Kusuoka and modifies the correlated random walks on a multi-dimensional binomial tree by adding a predictable process times $\sqrt{h}$ where h is the size of the time step. We then use this freedom to choose the predicable process together with an appropriate choice of the conditional probabilities to construct a Markov process that weakly converges to the stochastic volatility model. This construction is explained in Section 3. The weak convergence of our approximation is given in Section 4. Then the approximating martingale measures are constructed so that the modified tree under these measures asymptotically matches the first two conditional moments. This fact allows for a straightforward convergence proof. We also note that this approach was successively used by the last two authors [9] to prove convergence of a market with trading costs.

Our extensive numerical experimentation is reported in our final section. In general, weak convergence does not provide any error estimation. However, binomial-type approximations of diffusion models have a convergence rate of $(\Delta t)^{1/2}$ which we accept it to be true. We leave the detailed description of the computational studies to that section and here simply state that our algorithm works efficiently compared to all existing methods for the Heston model.

We emphasize that our tool can also be applied for a general type of stochastic volatility models (see Remark 4.2). There is also GARCH approach to stochastic volatility models that we refer the reader to Duan [10–13], Nelson [23], Ritchken and Trevor [27] and the references therein.

Clearly, there are several other successful computational approaches to stochastic models, including the ones based on partial differential equations, semi-analytic methods and Monte Carlo simulations. Here we do not survey all these results but compare our numerical results with the appropriate ones in the section that outlines our numerical experimentations.

In the literature, tree-based methods have also been considered. Beliaeva and Nawalkha [2] authored the most recent of these studies; see [2] and the references therein. However, our approach differs from these earlier studies in two fundamental ways. First, our approximation is recombinant by construction, while in the previous studies recombination is achieved through truncation. Also, our tree is arbitrage free, and we provide a proof of convergence.

2. The Heston model. Consider the Heston model,

$$dS_t = S_t(r\,dt + \sqrt{v_t}\,dW_t),$$
$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,d\widetilde{W}_t,$$

with initial conditions $S_0, v_0 > 0$, given positive parameters r, κ, θ, η and two Brownian motions $W, \widetilde{W}$ with a constant correlation $\rho \in (-1, 1)$. The constant $r > 0$ is the interest rate and S is the stock price process. As it is standard, we also assume that

$$2\kappa\theta > \eta^2.$$

Then, the Heston equation has a unique *positive* solution in $\mathbb{R}^2_+$; see, for instance, [7].

The main goal of this paper is to construct a discrete approximation of this model. For this purpose, it is more convenient to work with a transformed system of affine equations driven by independent Brownian motions. Therefore, we set

$$x_t := \ln S_t, \qquad y_t := \frac{v_t}{\eta} - \rho x_t,$$

so that

$$
\begin{aligned}
(2.1) \qquad & dx_t = \mu_x(x_t, y_t)\, dt + \sqrt{\eta}\sigma(x_t, y_t)\, dW_t, \\
& dy_t = \mu_y(x_t, y_t)\, dt + \sqrt{\eta(1 - \rho^2)}\sigma(x_t, y_t)\, dB_t,
\end{aligned}
$$

where

$$\mu_x(x, y) := r - \frac{1}{2}\eta(y + \rho x), \qquad \mu_y(x, y) := \frac{\kappa\theta}{\eta} - \rho r + \frac{1}{2}(\rho\eta - 2\kappa)(y + \rho x),$$

$$B_t := \frac{W_t - \rho\widetilde{W}_t}{\sqrt{1 - \rho^2}}, \qquad \sigma(x, y) := \sqrt{(y + \rho x)^+},$$

and $z^+ = \max(0, z)$. One may directly verify that B is also a standard Brownian motion independent of W.

3. Derivation of the approximation.
We fix a time horizon, or equivalently a maturity, $T > 0$ and a time discretization

$$h := \frac{T}{n},$$

with a large integer n. We then use two-dimensional correlated random walks to approximate the diffusion processes given by (2.1). Indeed, consider the random walks $\{X_k^{(n)}, Y_k^{(n)}\}_{k=0}^n$ of the form

$$(3.1) \qquad X_k^{(n)} := x_0 + \sqrt{h\eta}\sum_{i=1}^k \xi_i^X,$$

$$(3.2) \qquad Y_k^{(n)} := y_0 + \sqrt{h\eta(1 - \rho^2)}\sum_{i=1}^k \xi_i^Y,$$

where $x_0 := \ln(s_0)$, $y_0 := (v_0/\eta) - \rho x_0$ and (ξ^X, ξ^Y)'s are random variables with values in $\{-1, 1\}$. In the sequel, we always use the initial data

$$\xi_0^X = \xi_0^Y = 0.$$

We construct a probabilistic structure so that the four tuple $(X_k^{(n)}, Y_k^{(n)}, \xi_k^X, \xi_k^Y)$ forms a Markov chain weakly approximating the solution of (2.1). To achieve this we also need to introduce a modification of this discrete Markov chain. Indeed, for given *predictable* processes $\hat{\alpha}, \hat{\beta}$, we introduce

$$(3.3) \qquad \hat{X}_k^{(n)} := X_k^{(n)} + \sqrt{h}\eta\hat{\alpha}_k\xi_k^X,$$

$$(3.4) \qquad \hat{Y}_k^{(n)} := Y_k^{(n)} + \sqrt{h\eta(1-\rho^2)}\hat{\beta}_k\xi_k^Y, \qquad k = 1, \ldots, n.$$

Clearly, the convergence of (X, Y) is equivalent to that of $(\hat{X}, \hat{Y})$ as

$$\|\hat{X}^{(n)} - X^{(n)}\| = O(\sqrt{h}), \qquad \|\hat{Y}^{(n)} - Y^{(n)}\| = O(\sqrt{h}),$$

where for any exponent k, we use the standard notation $O(h^k)$ to denote a generic random variable of the order h^k and $o(h^k)$ denotes a random variable that converges to zero after divided by h^k.

Our goal is to construct a sequence of probability measures $\mathbb{P}^{(n)}$ and stochastic processes $\hat{\alpha}^{(n)}, \hat{\beta}^{(n)}$ such that

$$\{(\hat{X}_{[nt/T]}^{(n)}, \hat{Y}_{[nt/T]}^{(n)})\}_{t=0}^T \Rightarrow \{(x_t, y_t)\}_{t=0}^T,$$

where $\Rightarrow$ denotes weak convergence. We provide the definitions in the next section.

In view of the martingale convergence Theorem 7.4.1 in [14], to establish this convergence, it is essentially sufficient to match the first and the second conditional moments. Indeed, for a positive integer k, set

$$\mathcal{F}_k = \sigma\{\xi_1^X, \ldots, \xi_k^X, \xi_1^Y, \ldots, \xi_k^Y\},$$

and let $\mathbb{E}_k^{(n)}[\cdot]$ be the conditional expectation $\mathbb{E}^{(n)}[\cdot|\mathcal{F}_k]$ with respect to the probability measure $\mathbb{P}^{(n)}$. Then the moment matching conditions are the following equations:

$$(3.5) \qquad \mathbb{E}_{k-1}^{(n)}[\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)}] = \mu_x(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$(3.6) \qquad \mathbb{E}_{k-1}^{(n)}[\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)}] = \mu_y(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$(3.7) \qquad \mathbb{E}_{k-1}^{(n)}[(\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})^2] = \eta\sigma^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$(3.8) \qquad \mathbb{E}_{k-1}^{(n)}[(\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)})^2] = \eta(1-\rho^2)\sigma^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h).$$

We also need conditions on the covariances. However, since W and B in (2.1) are independent, this condition is simply reduced to the requirement that ξ_k^X and ξ_k^Y are conditionally independent given $\mathcal{F}_{k-1}$.

Observe that we need to solve four equations, and the number of unknowns or parameters to choose are four as well; the corrections $\hat{\alpha}$, $\hat{\beta}$ and two probabilities,

(3.9) $$p_k := \mathbb{P}^{(n)}_{k-1}(\xi^X_k = 1), \qquad q_k := \mathbb{P}^{(n)}_{k-1}(\xi^Y_k = 1).$$

This construction would provide a financial market which is asymptotically arbitrage free. However, a slight modification of the above procedure would also ensure that each discrete market itself is free of arbitrage. In our model, the discrete stochastic process

$$\left\{\exp(-rkh)\exp(\hat{X}^{(n)}_k)\right\}^n_{k=0},$$

is the approximation of the discounted price process. Hence, we replace the first order condition (3.5) by requiring that above process is a martingale, that is,

(3.10) $$\mathbb{E}^{(n)}_{k-1}\left[\exp(-rh)\exp(\hat{X}^{(n)}_k) - \exp(\hat{X}^{(n)}_{k-1})\right] = 0.$$

In fact, (3.5) and (3.10) are asymptotically equivalent and both would be sufficient to prove convergence. However, in our numerical experimentation we observe that this modification is substantially better than the nonmodified version. We continue by constructing $\mathbb{P}^{(n)}$ and $\hat{\alpha}^{(n)}$, $\hat{\beta}^{(n)}$ satisfying equations (3.10) and (3.6)–(3.8). Indeed, by (3.10) we directly calculate that

$$(1 + \hat{\alpha}_k)\mathbb{E}^{(n)}_{k-1}[\xi^X_k] - \hat{\alpha}_{k-1}\xi^X_{k-1} = o(h).$$

Hence

$$(1 + \hat{\alpha}^{(n)}_k)(\hat{\alpha}_{k-1})\mathbb{E}^{(n)}_{k-1}[\xi^X_k]\xi^X_{k-1} = (\hat{\alpha}_{k-1})^2 + o(h).$$

We use this and calculate that

$$\mathbb{E}^{(n)}_{k-1}\left((\hat{X}^{(n)}_k - \hat{X}^{(n)}_{k-1})^2\right) = \eta h\left((1 + \hat{\alpha}_k)^2 - (\hat{\alpha}^{(n)}_{k-1})^2 + o(h)\right) + o(h).$$

We expect that the difference $\hat{\alpha}_k - \hat{\alpha}_{k-1}$ to be of order h. Hence, the above expression simplifies to

$$\mathbb{E}^{(n)}_{k-1}\left((\hat{X}^{(n)}_k - \hat{X}^{(n)}_{k-1})^2\right) = \eta h(1 + 2\hat{\alpha}_k) + o(h).$$

We now compare the above equation with (3.7) to conclude that

$$1 + 2\hat{\alpha}_k = \sigma^2(X^{(n)}_{k-1}, Y^{(n)}_{k-1}) + o(h).$$

Using (3.6) and (3.8), we obtain the same equation for $\hat{\beta}$. Hence, we conclude that

$$\hat{\alpha}_k = \hat{\beta}_k = \frac{\sigma^2(X^{(n)}_{k-1}, Y^{(n)}_{k-1}) - 1}{2} + o(h).$$

We use the above identity and the freedom on the order $o(h)$ to define the processes $\hat{\alpha}$, $\hat{\beta}$ below. The below definition contains a certain truncation that is within the $o(h)$ margin. Although this correction is asymptotically small, it allows us to obtain

several bounds in the convergence proof and also enables to construct transition probabilities that always remain in the unit interval; see (3.12), below. So we now define

$$(3.11) \qquad \hat{\alpha}_k := \hat{\beta}_k := \frac{\max\{A_n, \sigma^2(X^{(n)}_{k-1}, Y^{(n)}_{k-1})\} - 1}{2}, \qquad 1 \le k \le n,$$

where

$$A_n = \left(\frac{\kappa\theta}{\eta} + |\rho|r\right)\sqrt{\frac{h}{\eta(1 - \rho^2)}},$$

and we set

$$\hat{\alpha}^{(n)}_0 = \hat{\beta}^{(n)}_0 = 0.$$

To reiterate once again, the function A_n is chosen to ensure that the probabilities that are defined in (3.12), below, remain in the unit interval. Although, this is clearly crucial for our analysis, in our numerical implementation we do not use this truncation and instead modify (3.12) to ensure that these are true probabilities.

The above construction together with the conditional independence of the increments ensure the second moment matching. We now use the first order conditions (3.10) and (3.6) to construct the transition probabilities. Indeed, recall that by (3.9),

$$p_k := \mathbb{P}^{(n)}_{k-1}(\xi^X_k = 1),$$

and rewrite (3.10) as

$$p_k \exp(\sqrt{h\eta}[(1 + \hat{\alpha}_k) - \hat{\alpha}_{k-1}\xi^X_{k-1}]) \\ + (1 - p_k)\exp(-\sqrt{h\eta}[(1 + \hat{\alpha}_k) + \hat{\alpha}_{k-1}\xi^X_{k-1}]) = \exp(rh).$$

This implies that p_k must be given by

$$(3.12) \qquad p_k = \frac{\exp(rh + \sqrt{\eta h}\hat{\alpha}_{k-1}\xi^X_{k-1}) - \exp(-\sqrt{\eta h}(1 + \hat{\alpha}_k))}{\exp(\sqrt{\eta h}(1 + \hat{\alpha}_k)) - \exp(-\sqrt{\eta h}(1 + \hat{\alpha}_k))}.$$

In view of the truncation introduced in (3.11), $p_k \in [0, 1]$ for all large n.

We now recall that

$$q_k := \mathbb{P}^{(n)}_{k-1}(\xi^Y_k = 1),$$

and use (3.6) to arrive at

$$q_k = \frac{1}{2} + \frac{\hat{\alpha}_{k-1}}{2(1 + \hat{\alpha}_k)}\xi^Y_{k-1} + \frac{\sqrt{h}\mu_y(X^{(n)}_{k-1}, Y^{(n)}_{k-1})}{2\sqrt{\eta(1 - \rho^2)}(1 + \hat{\alpha}_k)}.$$

Since q_k must take values in the unit interval, we modify it in the following way:

$$(3.13) \quad q_k = \left(\min\left\{ 1, \frac{1}{2} + \frac{\hat{\alpha}_{k-1}}{2(1+\hat{\alpha}_k)} \xi_{k-1}^Y + \frac{\sqrt{h}\mu_y(X_{k-1}^{(n)}, Y_{k-1}^{(n)})}{2\sqrt{\eta(1-\rho^2)(1+\hat{\alpha}_k)}} \right\} \right)^+.$$

Set

$$\Xi_k := \left(X_k^{(n)}, Y_k^{(n)}, \xi_k^X, \xi_k^Y \right).$$

Then, we claim that Ξ is a Markov process. Indeed, recall that the independence of the Brownian motions in (2.1) implies the conditional independence of the increments ξ^X and ξ^Y. Hence

$$(3.14) \qquad \mathbb{P}^{(n)}\left(\xi_k^X = a, \xi_k^Y = b | \Xi_{k-1} \right) = \mathbb{P}_{k-1}^{(n)}\left(\xi_k^X = a \right) \mathbb{P}_{k-1}^{(n)}\left(\xi_k^Y = b \right).$$

Moreover, in view of (3.1) and (3.2), the set

$$\{ X_k^{(n)} = X_{k-1}^{(n)} + c, Y_k^{(n)} = Y_{k-1}^{(n)} + d, \xi_k^X = a, \xi_k^Y = b \}$$

is empty unless $c = a\eta h$ and $d = b\eta h \sqrt{1-\rho^2}$, and in this case it is equal to $\{ \xi_k^X = a, \xi_k^Y = b \}$. Therefore, the transition probabilities of the process Ξ are determined by

$$\mathbb{P}^{(n)}\left(\xi_k^X = 1, \xi_k^Y = 1 | \Xi_{k-1} \right) = p_k q_k.$$

Moreover, there is a simple transformation between Ξ_k and

$$\hat{\Xi}_k := \left(\hat{X}_k^{(n)}, \hat{Y}_k^{(n)}, \xi_k^X, \xi_k^Y \right).$$

Hence, one may consider the process $\hat{\Xi}$ as the basic approximating Markov process.

4. Main convergence result.
In this section, we first briefly recall the concept of weak convergence of probability measures and then state our main convergence result. For more information on weak convergence, we refer the reader to the books of Billingsley [3] and Ethier and Kurtz [14].

For any *càdlàg* stochastic process $\{Z(t)\}_{t=0}^T$ with values in some Euclidean space $\mathbb{R}^d$, let $\mathbb{P}^Z$ be the distribution of Z on the canonical space $\mathbb{D}([0,T];\mathbb{R}^d)$ equipped with the Skorohod topology (for details see [3]), that is, for any Borel set $D \subset \mathbb{D}([0,T];\mathbb{R}^d)$, $\mathbb{P}^Z(D) = \mathbb{P}\{Z \in D\}$. For a sequence of $\mathbb{R}^d$-valued, stochastic processes $Z^{(n)}$ we use the notation $Z^{(n)} \Rightarrow Z$ to indicate that the probability measures $\mathbb{P}^{Z^{(n)}}$, converge vaguely to $\mathbb{P}^Z$ on the space $\mathbb{D}([0,T];\mathbb{R}^d)$.

We are now ready to state the main convergence theorem which is the main theoretical foundation of our numerical scheme. It will be proved in Section 6.

THEOREM 4.1.   *For any $n \in \mathbb{N}$, let $\mathbb{P}^{(n)}$ be the probability measure defined by* (3.14). *Consider the stochastic processes* $\{X_{[nt/T]}^{(n)}\}_{t=0}^{T}$, $\{\hat{X}_{[nt/T]}^{(n)}\}_{t=0}^{T}$ *and* $\{Y_{[nt/T]}^{(n)}\}_{t=0}^{T}$ *under* $\mathbb{P}^{(n)}$. *Let* (x, y) *be the unique solution of* (2.1). *Then*

$$(4.1) \qquad \{(X_{[nt/T]}^{(n)}, Y_{[nt/T]}^{(n)})\}_{t=0}^{T} \Rightarrow \{(x_t, y_t)\}_{t=0}^{T}$$

and

$$(4.2) \qquad \{(\hat{X}_{[nt/T]}^{(n)}, Y_{[nt/T]}^{(n)})\}_{t=0}^{T} \Rightarrow \{(x_t, y_t)\}_{t=0}^{T}$$

on the space $\mathbb{D}([0, T]) \times \mathbb{D}([0, T])$.

REMARK 4.2.   For the Heston model, one applies a transformation that decorrelates the Brownian motions. However, this decorrelation is not necessary and used only to simplify the procedure. Indeed, consider a general two-dimensional diffusion

$$dx_t = \mu_x(x_t, y_t)\, dt + \sigma_x(x_t, y_t)\, dW_t,$$
$$dy_t = \mu_y(x_t, y_t)\, dt + \sigma_y(x_t, y_t)\, d\widetilde{W}_t,$$

where $W, \widetilde{W}$ are two-standard Brownian motions with a correlation ρ. Introduce the two-dimensional correlated random walk $\{X_k^{(n)}, Y_k^{(n)}\}_{k=0}^{n}$ by

$$X_k^{(n)} := x_0 + \sqrt{h} \sum_{i=1}^{k} \xi_i^{X},$$

$$Y_k^{(n)} := y_0 + \sqrt{h} \sum_{i=1}^{k} \xi_i^{Y}.$$

As before, we consider a small modification of the correlated random walks

$$\hat{X}_k^{(n)} := X_k^{(n)} + \sqrt{h}\hat{\alpha}_k \xi_k^{X},$$
$$\hat{Y}_k^{(n)} := Y_k^{(n)} + \sqrt{h}\hat{\beta}_k \xi_k^{Y}, \qquad k = 1, \dots, n.$$

In this case, the moment matching conditions are the following equations:

$$\mathbb{E}_{k-1}^{(n)}[\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)}] = \mu_x(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$\mathbb{E}_{k-1}^{(n)}[\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)}] = \mu_y(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$\mathbb{E}_{k-1}^{(n)}[(\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})^2] = \sigma_x^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$\mathbb{E}_{k-1}^{(n)}[(\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)})^2] = \sigma_y^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)})h + o(h),$$

$$\mathbb{E}_{k-1}^{(n)}[(\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})(\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)})] = \sigma_x(X_{k-1}^{(n)}, Y_{k-1}^{(n)})\sigma_y(X_{k-1}^{(n)}, Y_{k-1}^{(n)})\rho h + o(h).$$

We solve these equations as in the Heston case and obtain that

$$\hat{\alpha}_k = \frac{\sigma_x^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)}) - 1}{2}, \qquad \hat{\beta}_k = \frac{\sigma_y^2(X_{k-1}^{(n)}, Y_{k-1}^{(n)}) - 1}{2}.$$

The transition probabilities are also given by

$$\mathbb{P}_{k-1}^{(n)}(\xi_k^X = 1, \xi_k^Y = 1) = \frac{1}{4} + \frac{\hat{\alpha}_{k-1}\xi_{k-1}^X + \mu_x\sqrt{h}}{4(1 + \hat{\alpha}_k)} + \frac{\hat{\beta}_{k-1}\xi_{k-1}^Y + \mu_y\sqrt{h}}{4(1 + \hat{\beta}_k)}$$
$$+ \frac{\rho\sigma_x\sigma_y + \hat{\alpha}_{k-1}\hat{\beta}_{k-1}\xi_{k-1}^X\xi_{k-1}^Y}{4(1 + \hat{\alpha}_k)(1 + \hat{\beta}_k)},$$

$$\mathbb{P}_{k-1}^{(n)}(\xi_k^X = 1, \xi_k^Y = -1) = \frac{1}{4} + \frac{\hat{\alpha}_{k-1}\xi_{k-1}^X + \mu_x\sqrt{h}}{4(1 + \hat{\alpha}_k)} - \frac{\hat{\beta}_{k-1}\xi_{k-1}^Y + \mu_y\sqrt{h}}{4(1 + \hat{\beta}_k)}$$
$$- \frac{\rho\sigma_x\sigma_y + \hat{\alpha}_{k-1}\hat{\beta}_{k-1}\xi_{k-1}^X\xi_{k-1}^Y}{4(1 + \hat{\alpha}_k)(1 + \hat{\beta}_k)},$$

$$\mathbb{P}_{k-1}^{(n)}(\xi_k^X = -1, \xi_k^Y = 1) = \frac{1}{4} - \frac{\hat{\alpha}_{k-1}\xi_{k-1}^X + \mu_x\sqrt{h}}{4(1 + \hat{\alpha}_k)} + \frac{\hat{\beta}_{k-1}\xi_{k-1}^Y + \mu_y\sqrt{h}}{4(1 + \hat{\beta}_k)}$$
$$- \frac{\rho\sigma_x\sigma_y + \hat{\alpha}_{k-1}\hat{\beta}_{k-1}\xi_{k-1}^X\xi_{k-1}^Y}{4(1 + \hat{\alpha}_k)(1 + \hat{\beta}_k)},$$

$$\mathbb{P}_{k-1}^{(n)}(\xi_k^X = -1, \xi_k^Y = -1) = \frac{1}{4} - \frac{\hat{\alpha}_{k-1}\xi_{k-1}^X + \mu_x\sqrt{h}}{4(1 + \hat{\alpha}_k)} - \frac{\hat{\beta}_{k-1}\xi_{k-1}^Y + \mu_y\sqrt{h}}{4(1 + \hat{\beta}_k)}$$
$$+ \frac{\rho\sigma_x\sigma_y + \hat{\alpha}_{k-1}\hat{\beta}_{k-1}\xi_{k-1}^X\xi_{k-1}^Y}{4(1 + \hat{\alpha}_k)(1 + \hat{\beta}_k)},$$

where in the above formulas, functions $\mu_x, \mu_y, \sigma_x, \sigma_y$ are all evaluated at $(X_{k-1}^{(n)}, Y_{k-1}^{(n)})$. However, the above terms do not necessarily lie in the interval $[0, 1]$. In that case, we apply a truncation of the form $\min(1, \max(0, \cdot))$.

REMARK 4.3. We emphasize that our approximation method using correlated random walks and the above convergence result can easily be extended to more general multidimensional diffusions. The key idea is the introduction of $\hat{X}$-type processes which differ from the original random walk X only by a predictable process $\hat{\alpha}$ times the increment ξ^X. We then use this freedom (viz., the function $\hat{\alpha}$) to construct transition probabilities that match the first and the second conditional moments of the original diffusion. The approximating process has essentially the same dimension as the original diffusion process. However, we need to augment the state space by adding the increments like ξ^X. But these increments take values in the discrete set $\{-1, +1\}$ so do not increase the complexity of the approximation.

Our next remark is toward American options.

REMARK 4.4. In general, the usual weak convergence is not sufficient for the convergence of American options prices. Indeed, the latter also requires the "good" behavior of the filtrations. In his unpublished manuscript (see [1], Sections 15–16), David Aldous introduced the concept of extended weak convergence to address this problem. Briefly his definition is as follows. A sequence $Z^{(n)} : \Omega_n \to \mathbb{D}([0, T]; \mathbb{R}^d)$, *extended weak converges* to a stochastic process $Z : \Omega \to \mathbb{D}([0, T]; \mathbb{R}^d)$, if for any k and continuous bounded functions $\psi_1, \ldots, \psi_k \in C(\mathbb{D}([0, T]; \mathbb{R}^d))$,

$$(Z^{(n)}, Z^{n,1}, \ldots, Z^{n,k}) \Rightarrow (Z, Z^{(1)}, \ldots, Z^{(k)}) \qquad \text{in } \mathbb{D}([0, T]; \mathbb{R}^{d+k}),$$

where for any $t \leq T$, $1 \leq i \leq k$ and $n \in \mathbb{N}$,

$$Z_t^{n,i} = \hat{E}^{(n)}\big(\psi_i\big(Z^{(n)}\big)|\mathcal{F}_t^{Z^{(n)}}\big), \qquad Z_t^{(i)} = \hat{E}\big(\psi_i(Z)|\mathcal{F}_t^Z\big),$$

$\hat{E}^{(n)}$ denotes the expectation on the probability space on which $Z^{(n)}$ is defined and $\hat{E}$ denotes the expectation on the probability space on which Z is defined. In the formulas above $\mathcal{F}^{Z^{(n)}}$ and $\mathcal{F}^Z$ are the filtrations which are generated by $Z^{(n)}$ and Z, respectively. The notion of extended weak convergence provides (in addition to the standard weak convergence of stochastic processes) convergence of filtrations. In particular, Aldous proved (see [1], Section 17) that under uniform integrability of the payoffs, extended weak convergence implies convergence of optimal stopping values. However, it is known that when the proof of weak convergence relies on martingale techniques (like our proof), then the standard weak convergence implies extended weak convergence. For details, we refer the reader to [1], Section 21.

5. Discrete pricing equations.
In this section, we apply the approximation developed in Section 3 to price American put and lookback options.

5.1. *American put.* Consider an American put option with a strike price K. We are interested in approximating its value given by

$$V = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}\big(e^{-r\tau}(K - S_\tau)^+\big),$$

where $\mathcal{T}_{[0,T]}$ is the set of all stopping times with respect to the filtration generated by S, with values in the set $[0, T]$. We approximate the discounted stock price by the discrete time martingales

$$\big\{e^{-rkh} e^{\hat{X}_k^{(n)}}\big\}_{k=0}^n, \qquad n \in \mathbb{N},$$

constructed in Section 3. For any $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the set of all stopping times with respect to the filtration $\mathcal{F}_k$ (again constructed in Section 3), with values in the set $\{0, 1, \ldots, n\}$. Define

$$V^{(n)} := \max_{\tau \in \mathcal{T}_n} \mathbb{E}^{(n)} \big(e^{-r\tau h} \big(K - S_0 e^{\hat{X}_\tau^{(n)}} \big)^+ \big).$$

In view of Theorem 4.1 and Remark 4.4, we directly conclude that

$$\lim_{n \to \infty} V^{(n)} = V.$$

Next, we describe a dynamical programming algorithm for the calculation of $V^{(n)}$. Observe that for a given $k \in \{0, \ldots, n\}$ the random variables $X_k^{(n)}$ and $Y_k^{(n)}$ take values on the grid

$$x_0 + (2l - k)\sqrt{\eta h}, \qquad 0 \le l \le k,$$
$$y_0 + (2m - k)\sqrt{\eta(1 - \rho^2)h}, \qquad 0 \le m \le k,$$

respectively. For nonnegative integers $m, l \le k \le n$ and $\xi_x, \xi_y \in \{-1, +1\}$, let

$$V_k^{(n)}(l, m, \xi_x, \xi_y)$$

be the value of the option at time k when the Markov process is given by

$$Z\Xi_k = \big(X_k^{(n)}, Y_k^{(n)}, \xi_k^X, \xi_k^Y \big) = F_k(l, m, \xi_x, \xi_y)$$
$$:= \big(x_0 + (2l - k)\sqrt{\eta h}, y_0 + (2m - k)\sqrt{\eta(1 - \rho^2)h}, \xi_x, \xi_y \big).$$

The above function F_k is invertible with an inverse F_k^{-1}. We sometimes, with an abuse of notation, write

$$V_{k-1}^{(n)}(\Xi) = V^{(n)}\big(F_{k-1}^{-1}(\Xi) \big)$$

for any four tuple Ξ given by $F_{k-1}(l, m, \xi_x, \xi_y)$ for some (l, m, ξ_x, ξ_y). With this convention, it is not straightforward to state the dynamic programming equation (see, e.g., [26], Chapter 1),

(5.1) $$V_{k-1}^{(n)}(\Xi) = \max\big\{ (K - S_0 \exp(\hat{X}_{k-1}))^+, \mathbb{E}^{(n)}\big[V_k^{(n)}(\Xi_k) | \Xi_{k-1} = \Xi \big] \big\}.$$

We continue by rewriting the dynamic programming equation in an algorithmic manner. In view of (3.11)–(3.13), for any $1 \le k \le n$ and $0 \le l, m \le k - 1$, we define

$$\mathcal{X}_k := x_0 + (2l - k)\sqrt{\eta h},$$
$$\mathcal{Y}_k := y_0 + (2m - k)\sqrt{\eta(1 - \rho^2)h},$$

where both of the above are functions of (l, m), but this dependence is suppressed in the notation. Similarly, we define two probabilities

$$p_k(l, m, \xi_x, \xi_y) := \frac{\exp(rh + \sqrt{\eta h}\Psi_{k-1}\xi_x) - \exp(-\sqrt{\eta h}\Psi_k)}{\exp(\sqrt{\eta h}\Psi_k) - \exp(-\sqrt{\eta h}\Psi_k)},$$

$$q_k(l, m, \xi_x, \xi_y) := \left(\min\left\{1, \frac{1}{2} + \frac{\alpha_{k-1}(l - \xi_x, m - \xi_y)\xi_y}{2\Psi_k} + \frac{\sqrt{h}\mu_{y,k}}{2\sqrt{\eta(1 - \rho^2)}\Psi_k}\right\}\right)^+,$$

where $\alpha_0^{(n)} \equiv 0$ and

$$\alpha_k(l, m) := \frac{\max(A_n, \sigma^2(\mathcal{X}_{k-1}, \mathcal{Y}_{k-1})) - 1}{2},$$

$$\Psi_k := 1 + \alpha_k^{(n)}(l, m),$$

$$\mu_{y,k} := \mu_y(\mathcal{X}_{k-1}, \mathcal{Y}_{k-1}).$$

As we remarked earlier, in our actual numerical codes, we simply define $\alpha = (\sigma^2 - 1)/2$ without the truncation with A_n and instead truncate p_k, above, to ensure that it stays within the unit interval.

Observe that

$$p_k(l, m, \xi_x, \xi_y) = \mathbb{P}^{(n)}(\xi_k^X = 1 | \Xi_{k-1} = F_{k-1}(l, m, \xi_x, \xi_y)),$$

$$q_k(l, m, \xi_x, \xi_y) = \mathbb{P}^{(n)}(\xi_k^Y = 1 | \Xi_{k-1} = F_{k-1}(l, m, \xi_x, \xi_y)).$$

Moreover,

$$\mathbb{P}_{k-1}^{(n)}(\xi_k^X = 1, \xi_k^Y = 1) = p_k(l, m, \xi_x, \xi_y)q_k(l, m, \xi_x, \xi_y).$$

One can easily obtain expressions for the other three probabilities as well.

We are now ready to restate the dynamic programming equation (5.1). Indeed, $V_k^{(n)}(l, m, \xi_x, \xi_y)$ is the unique solution of the following recursive relations:

$$V_n^{(n)}(l, m, \xi_x, \xi_y) = (K - \exp(\mathcal{X}_n + \sqrt{\eta h}\alpha_n\xi_x))^+,$$

and for $1 \leq k \leq n$,

$$V_{k-1}^{(n)}(l, m, \xi_x, \xi_y) = \max\{(K - \exp(\mathcal{X}_{k-1} + \sqrt{\eta h}\alpha_{k-1}\xi_x))^+, \mathcal{E}(V_k^{(n)})\},$$

where

$$\mathcal{E}(V_k^{(n)})$$

$$= \mathbb{E}^{(n)}[V_k^{(n)}(\Xi_k) | \Xi_{k-1} = F_{k-1}(l, m, \xi_x, \xi_y)]$$

$$= \sum_{i,j=0}^{1} \mathbb{P}_{k-1}^{(n)}(\xi_k^X = 2i - 1, \xi_k^Y = 2j - 1)V_k^{(n)}(l + i, m + j, 2i - 1, 2j - 1)$$

$$= \sum_{i,j=0}^{1} [1 - i + (2i - 1) p_k(l, m, \xi_x, \xi_y)][1 - j + (2j - 1) q_k(l, m, \xi_x, \xi_y)]$$
$$\times V_k^{(n)}(l + i, m + j, 2i - 1, 2j - 1).$$

Then our approximation is simply given by

$$V_n = V_0^{(n)}(0, 0, 0, 0).$$

5.2. *Lookback options.* Consider a lookback put option with a fixed strike K, that is, an option with payoff $(K - \min_{0 \le t \le T} S_t)^+$. Again, we want to approximate the price

$$\hat{V} = \mathbb{E}\Big(e^{-rT}\Big(K - \min_{0 \le t \le T} S_t\Big)^+\Big).$$

Since the running minimum of the processes

$$\{\exp(X_k^{(n)})\}_{k=0}^n, \qquad n \in \mathbb{N}$$

lies on a grid, we will use these processes instead of the martingale $\exp(\hat{X}_k^{(n)})$. The advantage of the processes $\exp(X_k^{(n)})$ becomes clear when we describe the dynamical programming algorithm below.

We set

(5.2) $$\hat{V}^{(n)} = \mathbb{E}^{(n)}\Big(e^{-rT}\Big(K - S_0 \exp\Big(\min_{0 \le i \le n} X_i^{(n)}\Big)\Big)^+\Big).$$

By Theorem 4.1 we conclude that $\hat{V}^{(n)}$ converges to $\hat{V}$.

First, we observe that the random variable

$$z_k := \min_{0 \le i \le k} \sum_{j=1}^{i} \xi_j^X$$

takes values on the grid $\{-k, 1 - k, \ldots, 0\}$.

Using the notation and the conventions of the previous subsection, for $0 \le k \le n$, we let $\hat{V}_k^{(n)}(l, m, z, \xi_x, \xi_y)$ to be the option price at time k. The extra state variable z denotes the value of the running minimum z_k at time k. Then, $\hat{V}^{(n)}$ is the unique solution of

$$\hat{V}_n^{(n)}(l, m, z, \xi_x, \xi_y) = (K - S_0 \exp(-\sqrt{\eta h} z))^+,$$

and for $1 \le k \le n$,

$$\hat{V}_{k-1}^{(n)}(l, m, z, \xi_x, \xi_y) = \max\{(K - S_0 \exp(-\sqrt{\eta h} z))^+, \hat{\mathcal{E}}(V_k^{(n)})\},$$

where

$$\hat{\mathcal{E}}(V_k^{(n)}) = \sum_{i,j=0}^{1} \mathbb{P}_{k-1}^{(n)}(\xi_k^X = 2i - 1, \xi_k^Y = 2j - 1)$$

$$\times \hat{V}_k^{(n)}(l + i, m + j, z + \chi_{\{i=0, z+2l=k-1\}}, 2i - 1, 2j - 1),$$

and χ_Q is the characteristic set of Q. Finally,

$$\hat{V}_n = \hat{V}_0^{(n)}(0, 0, 0, 0, 0).$$

6. Proof of Theorem 4.1.   In this section we provide a proof of Theorem 4.1. Our main tool is the martingale convergence result of Theorem 7.4.1 in [14].

In view of (3.1)–(3.4) and (3.11), we have the following inequality for all sufficiently large n:

$$|\hat{X}_k^{(n)}| \geq |X_k^{(n)}| - \tfrac{1}{3}(|X_k^{(n)}| + |Y_k^{(n)}| + 1),$$

$$|\hat{Y}_k^{(n)}| \geq |Y_k^{(n)}| - \tfrac{1}{3}(|X_k^{(n)}| + |Y_k^{(n)}| + 1).$$

Therefore,

$$(6.1) \qquad |X_k^{(n)}| + |Y_k^{(n)}| \leq 3(|\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}| + 1), \qquad k = 0, 1, \ldots, n.$$

This together with (3.3)–(3.4) and (3.11) imply that there exists a constant $c > 0$ satisfying

$$|X_k^{(n)} - \hat{X}_k^{(n)}| + |Y_k^{(n)} - \hat{Y}_k^{(n)}| \leq \frac{c(1 + |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}|)}{\sqrt{n}},$$

(6.2)

$$k = 0, 1, \ldots, n.$$

It is sufficient to establish that

$$(6.3) \qquad \{(\hat{X}_{[nt/T]}^{(n)}, \hat{Y}_{[nt/T]}^{(n)})\}_{t=0}^{T} \Rightarrow \{(x_t, y_t)\}_{t=0}^{T}.$$

Indeed, from (6.2) it follows that

$$\hat{X}_k^{(n)} - \frac{c(1 + |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}|)}{\sqrt{n}} \leq X_k^{(n)} \leq \hat{X}_k^{(n)} + \frac{c(1 + |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}|)}{\sqrt{n}},$$

$$\hat{Y}_k^{(n)} - \frac{c(1 + |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}|)}{\sqrt{n}} \leq Y_k^{(n)} \leq \hat{Y}_k^{(n)} + \frac{c(1 + |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}|)}{\sqrt{n}}.$$

From (6.3) it follows that the sequences

$$\left\{ \left(\hat{X}_{[nt/T]}^{(n)} - \frac{c(1 + |\hat{X}_{[nt/T]}^{(n)}| + |\hat{Y}_{[nt/T]}^{(n)}|)}{\sqrt{n}}, \hat{Y}_{[nt/T]}^{(n)} - \frac{c(1 + |\hat{X}_{[nt/T]}^{(n)}| + |\hat{Y}_{[nt/T]}^{(n)}|)}{\sqrt{n}} \right) \right\},$$

$$\left\{ \left(\hat{X}_{[nt/T]}^{(n)} + \frac{c(1 + |\hat{X}_{[nt/T]}^{(n)}| + |\hat{Y}_{[nt/T]}^{(n)}|)}{\sqrt{n}}, \hat{Y}_{[nt/T]}^{(n)} + \frac{c(1 + |\hat{X}_{[nt/T]}^{(n)}| + |\hat{Y}_{[nt/T]}^{(n)}|)}{\sqrt{n}} \right) \right\}$$

converge weakly to $\{(x_t, y_t)\}_{t=0}^T$. Thus Theorem 4.1 follows from (6.3). For any $0 \leq k \leq n$, set

$$A_k^{n,x} = \sum_{j=1}^{k} \mathbb{E}_{j-1}^{(n)}(\hat{X}_j^{(n)} - \hat{X}_{j-1}^{(n)}), \qquad A_k^{n,y} = \sum_{j=1}^{k} \mathbb{E}_{j-1}^{(n)}(\hat{Y}_j^{(n)} - \hat{Y}_{j-1}^{(n)}),$$

$$M_k^{n,x} = \hat{X}_k^{(n)} - A_k^{n,x}, \qquad M_k^{n,y} = \hat{Y}_k^{(n)} - A_k^{n,y},$$

$$A_k^{n,x,x} = \sum_{j=1}^{k} \mathbb{E}_{j-1}^{(n)}((M_j^{n,x} - M_{j-1}^{n,x})^2), \qquad A_k^{n,y,y} = \sum_{j=1}^{k} \mathbb{E}_{j-1}^{(n)}((M_j^{n,y} - M_{j-1}^{n,y})^2),$$

$$A_k^{n,x,y} = \sum_{j=1}^{k} \mathbb{E}_{j-1}^{(n)}((M_j^{n,x} - M_{j-1}^{n,x})(M_j^{n,y} - M_{j-1}^{n,y})).$$

Notice that the processes $A^{n,x}$, $A^{n,y}$, $A^{n,x,x}$, $A^{n,y,y}$, $A^{n,x,y}$ are predictable, and the processes $M^{n,x}$, $M^{n,y}$ are martingales.

We now fix a large $N > 0$ and define the stopping times by

$$\sigma_n = \min\{k : |\hat{X}_k^{(n)}| + |\hat{Y}_k^{(n)}| \geq N\} \wedge n, \qquad n \in \mathbb{N}.$$

Using (3.1), (3.2) and (6.2), we conclude that for all $k \leq \sigma_n$,

$$\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)} = O(1/\sqrt{n}) \quad \text{and} \quad \hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)} = O(1/\sqrt{n}),$$

where in this section $o(\cdot)$ and $O(\cdot)$ are defined uniformly in space, that is, $O(1/\sqrt{n})$ is a function which is bounded by a deterministic constant over $\sqrt{n}$, and $\sqrt{n}o(1/\sqrt{n})$ converges uniformly to zero as n tends to infinity.

By Theorem 7.4.1 in [14], (6.3) would result from the following relations:

$$(6.4) \qquad \lim_{n \to \infty} \max_{1 \leq k \leq \sigma_n} \left| A_k^{n,x} - h \sum_{i=0}^{k-1} \mu_x(\hat{X}_i^{(n)}, \hat{Y}_i^{(n)}) \right| = 0 \qquad \text{a.s.,}$$

$$(6.5) \qquad \lim_{n \to \infty} \max_{1 \leq k \leq \sigma_n} \left| A_k^{n,y} - h \sum_{i=0}^{k-1} \mu_y(\hat{X}_i^{(n)}, \hat{Y}_i^{(n)}) \right| = 0 \qquad \text{a.s.,}$$

$$(6.6) \qquad \lim_{n \to \infty} \max_{1 \leq k \leq \sigma_n} \left| A_k^{n,x,x} - \eta h \sum_{i=0}^{k-1} \sigma^2(\hat{X}_i^{(n)}, \hat{Y}_i^{(n)}) \right| = 0 \qquad \text{a.s.,}$$

$$(6.7) \quad \lim_{n \to \infty} \max_{1 \leq k \leq \sigma_n} \left| A_k^{n,y,y} - \eta(1 - \rho^2) h \sum_{i=0}^{k-1} \sigma^2(\hat{X}_i^{(n)}, \hat{Y}_i^{(n)}) \right| = 0 \qquad \text{a.s.,}$$

$$(6.8) \qquad\qquad\qquad\qquad \lim_{n \to \infty} \max_{1 \leq k \leq \sigma_n} |A_k^{n,x,y}| = 0 \qquad \text{a.s.}$$

The rest of the proof is devoted to the verification of the above identities.

We start with a proof of (6.4). Since $\sigma^2(x, y)$ is Lipschitz continuous, (3.1), (3.2) and (3.11) imply that

$$|\hat{\alpha}_k - \hat{\alpha}_{k-1}| = O(\sqrt{h}).$$

In view of (6.1), for $k < \sigma_n$, we have

$$-\tfrac{1}{2} \le \alpha_k \le \hat{c}(N+1)$$

for some constant $\hat{c}$. Since the event $k < \sigma_n$ is $\mathcal{F}_{k-1}$-measurable,

$$\mathbb{P}^{(n)}\big(\xi_k^X = 1 \text{ and } k < \sigma_n | \Xi_{k-1}\big) = \chi_{\{k < \sigma_n\}} \mathbb{P}^{(n)}\big(\xi_k^X = 1 | \Xi_{k-1}\big) = \chi_{\{k < \sigma_n\}} p_k.$$

We now use the above estimates, the definition (3.12) of the transition probability p_k and Taylor expansion. Then, on the set $k < \sigma_n$,

$$
\begin{aligned}
(6.9) \quad & \mathbb{P}^{(n)}\big(\xi_k^X = 1 | \Xi_{k-1}\big) \\
&= \frac{rh + \sqrt{\eta h}(1 + \hat{\alpha}_{k-1}\xi_{k-1}^X + \hat{\alpha}_k) - \eta h(1/2 + \hat{\alpha}_k) + o(h)}{2\sqrt{\eta h}(1 + \hat{\alpha}_k) + o(h)} \\
&= \frac{rh + \sqrt{\eta h}(1 + \hat{\alpha}_{k-1}\xi_{k-1}^X + \hat{\alpha}_k) - \eta h(1/2 + \hat{\alpha}_k)}{2\sqrt{\eta h}(1 + \hat{\alpha}_k)} + o(\sqrt{h}) \\
&= \frac{1}{2} + \frac{\hat{\alpha}_{k-1}}{2(1 + \hat{\alpha}_k)}\xi_{k-1}^X + \frac{rh - \eta(1/2 + \hat{\alpha}_k)h}{2(1 + \hat{\alpha}_k)} + o(h).
\end{aligned}
$$

We thus conclude that on the event $k < \sigma_n$, the following estimate holds:

$$
\begin{aligned}
\mathbb{E}_{k-1}^{(n)}\big[\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)}\big] \\
&= \sqrt{\eta h}\,\mathbb{E}_{k-1}^{(n)}\big[(1 + \hat{\alpha}_k)\xi_k^X - \hat{\alpha}_{k-1}\xi_{k-1}^X\big] \\
&= \sqrt{\eta h}\big[(1 + \hat{\alpha}_k)\big(2\mathbb{P}^{(n)}\big(\xi_k^X = 1 | \Xi_{k-1}\big) - 1\big) - \hat{\alpha}_{k-1}\xi_{k-1}^X\big] \\
&= rh - \eta\big(\tfrac{1}{2} + \hat{\alpha}_k\big)h + o(h) \\
&= \mu_x\big(\hat{X}_{k-1}^{(n)}, \hat{Y}_{k-1}^{(n)}\big)h + o(h),
\end{aligned}
$$

where the last equality follows from the definition of $\hat{\alpha}$, the Lipschitz continuity of $\mu(x, y)$ and (6.2). Then (6.4) follows directly from the above estimate.

We continue with a proof of (6.5). We start with the definition of q_k and use the truncation introduced in (3.11). On $k < \sigma_n$, this fields the following estimate:

$$2 \times \mathbb{P}^{(n)}\big(\xi_k^Y = 1 | \Xi_{k-1}\big) - 1 = \frac{\hat{\alpha}_{k-1}}{1 + \hat{\alpha}_k}\xi_{k-1}^Y + \frac{\sqrt{h}\mu_y(X_{k-1}^{(n)}, Y_{k-1}^{(n)})}{\sqrt{\eta(1 - \rho^2)}(1 + \hat{\alpha}_k)}.$$

As before we directly estimate the on $k - 1 \le \sigma_n$,

$$
\begin{aligned}
\mathbb{E}_{k-1}^{(n)} & (\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)}) \\
&= \sqrt{\eta(1 - \rho^2)h}\big((1 + \hat{\alpha}_k)(2\mathbb{P}^{(n)}(\xi_k^Y = 1|\Xi_{k-1}) - 1) - \hat{\alpha}_k\xi_{k-1}^Y\big) \\
&= \mu_y(\hat{X}_{k-1}^{(n)}, \hat{Y}_{k-1}^{(n)})h + o(h).
\end{aligned}
$$

Again, the last equality follows from (6.2) and the fact that $\mu_y(x, y)$ is Lipschitz continuous. This completes the proof of (6.5).

We continue with the quadratic estimates. Indeed, by (6.9), on $k < \sigma_n$,

$$
2 \times \mathbb{P}^{(n)}(\xi_k^X = 1|\Xi_{k-1}) - 1 = \frac{\hat{\alpha}_{k-1}}{1 + \hat{\alpha}_k}\xi_{k-1}^X + o(\sqrt{h}).
$$

Since $A^{n,x}$ is predictable, on $k < \sigma_n$,

$$
\mathbb{E}_{k-1}^{(n)}((M_k^{n,x} - M_{k-1}^{n,x})^2) = \mathbb{E}_{k-1}^{(n)}((\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})^2) + o(h)
$$

and

$$
\begin{aligned}
\mathbb{E}_{k-1}^{(n)} & ((\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})^2) \\
&= \eta h\big((1 + \hat{\alpha}_k)^2 + (\hat{\alpha}_{k-1})^2 - 2\hat{\alpha}_{k-1}(1 + \hat{\alpha}_k)\xi_{k-1}^X(2\mathbb{P}^{(n)}(\xi_k^X = 1|\Xi_{k-1}) - 1)\big) \\
&= \eta h(1 + 2\hat{\alpha}_k^{(n)}) \\
&= \eta h\sigma^2(\hat{X}_{k-1}^{(n)}, \hat{Y}_{k-1}^{(n)}),
\end{aligned}
$$

and (6.6) follows. Relation (6.7) is proved similarly.

It remains to establish (6.8). The processes $A^{n,x}$, $A^{n,y}$ are predictable. Thus, from (3.14) it follows that, on $k < \sigma_n$,

$$
\begin{aligned}
\mathbb{E}_{k-1}^{(n)} & ((M_k^{n,x} - M_{k-1}^{n,x})(M_k^{n,y} - M_{k-1}^{n,y})) \\
&= \mathbb{E}_{k-1}^{(n)}((\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})(\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)})) + o(h) \\
&= \mathbb{E}_{k-1}^{(n)}(\hat{X}_k^{(n)} - \hat{X}_{k-1}^{(n)})\mathbb{E}_{k-1}^{(n)}(\hat{Y}_k^{(n)} - \hat{Y}_{k-1}^{(n)}) + o(h) \\
&= o(h),
\end{aligned}
$$

where we used the fact that ξ_k^X and ξ_k^Y are conditionally independent.

7. Numerical results. In this section, we present numerical results from our model for European and American vanilla, lookback, geometric and arithmetic Asian options under the Heston dynamics. Our computations are obtained by a direct implementation of the methodology described in the previous sections. In particular, we explicitly refrained from using known numerical techniques that improve the performance of the trees. This is done to ensure the replicability of our reported results.

7.1. *Vanillas.* In Tables 1, 2 and 3, we use the same parameter sets as in Beliaeva and Nawalkha [2], that is, for European call and put options: strike $K = 100$; initial stock prices: $S_0 = 90, 95, 100, 105, 110$; maturities: $T = 1$ month, 3 months and 6 months; initial volatility values: $\sqrt{v_0} = 0.2, 0.3, 0.4$; interest rate: $r = 0.05$; vol of vol: $\eta = 0.1$; mean reversion rate: $\kappa = 3$; long run vol: $\theta = 0.04$; and correlation: $\rho = -0.7$. For American put options: $K = 100$, $S_0 = 90, 100, 110$; $T = 1$ month, 3 months and 6 months; $\sqrt{v_0} = 0.2, 0.4$; $\rho = -0.1, -0.7$; $r = 0.05$; $\eta = 0.1$; $\kappa = 3$, $\theta = 0.04$.

Tables 1 and 2 show the convergence of European put and call prices computed by our method compared to the closed form solutions of Heston [17]. In the European case, one can calculate errors as Heston's solution is available in closed form. The option prices computed for the number of time steps $N = 200, 350$ and 500 illustrate very good convergence to the closed form solutions as reported in Tables 1 and 2. Furthermore, one can verify that the put-call parity holds exactly for option prices at each of these time steps sizes. Clearly, this is the outcome of the fact that our price process in any step size is a martingale.

Table 3 reports the difference between the American put prices obtained from our method and those obtained by the control variate (CV) technique of [2]. The table shows that our numbers are in good agreement with those obtained by the CV method. The first three largest differences between the models are $(0.27\%, 0.26\%, 0.22\%)$, and on average there is a difference of 0.10% per option. We should point out to the reader that the CV technique computes the value of the put option via the formula

$$\text{CV American Price} = \text{Tree American} + (\text{Closed Form Euro} - \text{Tree Euro}).$$

According to Beliaeva and Nawalkha [2], this method is particularly useful for longer maturity options.

Chockalingam and Muthuraman [6] develop a partial differential equations (PDE) based finite difference method to price American options under stochastic volatility. More specifically, they transform the free boundary problem resulting from the pricing of American options into a sequence of fixed-boundary problems of European type. The prices listed in Tables 4 and 5 are taken from [6] as a benchmark for our tree-based method. The authors provide the values arising from the projected successive over relaxation (PSOR) method and the component-wise splitting (CS) method. They state that other PDE-based methods (see Ikonen and Toivanen [20] for a detailed analysis) fall between these two in terms of speed/accuracy and ease of implementation. As test parameters, they use the most common parameter values for American options under the Heston dynamics in the PDE-based literature: $K = 10$, $r = 0.1$, $\eta = 0.9$, $\kappa = 5.0$, $\theta = 0.16$ and $\rho = 0.1$, $T = 0.25$, $\sqrt{v_0} = 0.25, 0.5$. Following [6], we take the prices computed by Ikonen and Toivanen [20] (using the CS method together with a very fine grid) as the reference values. From Tables 4 and 5, one can clearly conclude that our results for both $N = 250$ and $N = 350$ are very close to reference values.

TABLE 1
Convergence of European put prices versus analytical solution of Heston [17]. *Parameters*:
$K = 100$, $r = 0.05$, $\eta = 0.1$, $\kappa = 3.0$, $\theta = 0.04$ *and* $\rho = -0.7$

$S(0)$	$\sqrt{v_0}$	T	Tree			Analytical solution	Error %		
			$N = 200$	$N = 350$	$N = 500$		$N = 200$	$N = 350$	$N = 500$
90	0.2	0.0833	9.6541	9.6533	9.6533	9.6533	0.01	0.00	0.00
95	0.2	0.0833	5.2059	5.2084	5.2077	5.2074	−0.03	0.02	0.01
100	0.2	0.0833	2.0953	2.0960	2.0965	2.0971	−0.08	−0.05	−0.03
105	0.2	0.0833	0.6082	0.6047	0.6050	0.6053	0.48	−0.10	−0.06
110	0.2	0.0833	0.1267	0.1271	0.1270	0.1265	0.11	0.48	0.35
90	0.3	0.0833	9.9913	9.9900	9.9900	9.9905	0.01	0.00	0.00
95	0.3	0.0833	6.0147	6.0170	6.0162	6.0155	−0.01	0.02	0.01
100	0.3	0.0833	3.1308	3.1288	3.1290	3.1302	0.02	−0.05	−0.04
105	0.3	0.0833	1.4001	1.3955	1.3955	1.3967	0.25	−0.08	−0.09
110	0.3	0.0833	0.5365	0.5374	0.5372	0.5367	−0.05	0.13	0.09
90	0.4	0.0833	10.5687	10.5670	10.5668	10.5668	0.02	0.00	0.00
95	0.4	0.0833	6.9357	6.9363	6.9352	6.9335	0.03	0.04	0.02
100	0.4	0.0833	4.1893	4.1864	4.1861	4.1852	0.10	0.03	0.02
105	0.4	0.0833	2.3280	2.3232	2.3229	2.3222	0.25	0.04	0.03
110	0.4	0.0833	1.1893	1.1897	1.1893	1.1882	0.09	0.13	0.09
90	0.2	0.25	9.5736	9.5693	9.5694	9.5698	0.04	0.00	0.00
95	0.2	0.25	5.9691	5.9685	5.9693	5.9692	0.00	−0.01	0.00
100	0.2	0.25	3.3742	3.3774	3.3794	3.3770	−0.08	0.01	0.07
105	0.2	0.25	1.7420	1.7393	1.7402	1.7410	0.06	−0.10	−0.05
110	0.2	0.25	0.8290	0.8249	0.8253	0.8259	0.37	−0.13	−0.08
90	0.3	0.25	10.5941	10.5879	10.5882	10.5893	0.04	−0.01	−0.01
95	0.3	0.25	7.3343	7.3327	7.3329	7.3316	0.04	0.02	0.02
100	0.3	0.25	4.8279	4.8331	4.8340	4.8310	−0.06	0.04	0.06
105	0.3	0.25	3.0420	3.0379	3.0391	3.0388	0.11	−0.03	0.01
110	0.3	0.25	1.8368	1.8320	1.8319	1.8325	0.23	−0.03	−0.03
90	0.4	0.25	11.8375	11.8281	11.8288	11.8287	0.07	0.00	0.00
95	0.4	0.25	8.8120	8.8081	8.8070	8.8035	0.10	0.05	0.04
100	0.4	0.25	6.3762	6.3790	6.3786	6.3735	0.04	0.09	0.08
105	0.4	0.25	4.5066	4.5005	4.5004	4.4976	0.20	0.06	0.06
110	0.4	0.25	3.1099	3.1035	3.1025	3.1011	0.28	0.08	0.05
90	0.2	0.5	9.7547	9.7545	9.7606	9.7572	−0.03	−0.03	0.04
95	0.2	0.5	6.7258	6.7248	6.7185	6.7199	0.09	0.07	−0.02
100	0.2	0.5	4.4355	4.4369	4.4320	4.4312	0.10	0.13	0.02
105	0.2	0.5	2.8077	2.8159	2.8100	2.8107	−0.11	0.18	−0.02
110	0.2	0.5	1.7286	1.7289	1.7275	1.7240	0.27	0.28	0.20
90	0.3	0.5	11.0786	11.0792	11.0845	11.0807	−0.02	−0.01	0.03
95	0.3	0.5	8.2445	8.2422	8.2367	8.2363	0.10	0.07	0.00
100	0.3	0.5	5.9835	5.9830	5.9784	5.9763	0.12	0.11	0.04
105	0.3	0.5	4.2450	4.2504	4.2449	4.2443	0.02	0.15	0.02
110	0.3	0.5	2.9647	2.9640	2.9623	2.9582	0.22	0.20	0.14
90	0.4	0.5	12.6195	12.6199	12.6231	12.6171	0.02	0.02	0.05
95	0.4	0.5	9.9373	9.9318	9.9260	9.9223	0.15	0.10	0.04
100	0.4	0.5	7.7110	7.7069	7.7017	7.6965	0.19	0.13	0.07
105	0.4	0.5	5.9065	5.9075	5.9015	5.8978	0.15	0.17	0.06
110	0.4	0.5	4.4841	4.4806	4.4779	4.4716	0.28	0.20	0.14

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TABLE 2

Convergence of European call prices versus analytical solution of Heston [17]. *Parameters*:
$K = 100, r = 0.05, \eta = 0.1, \kappa = 3.0, \theta = 0.04,$ *and* $\rho = -0.7$

			Tree			Analytical	Error %		
$S(0)$	$\sqrt{v_0}$	T	$N = 200$	$N = 350$	$N = 500$	solution	$N = 200$	$N = 350$	$N = 500$
90	0.2	0.0833	0.0699	0.0691	0.0691	0.0691	1.13	−0.05	0.02
95	0.2	0.0833	0.6217	0.6242	0.6235	0.6232	−0.23	0.17	0.06
100	0.2	0.0833	2.5111	2.5118	2.5122	2.5129	−0.07	−0.04	−0.02
105	0.2	0.0833	6.0240	6.0205	6.0208	6.0211	0.05	−0.01	−0.01
110	0.2	0.0833	10.5425	10.5429	10.5428	10.5423	0.00	0.01	0.00
90	0.3	0.0833	0.4071	0.4058	0.4058	0.4063	0.20	−0.12	−0.12
95	0.3	0.0833	1.4305	1.4328	1.4320	1.4313	−0.06	0.10	0.05
100	0.3	0.0833	3.5466	3.5446	3.5448	3.5460	0.02	−0.04	−0.04
105	0.3	0.0833	6.8159	6.8113	6.8113	6.8125	0.05	−0.02	−0.02
110	0.3	0.0833	10.9523	10.9532	10.9530	10.9525	0.00	0.01	0.00
90	0.4	0.0833	0.9845	0.9828	0.9826	0.9826	0.19	0.02	0.00
95	0.4	0.0833	2.3515	2.3521	2.3510	2.3493	0.10	0.12	0.07
100	0.4	0.0833	4.6051	4.6022	4.6019	4.6010	0.09	0.03	0.02
105	0.4	0.0833	7.7438	7.7390	7.7387	7.7380	0.08	0.01	0.01
110	0.4	0.0833	11.6051	11.6055	11.6051	11.6040	0.01	0.01	0.01
90	0.2	0.25	0.8158	0.8115	0.8116	0.8120	0.47	−0.05	−0.05
95	0.2	0.25	2.2113	2.2107	2.2116	2.2114	−0.01	−0.03	0.01
100	0.2	0.25	4.6164	4.6196	4.6216	4.6192	−0.06	0.01	0.05
105	0.2	0.25	7.9842	7.9815	7.9824	7.9832	0.01	−0.02	−0.01
110	0.2	0.25	12.0712	12.0671	12.0675	12.0682	0.03	−0.01	−0.01
90	0.3	0.25	1.8363	1.8301	1.8305	1.8316	0.26	−0.08	−0.06
95	0.3	0.25	3.5766	3.5750	3.5751	3.5738	0.08	0.03	0.03
100	0.3	0.25	6.0701	6.0753	6.0762	6.0732	−0.05	0.03	0.05
105	0.3	0.25	9.2842	9.2802	9.2813	9.2810	0.04	−0.01	0.00
110	0.3	0.25	13.0790	13.0742	13.0741	13.0747	0.03	0.00	0.00
90	0.4	0.25	3.0797	3.0703	3.0710	3.0709	0.29	−0.02	0.00
95	0.4	0.25	5.0542	5.0503	5.0493	5.0457	0.17	0.09	0.07
100	0.4	0.25	7.6184	7.6212	7.6208	7.6157	0.04	0.07	0.07
105	0.4	0.25	10.7488	10.7428	10.7426	10.7399	0.08	0.03	0.03
110	0.4	0.25	14.3521	14.3457	14.3447	14.3433	0.06	0.02	0.01
90	0.2	0.5	2.2237	2.2235	2.2296	2.2262	−0.11	−0.12	0.15
95	0.2	0.5	4.1948	4.1938	4.1875	4.1889	0.14	0.12	−0.03
100	0.2	0.5	6.9045	6.9060	6.9010	6.9002	0.06	0.08	0.01
105	0.2	0.5	10.2767	10.2849	10.2790	10.2797	−0.03	0.05	−0.01
110	0.2	0.5	14.1976	14.1979	14.1965	14.1930	0.03	0.03	0.02
90	0.3	0.5	3.5476	3.5483	3.5535	3.5497	−0.06	−0.04	0.11
95	0.3	0.5	5.7135	5.7112	5.7057	5.7053	0.14	0.10	0.01
100	0.3	0.5	8.4525	8.4520	8.4474	8.4453	0.09	0.08	0.03
105	0.3	0.5	11.7140	11.7194	11.7140	11.7133	0.01	0.05	0.01
110	0.3	0.5	15.4337	15.4330	15.4313	15.4272	0.04	0.04	0.03
90	0.4	0.5	5.0885	5.0889	5.0921	5.0861	0.05	0.06	0.12
95	0.4	0.5	7.4063	7.4008	7.3950	7.3913	0.20	0.13	0.05
100	0.4	0.5	10.1800	10.1759	10.1707	10.1655	0.14	0.10	0.05
105	0.4	0.5	13.3755	13.3765	13.3705	13.3668	0.07	0.07	0.03
110	0.4	0.5	16.9532	16.9496	16.9469	16.9406	0.07	0.05	0.04

TABLE 3
Comparison of American put prices calculated with our method and with the control variate technique of Beliaeva and Nawalkha [2]. Parameters: $K = 100$, $r = 0.05$, $\eta = 0.1$, $\kappa = 3.0$, $\theta = 0.04$, and $\rho = -0.7$

$S(0)$	ρ	$\sqrt{v_0}$	T	Tree $N = 250$	Control variate $N = 200$	Difference %
90	−0.1	0.2	0.0833	10.0000	10.0000	0.00
100	−0.1	0.2	0.0833	2.1236	2.1254	−0.08
110	−0.1	0.2	0.0833	0.1090	0.1091	−0.05
90	−0.7	0.2	0.0833	10.0000	9.9997	0.00
100	−0.7	0.2	0.0833	2.1249	2.1267	−0.08
110	−0.7	0.2	0.0833	0.1273	0.1274	−0.07
90	−0.1	0.4	0.0833	10.7123	10.7100	0.02
100	−0.1	0.4	0.0833	4.2194	4.2158	0.08
110	−0.1	0.4	0.0833	1.1666	1.1667	−0.01
90	−0.7	0.4	0.0833	10.6843	10.6804	0.04
100	−0.7	0.4	0.0833	4.2183	4.2140	0.10
110	−0.7	0.4	0.0833	1.1942	1.1939	0.02
90	−0.1	0.2	0.25	10.1713	10.1706	0.01
100	−0.1	0.2	0.25	3.4729	3.4747	−0.05
110	−0.1	0.2	0.25	0.7726	0.7736	−0.13
90	−0.7	0.2	0.25	10.1222	10.1206	0.02
100	−0.7	0.2	0.25	3.4790	3.4807	−0.05
110	−0.7	0.2	0.25	0.8405	0.8416	−0.13
90	−0.1	0.4	0.25	12.1880	12.1819	0.05
100	−0.1	0.4	0.25	6.5023	6.4964	0.09
110	−0.1	0.4	0.25	3.0952	3.0914	0.12
90	−0.7	0.4	0.25	12.1245	12.1122	0.10
100	−0.7	0.4	0.25	6.4989	6.4899	0.14
110	−0.7	0.4	0.25	3.1512	3.1456	0.18
90	−0.1	0.2	0.5	10.6521	10.6478	0.04
100	−0.1	0.2	0.5	4.6531	4.6473	0.12
110	−0.1	0.2	0.5	1.6857	1.6832	0.15
90	−0.7	0.2	0.5	10.5682	10.5637	0.04
100	−0.7	0.2	0.5	4.6691	4.6636	0.12
110	−0.7	0.2	0.5	1.7899	1.7874	0.14
90	−0.1	0.4	0.5	13.3279	13.3142	0.10
100	−0.1	0.4	0.5	8.0231	8.0083	0.18
110	−0.1	0.4	0.5	4.5554	4.5454	0.22
90	−0.7	0.4	0.5	13.2431	13.2172	0.20
100	−0.7	0.4	0.5	8.0204	7.9998	0.26
110	−0.7	0.4	0.5	4.6328	4.6201	0.27

7.2. *Exotics.* Our numerical experimentation confirms that backward recursion yields quite fast and accurate results for the two-dimensional problems like European and American vanilla option pricing problems. However, our numerical

TABLE 4

American put prices determined with our tree approach and finite difference methods. Parameters:
$K = 10, r = 0.1, \eta = 0.9, \kappa = 5.0, \theta = 0.16,$ *and* $\rho = 0.1, T = 0.25, \sqrt{v_0} = \mathbf{0.25}$

Method	Grid size	S_0				
		8	9	10	11	12
PSOR	(40, 16, 8)	2.0000	1.0952	0.4966	0.2042	0.0838
	(60, 32, 66)	2.0000	1.1037	0.5142	0.2105	0.0815
	(120, 64, 130)	2.0000	1.1064	0.5182	0.2126	0.0819
	(240, 128, 258)	2.0000	1.1071	0.5193	0.2133	0.0820
Componentwise	(40, 16, 8)	2.0004	1.1003	0.4991	0.2035	0.0828
splitting	(60, 32, 66)	2.0000	1.1043	0.5147	0.2104	0.0813
	(120, 64, 130)	2.0000	1.1066	0.5183	0.2126	0.0819
	(240, 128, 258)	2.0000	1.1073	0.5194	0.2133	0.0820
Transformation	(40, 16, 8)	2.0000	1.0952	0.4966	0.2042	0.0838
procedure	(60, 32, 66)	2.0000	1.1035	0.5142	0.2105	0.0815
	(120, 64, 130)	2.0000	1.1063	0.5181	0.2126	0.0819
	(240, 128, 258)	2.0000	1.1071	0.5193	0.2133	0.0820
Our tree method	N					
	150	2.0000	1.1086	0.5155	0.2140	0.0825
	250	2.0000	1.1079	0.5190	0.2140	0.0822
	350	2.0000	1.1074	0.5193	0.2134	0.0828
Reference value		2.0000	1.1076	0.5200	0.2137	0.0820

experimentation also reveals that the straightforward application of the recursive method takes too long on a personal computer when another continuous variable is introduced to price an exotic option. Hence, in order to substantially speed up the computations, we use our discrete equations as a discretization scheme for our Monte Carlo (MC) simulation. In other words, we carry out the MC simulation on the tree.

It is also important to note that our main concern in this section is to show the pure application of our computation method. There are many well-known techniques in the literature which improve the speed and the accuracy of tree and MC methods. However, as in the backward recursion we refrain from using any of these techniques.

Below we outline results for the geometric, arithmetic Asian and for lookback options.

We start with the geometric Asian and let

$$G_T = \exp\left(\frac{1}{T} \int_0^T \ln(S_t)\, dt\right)$$

TABLE 5
American put prices determined with our tree approach and finite difference methods. Parameters:
$K = 10$, $r = 0.1$, $\eta = 0.9$, $\kappa = 5.0$, $\theta = 0.16$, and $\rho = 0.1$, $T = 0.25$, $\sqrt{v_0} = \mathbf{0.5}$

Method	Grid size	S_0				
		8	9	10	11	12
PSOR	(40, 16, 8)	2.0691	1.3139	0.7720	0.4293	0.2324
	(60, 32, 66)	2.0760	1.3292	0.7908	0.4442	0.2405
	(120, 64, 130)	2.0775	1.3320	0.7940	0.4467	0.2419
	(240, 128, 258)	2.0779	1.3329	0.7951	0.4476	0.2424
Componentwise	(40, 16, 8)	2.0676	1.3094	0.7646	0.4232	0.2297
splitting	(60, 32, 66)	2.0758	1.3287	0.7900	0.4435	0.2401
	(120, 64, 130)	2.0774	1.3317	0.7936	0.4463	0.2417
	(240, 128, 258)	2.0780	1.3328	0.7949	0.4474	0.2423
Transformation	(40, 16, 8)	2.0691	1.3140	0.7721	0.4294	0.2325
procedure	(60, 32, 66)	2.0760	1.3291	0.7908	0.4442	0.2405
	(120, 64, 130)	2.0775	1.3319	0.7940	0.4467	0.2419
	(240, 128, 258)	2.0780	1.3329	0.7951	0.4476	0.2424
Our tree method	N					
	150	2.0791	1.3362	0.7957	0.4495	0.2435
	250	2.0786	1.3338	0.7964	0.4501	0.2435
	350	2.0790	1.3339	0.7964	0.4485	0.2440
Reference value		2.0784	1.3336	0.7960	0.4483	0.2428

be the geometric mean of S_t over time t during $[0, T]$. Then the payoff of a fixed strike geometric Asian call is given by $\max(G_T - K, 0)$. Kim and Wee [21] provide semi-closed solutions for the price of geometric Asian options under the Heston model. We compare our results with theirs.

Table 6 displays a comparison between prices from the semi-closed solution and those from our MC simulation on tree with $N = 300$ and number of simulations (NumSim) $= 10^5, 5 * 10^5, 10^6$. As benchmark prices, we use the values given in Table 5 from [21] for the parameter values: $S_0 = 100$, $v_0 = 0.09$, $r = 0.05$, $\kappa = 1.15$, $\theta = 0.348$, $\rho = -0.64$, $\eta = 0.39$. As it is clear from the table, our numerical scheme provides a very good approximation for the analytical prices. For NumSim $= 10^6$, we get the three largest percentage errors as $(0.40\%, 0.34\%, 0.28\%)$ and the average percentage error is 0.11%. Table 7 shows the 95% confidence intervals for the prices computed for different numbers of simulations.

Table 8 includes our results for arithmetic Asian options under the Heston model. We carry out the simulations as in the same way described previously. Let

$$A_T = \exp\left(\frac{1}{T} \int_0^T S_t \, dt\right)$$

TABLE 6

Comparison of our method and the semi-closed solution for fixed-strike geometric Asian call options for: $S_0 = 100$, $v_0 = 0.09$, $r = 0.05$, $\kappa = 1.15$, $\theta = 0.348$, $\rho = -0.64$, $\eta = 0.39$

		MC on tree with $N = 300$				Difference %		
		NumSim				NumSim		
T	K	10^5	$5 * 10^5$	10^6	Semi-closed solution	10^5	$5 * 10^5$	10^6
0.2	90	10.6598	10.6551	10.6562	10.6571	0.02	−0.02	−0.01
0.2	95	6.6006	6.5970	6.5888	6.5871	0.20	0.15	0.03
0.2	100	3.4699	3.4564	3.4510	3.4478	0.64	0.25	0.09
0.2	105	1.4697	1.4610	1.4611	1.4552	1.00	0.40	0.40
0.2	110	0.4730	0.4742	0.4719	0.4724	0.14	0.38	−0.10
0.4	90	11.7310	11.7111	11.7077	11.7112	0.17	0.00	−0.03
0.4	95	8.0988	8.1067	8.0877	8.0894	0.12	0.21	−0.02
0.4	100	5.1480	5.1746	5.1641	5.1616	−0.26	0.25	0.05
0.4	105	3.0414	3.0060	3.0040	3.0018	1.32	0.14	0.07
0.4	110	1.5555	1.5776	1.5679	1.5715	−1.02	0.39	−0.23
0.5	90	12.2974	12.2495	12.2330	12.2329	0.53	0.14	0.00
0.5	95	8.7711	8.7668	8.7753	8.7553	0.18	0.13	0.23
0.5	100	5.9036	5.9151	5.9008	5.8971	0.11	0.31	0.06
0.5	105	3.7150	3.7120	3.7165	3.7072	0.21	0.13	0.25
0.5	110	2.1622	2.1692	2.1595	2.1589	0.15	0.48	0.03
1	90	14.5646	14.6087	14.5937	14.5779	−0.09	0.21	0.11
1	95	11.6287	11.5518	11.5474	11.5551	0.64	−0.03	−0.07
1	100	8.9708	8.9378	8.9530	8.9457	0.28	−0.09	0.08
1	105	6.8003	6.7392	6.7505	6.7559	0.66	−0.25	−0.08
1	110	5.0161	4.9878	4.9704	4.9722	0.88	0.31	−0.04
1.5	90	16.3889	16.4588	16.5200	16.5030	−0.69	−0.27	0.10
1.5	95	13.7324	13.7764	13.7690	13.7625	−0.22	0.10	0.05
1.5	100	11.3599	11.3247	11.3304	11.3374	0.20	−0.11	−0.06
1.5	105	9.2487	9.2187	9.2076	9.2245	0.26	−0.06	−0.18
1.5	110	7.4342	7.3959	7.4019	7.4122	0.30	−0.22	−0.14
2	90	18.0757	18.1112	18.0816	18.0914	−0.09	0.11	−0.05
2	95	15.6133	15.6021	15.5211	15.5640	0.32	0.24	−0.28
2	100	13.3624	13.3245	13.2833	13.2933	0.52	0.24	−0.08
2	105	11.2855	11.2862	11.2627	11.2728	0.11	0.12	−0.09
2	110	9.4243	9.4840	9.4901	9.4921	−0.71	−0.09	−0.02
3	90	20.6523	20.4276	20.5149	20.5102	0.69	−0.40	0.02
3	95	18.3985	18.2361	18.2884	18.3060	0.51	−0.38	−0.10
3	100	16.2151	16.2555	16.2609	16.2895	−0.46	−0.21	−0.18
3	105	14.5000	14.4330	14.4046	14.4531	0.32	−0.14	−0.34
3	110	12.6065	12.8177	12.7982	12.7882	−1.42	0.23	0.08

be the arithmetic average of S_t over time t during $[0, T]$. Then the payoff of a fixed strike arithmetic Asian call is given by $\max(A_T - K, 0)$. Pages and Printems [25] use the functional quantization based quadrature formula to price vanilla calls

TABLE 7
Confidence intervals for fixed-strike geometric Asian call options for: $S_0 = 100$, $v_0 = 0.09$, $r = 0.05$, $\kappa = 1.15$, $\theta = 0.348$, $\rho = -0.64$, $\eta = 0.39$

Confidence intervals 95%		
NumSim = 10^5	NumSim = $5 * 10^5$	NumSim = 10^6
(10.6135, 10.7060)	(10.6345, 10.6758)	(10.6416, 10.6708)
(6.5609, 6.6402)	(6.5793, 6.6147)	(6.5763, 6.6014)
(3.4397, 3.5001)	(3.4429, 3.4699)	(3.4415, 3.4605)
(1.4501, 1.4894)	(1.4522, 1.4698)	(1.4548, 1.4673)
(0.4623, 0.4838)	(0.4694, 0.4790)	(0.4685, 0.4753)
(11.6678, 11.7941)	(11.6829, 11.7394)	(11.6877, 11.7277)
(8.0438, 8.1538)	(8.0820, 8.1313)	(8.0703, 8.1051)
(5.1027, 5.1932)	(5.1543, 5.1948)	(5.1498, 5.1784)
(3.0065, 3.0764)	(2.9904, 3.0216)	(2.9930, 3.0150)
(1.5308, 1.5803)	(1.5665, 1.5887)	(1.5601, 1.5758)
(12.2270, 12.3679)	(12.2181, 12.2808)	(12.2108, 12.2552)
(8.7094, 8.8328)	(8.7391, 8.7944)	(8.7557, 8.7949)
(5.8516, 5.9556)	(5.8919, 5.9384)	(5.8843, 5.9172)
(3.6735, 3.7566)	(3.6934, 3.7306)	(3.7034, 3.7297)
(2.1305, 2.1938)	(2.1551, 2.1833)	(2.1495, 2.1694)
(14.4642, 14.6650)	(14.5638, 14.6536)	(14.5619, 14.6255)
(11.5367, 11.7208)	(11.5109, 11.5927)	(11.5186, 11.5763)
(8.8888, 9.0528)	(8.9013, 8.9744)	(8.9272, 8.9789)
(6.7282, 6.8724)	(6.7072, 6.7713)	(6.7278, 6.7732)
(4.9538, 5.0784)	(4.9601, 5.0154)	(4.9508, 4.9899)
(16.2635, 16.5144)	(16.4023, 16.5152)	(16.4800, 16.5599)
(13.6150, 13.8498)	(13.7239, 13.8289)	(13.7319, 13.8061)
(11.2523, 11.4676)	(11.2765, 11.3729)	(11.2963, 11.3645)
(9.1503, 9.3471)	(9.1749, 9.2626)	(9.1766, 9.2387)
(7.3457, 7.5226)	(7.3563, 7.4355)	(7.3739, 7.4299)
(17.9261, 18.2253)	(18.0442, 18.1782)	(18.0342, 18.1289)
(15.4721, 15.7544)	(15.5392, 15.6651)	(15.4767, 15.5654)
(13.2303, 13.4945)	(13.2656, 13.3835)	(13.2416, 13.3249)
(11.1626, 11.4084)	(11.2315, 11.3409)	(11.2240, 11.3013)
(9.3113, 9.5373)	(9.4334, 9.5345)	(9.4544, 9.5258)
(20.4610, 20.8436)	(20.3429, 20.5123)	(20.4547, 20.5750)
(18.2156, 18.5814)	(18.1549, 18.3174)	(18.2310, 18.3459)
(16.0417, 16.3885)	(16.1781, 16.3329)	(16.2062, 16.3157)
(14.3343, 14.6656)	(14.3594, 14.5067)	(14.3525, 14.4566)
(12.4514, 12.7617)	(12.7476, 12.8878)	(12.7489, 12.8476)

and Asian calls in the Heston model. The numbers computed from MC method, Romberg log-extrapolation and K-interpolation of Romberg and their standard deviations in the parenthesis are tabulated for comparison; see Table 4 in [25] for a more detailed explanation of the results. We test our model using the numbers reported in their paper. As one can observe from Table 8, our prices together with the

TABLE 8

Comparison of our method and the functional quantization method by Pages and Printems [25] for arithmetic Asian options. Parameters: $S_0 = 50$, $v_0 = 0.01$, $r = 0.05$, $\kappa = 2$, $\theta = 0.01$, $\rho = 0.5$, $\eta = 0.1$

								Our method	
								$N = 300$, NumSim $= 10^6$	
K	10^8-MC	**Crude MC reference**		**Romberg on crude FQ**		**K-interpol. of Romberg**		**Price**	**Conf. int.**
44	6.92	(0.08%)	6.92	(0.01%)	6.92	(0.01%)	6.9196	(6.9139, 6.9252)
45	5.97	(0.10%)	5.97	(0.04%)	5.97	(0.02%)	5.9768	(5.9712, 5.9825)
46	5.03	(0.11%)	5.03	(0.05%)	5.03	(0.02%)	5.0334	(5.0278, 5.0390)
47	4.11	(0.14%)	4.12	(0.09%)	4.11	(0.04%)	4.1117	(4.1062, 4.1172)
48	3.245	(0.16%)	3.25	(0.17%)	3.24	(0.05%)	3.2506	(3.2453, 3.2559)
49	2.46	(0.20%)	2.47	(0.32%)	2.46	(0.04%)	2.4673	(2.4624, 2.4723)
50	1.79	(0.26%)	1.80	(0.63%)	1.79	(0.03%)	1.7926	(1.7882, 1.7970)
51	1.25	(0.31%)	1.26	(1.16%)	1.25	(0.17%)	1.2541	(1.2503, 1.2580)
52	0.84	(0.39%)	0.85	(2.06%)	0.84	(0.37%)	0.8430	(0.8398, 0.8463)
53	0.54	(0.50%)	0.56	(3.73%)	0.545	(0.78%)	0.5502	(0.5475, 0.5529)
54	0.34	(0.63%)	0.36	(6.58%)	0.34	(1.37%)	0.3485	(0.3464, 0.3506)
55	0.21	(0.81%)	0.23	(11.53%)	0.21	(2.15%)	0.2159	(0.2142, 0.2176)
56	0.125	(1.04%)	0.15	(19.96%)	0.125	(2.84%)	0.1317	(0.1303, 0.1330)

confidence intervals are in accordance with the only reference values for arithmetic Asian options under the Heston dynamics which can be found in the literature.

It is clear that when we price a lookback option using backward recursion, we also need another continuous variable holding the running max or min. But in this case, we can constrain this variable to take values on a tree as well. However, it still remains more efficient to apply our MC method on the tree. Table 9 presents numerical results obtained by the standard MC method and our numerical method for fixed strike lookback call options. As comparison we used simple Monte Carlo simulations based on a Euler method. The table contains prices for $N = 3000$ and NumSim $= 10^5$. As one can see from the last column, the numbers obtained from our numerical method differ only slightly from the prices computed by the Euler MC method.

In terms of the theoretical complexity, we require n^3 many computations for n many time steps in the difference equations case. This is similar to that of PDE approach. More precisely, Table 10 provides average running times for the options in Tables 1–9. The computer used is a standard laptop with an Intel Core i7 M620@2.67 GHz CPU and a 4 GB memory. The algorithm was implemented in MATLAB.

8. Concluding remarks. In this paper, we have developed a recombining tree approximation of the Heston model. Our approach is very general and applies to

TABLE 9

Comparison of our method and Euler simulation for lookback call option with fixed strike.
Parameters: $S_0 = 100$, $v_0 = 0.16$, $r = 0.05$, $\kappa = 3$, $\theta = 0.04$, $\rho = -0.7$, $\eta = 0.1$

| | | Euler simulation | | Our method | | |
| | | $N = 3000$, NumSim $= 10^5$ | | $N = 3000$, NumSim $= 10^5$ | | |
T	*K*	Price	Confidence interval	Price	Confidence interval	Difference %
0.2	90	23.4527	(23.3844, 23.5210)	23.4679	(23.3996, 23.5362)	0.06
0.2	95	18.5511	(18.4827, 18.6196)	18.5459	(18.4776, 18.6142)	0.03
0.2	100	13.5145	(13.4464, 13.5825)	13.6562	(13.5878, 13.7246)	1.05
0.2	105	9.2629	(9.1987, 9.3272)	9.2620	(9.1978, 9.3262)	0.01
0.2	110	6.0746	(6.0185, 6.1306)	6.0899	(6.0340, 6.1457)	0.25
0.4	90	27.7252	(27.6333, 27.8172)	27.7378	(27.6461, 27.8296)	0.05
0.4	95	22.7931	(22.7015, 22.8846)	22.7784	(22.6869, 22.8698)	0.06
0.4	100	17.8937	(17.8017, 17.9857)	17.9052	(17.8136, 17.9969)	0.06
0.4	105	13.5301	(13.4415, 13.6187)	13.6541	(13.5649, 13.7434)	0.92
0.4	110	10.0038	(9.9224, 10.0852)	10.0978	(10.0160, 10.1796)	0.94
0.5	90	29.1737	(29.0738, 29.2735)	29.2407	(29.1405, 29.3409)	0.23
0.5	95	24.2728	(24.1733, 24.3722)	24.3094	(24.2095, 24.4093)	0.15
0.5	100	19.4547	(19.3542, 19.5552)	19.5036	(19.4033, 19.6038)	0.25
0.5	105	15.1074	(15.0099, 15.2049)	15.0772	(14.9801, 15.1742)	0.20
0.5	110	11.4637	(11.3730, 11.5544)	11.4401	(11.3498, 11.5305)	0.21
1	90	34.1211	(33.9910, 34.2511)	34.1944	(34.0646, 34.3242)	0.21
1	95	29.4579	(29.3273, 29.5886)	29.4015	(29.2720, 29.5311)	0.19
1	100	24.6878	(24.5573, 24.8184)	24.7163	(24.5855, 24.8470)	0.12
1	105	20.1960	(20.0686, 20.3234)	20.3721	(20.2443, 20.4999)	0.87
1	110	16.5429	(16.4206, 16.6652)	16.4579	(16.3367, 16.5791)	0.51
1.5	90	37.6113	(37.4587, 37.7640)	37.8563	(37.7035, 38.0091)	0.65
1.5	95	33.2861	(33.1314, 33.4408)	33.0959	(32.9428, 33.2491)	0.57
1.5	100	28.5915	(28.4380, 28.7451)	28.3913	(28.2386, 28.5440)	0.70
1.5	105	24.2427	(24.0913, 24.3941)	24.1616	(24.0107, 24.3124)	0.33
1.5	110	20.4593	(20.3131, 20.6054)	20.4385	(20.2919, 20.5850)	0.10
2	90	41.0722	(40.8963, 41.2481)	41.0605	(40.8861, 41.2350)	0.03
2	95	36.6204	(36.4454, 36.7953)	36.5932	(36.4185, 36.7680)	0.07
2	100	31.9362	(31.7612, 32.1112)	32.0618	(31.8874, 32.2361)	0.39
2	105	27.8954	(27.7220, 28.0688)	27.7302	(27.5578, 27.9026)	0.59
2	110	24.0406	(23.8719, 24.2093)	23.8907	(23.7223, 24.0591)	0.62
3	90	47.0043	(46.7881, 47.2205)	47.0854	(46.8698, 47.3010)	0.17
3	95	42.6606	(42.4453, 42.8759)	42.5750	(42.3599, 42.7901)	0.20
3	100	38.6746	(38.4588, 38.8903)	38.3630	(38.1469, 38.5790)	0.81
3	105	34.5038	(34.2898, 34.7177)	34.2793	(34.0657, 34.4929)	0.65
3	110	30.7339	(30.5229, 30.9449)	30.4407	(30.2312, 30.6502)	0.95

all stochastic volatility models with a factor equation. Low-dimensional European and American option equations can be solved by a straightforward backward recursion. We have done extensive numerical experimentation with the resulting pricing

TABLE 10
Average running times for options in Tables 1–9

		Time in seconds
European put and call	$N = 200$	5.71
(Tables 1 and 2)	$N = 350$	30.37
	$N = 500$	89.27
American put	$N = 250$	13.97
(Table 3)		
American put	$N = 150$	3.15
(Tables 4 and 5)	$N = 250$	14.66
	$N = 350$	40.50
Geometric Asian	NumSim $= 10^5$	8.17
(Tables 6 and 7)	NumSim $= 5 * 10^5$	40.91
	NumSim $= 10^6$	81.79
Arithmetic Asian	$N = 300$	98.65
(Table 8)	NumSim $= 10^6$	
Lookback	$N = 3000$	94.54
(Table 9)	NumSim $= 10^5$	

equations. These results, reported in the previous section, confirm the efficiency of the method.

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