A Set Optimization Technique for Domain Reconstruction from Single-Measurement Electrical Impedance Tomography Data

Bastian Harrach and Janosch Rieger

Abstract We propose and test a numerical method for the computation of the convex source support from single-measurement electrical impedance tomography data. Our technique is based on the observation that the convex source support is the unique minimum of an optimization problem in the space of all convex and compact subsets of the imaged body.

1 Introduction

Electrical impedance tomography is a modern non-invasive imaging technology with the potential to complement computerized tomography in treatments like pulmonary function diagnostics and breast cancer screening. From a mathematical perspective, the reconstruction of the exact conductivity within the imaged body from electrical impedance tomography data amounts to solving a strongly ill-posed inverse problem.

The difficulty of this problem can partly be avoided by noting that one is usually not interested in the conductivity as such, but only in the domain where it differs from the conductivity of healthy tissue. A technique introduced in [5] for scattering problems and adapted to electrical impedance tomography later in [3] takes this approach one step further by considering a convex set, called the convex source support, which contains information on the desired domain, but can be computed from a single measurement.

B. Harrach
Goethe University Frankfurt, Institute for Mathematics, Frankfurt am Main, Germany
e-mail: harrach@math.uni-frankfurt.de

J. Rieger (✉)
Monash University, School of Mathematical Sciences, Clayton, VIC, Australia
e-mail: janosch.rieger@monash.edu

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We propose a numerical method for the computation of the convex source support from electrical impedance tomography data. It is based on the observation that this particular set is the unique minimum of an optimization problem in the space of all convex and compact subsets of the imaged body. In Sect. 2, we recall the notion of the convex source support, and in Sect. 3, we formulate the above-mentioned optimization problem and manipulate its constraint into a convenient form. In Sect. 4, we introduce Galerkin approximations, which are spaces of polytopes with fixed outer normals, to the space of all convex and compact subsets of a given Euclidean vector space. These spaces possess a sparse and unique representation in terms of coordinates. In Sect. 5, we discuss how the derivatives of the objective function and the constraint of our optimization problem can be computed efficiently. In Sect. 6, we gather all the above ingredients and solve the optimization problem numerically using a standard interior point method on the Galerkin approximation, which yields a numerical approximation of the convex source support.

This paper is a report on work in progress, which aims to present ideas rather than a complete solution of the problem. In particular, we assume that we can measure the potential on the entire boundary of the imaged body, which is not possible in real-world applications, and we neither include an error analysis nor stability results for the proposed algorithm.

2 The Convex Source Support in Electrical Impedance Tomography

We consider the following idealistic model of the EIT problem. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a smoothly bounded domain describing the imaged body, let $\sigma \in L^\infty(\Omega)$ be the conductivity within Ω, and let $g \in L^2_\diamond(\partial\Omega)$ be the electric current applied to $\partial\Omega$, where $L^2_\diamond(\partial\Omega)$ denotes the subspace of $L^2(\partial\Omega)$ with vanishing integral mean on $\partial\Omega$. Then the electrical potential $u \in H^1_\diamond(\Omega)$ solves

$$\begin{aligned} \nabla \cdot (\sigma(x)\nabla u(x)) &= 0, \quad x \in \Omega, \\ \sigma\, \partial_\nu u|_{\partial\Omega}(x) &= g(x), \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where ν is the outer normal on $\partial\Omega$, and $H^1_\diamond(\partial\Omega)$ is the subspace of $H^1(\Omega)$-functions with vanishing integral mean on $\partial\Omega$.

Our aim is to find inclusions or anomalies in Ω where the conductivity σ differs from a reference conductivity value σ_0 (e.g. that of healthy tissue) from measuring the electric potential $u|_{\partial\Omega}$ on $\partial\Omega$. To simplify our exposition we assume $\sigma_0 \equiv 1$ throughout this work. More precisely, we aim to find information on $\operatorname{supp}(\sigma - \sigma_0)$ from the data $(u - u_0)_{|\partial\Omega}$, where u_0 solves (1) with the same Neumann boundary data g, and σ replaced by σ_0. This is usually referred to as the problem of single measurement EIT since only one current g is applied to the patient.

Now we introduce the convex source support, following [3] and [5]. First note that since u and u_0 are solutions of (1) with conductivities σ and σ_0 and identical Neumann data $g \in L^2_\diamond(\partial\Omega)$, their difference $w := u - u_0$ solves the equation

$$\Delta w = \operatorname{div}((1-\sigma)\nabla u) \text{ in } \Omega, \quad \partial_\nu w = 0 \text{ on } \partial\Omega, \tag{2}$$

with a source term satisfying

$$\operatorname{supp}((1-\sigma)\nabla u) \subset \operatorname{supp}(1-\sigma). \tag{3}$$

This motivates the following construction of the convex source support. Let us define the virtual measurement operator

$$L : L^2(\Omega)^d \to L^2_\diamond(\partial\Omega), \quad F \mapsto w_{|\partial\Omega},$$

where $w \in H^1_\diamond(\Omega)$ solves

$$\Delta w = \operatorname{div} F \text{ in } \Omega, \quad \partial_\nu w = 0 \text{ on } \partial\Omega.$$

Given a measurement $f = (u - u_0)_{|\partial\Omega} \in L^2_\diamond(\partial\Omega)$, the convex source support of problem (1) is defined by

$$\mathscr{C} f := \bigcap_{LF=f} \operatorname{co}(\operatorname{supp}(F)),$$

which is the intersection of the convex hulls of all supports of sources that could possibly generate the measurement f. By Eqs. (2) and (3),

$$\mathscr{C}(u-u_0)_{|\partial\Omega} \subset \operatorname{co}(\operatorname{supp}((1-\sigma)\nabla u)) \subset \operatorname{co}(\operatorname{supp}(\sigma - \sigma_0)),$$

which means that the convex source support provides coarse, but reliable information about the position of the set $\operatorname{supp}(\sigma - \sigma_0)$. In fact, much more is known. The following theorem, e.g., can be found in [3].

Theorem 1 *We have $\mathscr{C} f = \emptyset$ if and only if $f = 0$, and for every $\epsilon > 0$, there exists $F_\epsilon \in L^2(\Omega)^d$ such that $LF_\epsilon = f$ and dist$(co(\operatorname{supp}(F_\epsilon)), \mathscr{C} f) < \epsilon$.*

3 An Optimization Problem in $\mathscr{K}_c(\mathbb{R}^d)$

For given data $f \in L^2_\diamond(\partial\Omega)$, we recast the computation of the convex source support as a minimization problem

$$\operatorname{vol}(D) = \min! \quad \text{subject to} \quad D \in \mathscr{K}_c(\mathbb{R}^d),\ \mathscr{C} f \subset D \subset \Omega \tag{4}$$

in the space $\mathscr{K}_c(\mathbb{R}^d)$ of all nonempty convex and compact subsets of $\mathbb{R}^d$, which obviously has the unique solution $D^* = \mathscr{C} f$. To solve the problem (4), we mainly need a handy criterion to check whether $\mathscr{C} f \subset D$.

By Theorem 1, we have $\mathscr{C} f \subset \operatorname{int} D$ if and only if there exists $F \in L^2(\Omega)^d$ with $\operatorname{supp}(F) \subset D$ and $LF = f$. In other words, we have to check whether $f \in \mathscr{R}(L_D)$, i.e. whether f is in the range of the operator L_D, where

$$L_D : L^2(D)^d \to L^2_\diamond(\partial\Omega), \quad F \mapsto w_{|\partial\Omega},$$

and $w \in H^1_\diamond(\Omega)$ solves

$$\Delta w = \operatorname{div} F \text{ in } \Omega, \quad \partial_\nu w = 0 \text{ on } \partial\Omega.$$

Proposition 1 *If the interior of $D \subseteq \Omega$ is not empty, then L_D is a compact linear operator with dense range, and*

$$f \notin \mathscr{R}(L_D) \quad \text{if and only if} \quad \lim_{\alpha\to 0} \|R^D_\alpha f\| = \infty.$$

where $R^D_\alpha f := (L^*_D L_D + \alpha I)^{-1} L^*_D f$.

Proof L_D is the concatenation of the linear bounded solution operator and the linear compact trace operator from $H^1_\diamond(\Omega)$ to $L^2_\diamond(\partial\Omega)$ and thus linear and compact. The adjoint of L_D is given by (see [4, Lemma 2])

$$L^*_D : L^2_\diamond(\partial\Omega) \to L^2(D)^n, \quad \varphi \mapsto \nabla v_0|_D,$$

where $v_0 \in H^1_\diamond(D)$ solves

$$\Delta v_0 = 0 \text{ in } \Omega, \quad \partial_\nu v_0|_{\partial\Omega} = \varphi \text{ on } \partial\Omega.$$

By unique continuation, L^*_D is injective and thus L_D has dense range. This also implies that the domain of definition of the Moore-Penrose inverse L^+_D (cf. [2, Def. 2.2]) is given by

$$\mathscr{D}(L^+_D) = \mathscr{R}(L_D) + \mathscr{R}(L_D)^\perp = \mathscr{R}(L_D).$$

Since R^D_α is a linear regularization (the Tikhonov regularization, cf. [2, Section 5]), and a simple computation shows that

$$\sup_{\alpha>0} \|L_D R^D_\alpha\| \le 1,$$

it follows from standard regularization theory (cf., e.g., [2, Prop. 3.6]) that

$$\lim_{\alpha\to 0} R^D_\alpha f = L^+_D f \quad \text{if } f \in \mathscr{D}(L^+_D) = \mathscr{R}(L_D),$$

and that

$$\lim_{\alpha\to 0} \|R_\alpha^D f\| = \infty \quad \text{if } f \notin \mathscr{D}(L_D^+) = \mathscr{R}(L_D).$$

This proves the assertion.

To implement Proposition 1, we therefore have to check whether the quantity

$$\begin{aligned}\|R_\alpha^D f\|^2 &= \|(L_D^* L_D + \alpha I)^{-1} L_D^* f\|^2 = \|L_D^*(L_D L_D^* + \alpha I)^{-1} f\|^2 \\ &= ((L_D L_D^* + \alpha I)^{-1} L_D L_D^* (L_D L_D^* + \alpha I)^{-1} f, f)\end{aligned}$$

remains bounded as $\alpha \to 0$. Writing $M_D := L_D L_D^* : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega)$, we obtain the convenient representation

$$\|R_\alpha^D f\|^2 = ((M_D + \alpha I)^{-1} M_D (M_D + \alpha I)^{-1} f, f). \tag{5}$$

Fix an orthonormal basis $(\varphi_j)_j$ of $L^2_\diamond(\partial\Omega)$. The characterization of L_D^* in [4, Lemma 2] shows that

$$(M_D \varphi_j, \varphi_k) = (L_D^* \varphi_j, L_D^* \varphi_k) = \int_D \nabla u_0^j \cdot \nabla u_0^k \, dx, \tag{6}$$

where u_0^j solves

$$\Delta u_0^j = 0 \text{ in } \Omega, \quad \partial_\nu u_0^j = \varphi_j \text{ on } \partial\Omega. \tag{7}$$

Note that the integrands $\nabla u_0^j \cdot \nabla u_0^k$ do not depend on D and hence can be precomputed. Since

$$\int_D \nabla u_0^j \cdot \nabla u_0^k \, dx = \int_{\partial D} \partial_\nu u_0^j \cdot u_0^k \, ds$$

by the Gauß-Green theorem, even more computational effort can be shifted to the offline phase, provided the sets under consideration possess essentially finitely many different normals, which is the situation we consider in Sect. 4 and what follows.

Proposition 1 gives a mathematically rigorous criterion to check whether a set D contains the convex source support. In the following we describe a heuristic numerical implementation of this criterion and test it on a simple test example. Let us stress that we do not have any theoretical results on convergence or stability of the proposed numerical implementation, and that it is completely unclear whether such an implementation exists. Checking whether a function lies in the dense range of an infinite-dimensional operator seems intrinsically unstable to discretization errors and errors in the function or the operator. Likewise, it is unclear how to numerically check whether the sequence in Proposition 1 diverges or not.

In other words, the following heuristic numerical algorithm is motivated by a rigorous theoretical result but it is completely heuristic and we do not have any theoretical justification for this algorithm. Since, to the knowledge of the authors, no convergent numerical methods are known for the considered problem, we believe that this algorithm might still be of interest and serve as a first step towards mathematically rigorously justified algorithms.

To heuristically check, whether $\|R_\alpha^D f\| \to \infty$, we fix suitable constants $\alpha, C > 0$ and $N \in \mathbb{N}$. Consider the finite-dimensional subspace $V_N := \operatorname{span}(\varphi_1, \ldots, \varphi_N)$ of $L^2_\diamond(\partial\Omega)$ and the corresponding L^2 orthogonal projector $P_N : L^2_\diamond(\partial\Omega) \to V_N$. Instead of M_D, we consider the truncated operator $M_D^N := P_N M|_{V_N} : V_N \to V_N$, which satisfies

$$(M_D^N \varphi_j, \varphi_k) = (P_N M_D \varphi_j, \varphi_k) = (M_D \varphi_j, \varphi_k) \quad \text{for} \quad 1 \le j, k \le N,$$

so that formula (6) holds for M_D^N as well. We define

$$\|R_{\alpha,N}^D v\|^2 := ((M_D^N + \alpha I)^{-1} M_D^N (M_D^N + \alpha I)^{-1} P_N v, P_N v) \quad \text{for all } v \in L^2_\diamond(\partial\Omega)$$

and note that

$$R_{\alpha,N}^D f \to R_\alpha^D f \quad \text{as} \quad N \to \infty$$

follows from a discussion, which involves a variant of the Banach Lemma. Therefore, the use of the criterion

$$\|R_{\alpha,N}^D f\|^2 \le C$$

instead of $\|R_\alpha^D f\| \to \infty$ is well-motivated, and we solve

$$\operatorname{vol}(D) = \min! \quad \text{subject to} \quad D \in \mathscr{K}_c(\mathbb{R}^d),\ D \subset \Omega,\ \|R_{\alpha,N}^D f\|^2 \le C. \tag{8}$$

with, e.g., $\alpha = 10^{-4}$ and $C = 10^6$ instead of (4).

4 Galerkin Approximations to $\mathscr{K}_c(\mathbb{R}^2)$

We outline a setting for a first-discretize-then-optimize approach to numerical optimization in the space $\mathscr{K}_c(\mathbb{R}^2)$, which we use to solve problem (8). To this end, we define Galerkin subspaces of $\mathscr{K}_c(\mathbb{R}^2)$ in terms of polytopes with prescribed sets of outer normals. These spaces have good global approximation properties (see Proposition 2), they possess a unique representation in terms of few coordinates, and their sets of admissible coordinates are characterized by sparse linear inequalities. A theory of these spaces in arbitrary dimension is work in progress.

Fix a matrix $A \in \mathbb{R}^{m\times 2}$ with rows a_i^T, $i = 1, \ldots, m$, where $a_i \in \mathbb{R}^2$, $\|a_i\|_2 = 1$ for all i and $a_i \neq a_j$ for all i, j with $i \neq j$. For every $b \in \mathbb{R}^m$, we consider the convex polyhedron

$$Q_{A,b} := \{x \in \mathbb{R}^2 : Ax \leq b\},$$

and we define a space $\mathcal{G}_A \subset \mathcal{K}_c(\mathbb{R}^2)$ of convex polyhedra by setting

$$\mathcal{G}_A := \{Q_{A,b} : b \in \mathbb{R}^m\} \setminus \{\emptyset\}.$$

The choice of these spaces is motivated by an approximation result from [8]. Recall the definition of the one-sided Hausdorff distance

$$\text{dist} : \mathcal{K}_c(\mathbb{R}^2) \times \mathcal{K}_c(\mathbb{R}^2) \to \mathbb{R}_+, \quad \text{dist}(D, D') := \sup_{x\in D} \inf_{x'\in D'} \|x - x'\|_2.$$

Proposition 2 *Assume that the matrix $A \in \mathbb{R}^{m\times 2}$ satisfies*

$$\delta := \max_{x\in\mathbb{R}^2,\ \|x\|_2=1} \text{dist}(\{x\}, \{a_1^T, \ldots, a_m^T\}) < 1. \tag{9}$$

Then the associated space $\mathcal{G}_A$ consists of convex polytopes, and for all $D \in \mathcal{K}_c(\mathbb{R}^2)$, there exists $Q_{A,b} \in \mathcal{G}_A$ such that $D \subset Q_{A,b}$ and

$$\text{dist}(Q_{A,b}, D) \leq \frac{2\delta - \delta^2}{1 - \delta} \text{dist}(D, \{0\}).$$

Hence, if the matrix A is augmented in such a way that $\delta \to 0$ as $m \to \infty$, then $\mathcal{G}_A$ converges to $\mathcal{K}_c(\mathbb{R}^2)$ uniformly on every bounded subset of $\mathcal{K}_c(\mathbb{R}^2)$.

It is, at present, not entirely clear how to represent the spaces $\mathcal{G}_A$ in terms of coordinates. There are $b \in \mathbb{R}^m$ with $Q_{A,b} = \emptyset$, and two different vectors $b, b' \in \mathbb{R}^m$ may encode the same polytope $Q_{A,b} = Q_{A,b'}$. In our concrete optimization problem, the constraint $\mathcal{C}f \subset D$ will enforce $Q_{A,b} \neq \emptyset$. For the time being, we treat the second issue by forcing all hyperplanes $\{x \in \mathbb{R}^2 : a_k^T x = b_k\}$, $k = 1, \ldots, m$, to possess at least one common point with $Q_{A,b}$. This approach will be made rigorous in the future.

Definition 1 We call the set $\mathcal{C}_A$ of all $b \in \mathbb{R}^m$ satisfying

$$p_1 \begin{pmatrix} b_m \\ b_2 \end{pmatrix} \geq b_1, \quad p_k \begin{pmatrix} b_{k-1} \\ b_{k+1} \end{pmatrix} \geq b_k, \quad k = 2, \ldots, m-1, \quad \text{and} \quad p_m \begin{pmatrix} b_{m-1} \\ b_1 \end{pmatrix} \geq b_m$$

with

$$p_1 := a_1^T \begin{pmatrix} a_m^T \\ a_2^T \end{pmatrix}^{-1}, \quad p_m := a_m^T \begin{pmatrix} a_{m-1}^T \\ a_1^T \end{pmatrix}^{-1},$$

$$p_k := a_k^T \begin{pmatrix} a_{k-1}^T \\ a_{k+1}^T \end{pmatrix}^{-1}, \quad k = 2, \ldots, m-1$$

the set of admissible coordinates.

Note that the inverse matrices above exist when $\delta < 1$ as required in Proposition 2. Hence it is easy to assemble the sparse matrix

$$H_A = \begin{pmatrix} 1 & -p_{1,2} & & & -p_{1,1} \\ -p_{2,1} & 1 & -p_{2,2} & & \\ & \ddots & \ddots & \ddots & \\ & & -p_{m-1,1} & 1 & -p_{m-1,2} \\ -p_{m,2} & & & -p_{m,1} & 1 \end{pmatrix},$$

which gives rise to the following characterization of the set $\mathscr{C}_A \subset \mathbb{R}^m$.

Lemma 1 *The set of admissible coordinates can be written as*

$$\mathscr{C}_A = \{b \in \mathbb{R}^m : H_A b \leq 0\}.$$

All in all, we replaced the relatively inaccessible space $\mathscr{K}_c(\mathbb{R}^2)$ with a Galerkin subspace $\mathscr{G}_A$ that is parametrized over a set $\mathscr{C}_A \subset \mathbb{R}^m$ of coordinates, which, in turn, is described by a sparse linear inequality. For the practical computations in this paper, we fix the matrix $A = (a_1, \ldots, a_m)^T$ given by

$$a_k = (\cos(2k\pi/m), \sin(2k\pi/m))^T, \quad k = 1, \ldots, m,$$

which is probably the best choice in the absence of detailed information on the set to be approximated. As we will have $\Omega = B_1(0)$ in our computational example, we will replace problem (4) with the fully discrete optimization problem

$$\left.\begin{aligned} &\mathrm{vol}(Q_{A,b}) = \min! \\ &\text{subject to} \quad b \in \mathbb{R}^m,\ H_A b \leq 0,\ b \leq 1,\ \|R_{\alpha,N}^{Q_{A,b}} f\|^2 \leq C. \end{aligned}\right\} \tag{10}$$

5 Gradients of Functions on $\mathscr{G}_A$

The objective function $D \mapsto \mathrm{vol}(D)$ and the constraint $D \mapsto \|R^D_{\alpha,N} f\|^2$ are both given in terms of integrals of a real-valued function over the set D. The evaluation of these integrals is straight-forward and efficient. The efficient evaluations of the gradients of both integrals with respect to coordinates requires some preparation. We follow [6, Lemma 2.2] and [7, Theorem 1].

Proposition 3 *Let $b \in \mathscr{C}_A$, let $k \in \{1, \ldots, m\}$, and let $Q^k_{A,b}$ be the facet*

$$Q^k_{A,b} := Q_{A,b} \cap \{x \in \mathbb{R}^2 : a_k^T x = b_k\}.$$

If we assume that $vol_2(Q_{A,b}) > 0$ and $vol_1(Q^k_{A,b}) > 0$, then for any continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\frac{d}{db_k} \int_{Q_{A,b}} h(x)\, dx = \int_{Q^k_{A,b}} h(\xi) d\xi.$$

The above proposition shows that whenever $Q_{A,b}$ is not degenerate, we have

$$\nabla_b \mathrm{vol}_2(Q_{A,b}) = (\mathrm{vol}_1(Q^1_{A,b}), \ldots, \mathrm{vol}_1(Q^m_{A,b}))^T.$$

To compute $\nabla_b \|R^{Q_{A,b}}_{\alpha,N} f\|^2$, we need the following lemma. The construction of the matrices P, S and U reduces the costs for the computation of the desired derivative.

Lemma 2 *Let $\varepsilon > 0$, let $M : (-\varepsilon, \varepsilon) \to \mathbb{R}^{N \times N}$, $\gamma \mapsto M(\gamma)$, be differentiable with $M(\gamma)$ symmetric and $M(\gamma) + \alpha I$ invertible for all $\gamma \in (-\varepsilon, \varepsilon)$. Using the abbreviations*

$$X := M + \alpha I,\ Y := X^{-1} M' X^{-1} \text{ and } Z := M X^{-1},$$

we find that

$$(X^{-1} M X^{-1})' = -YZ + Y - (YZ)^T.$$

The proof is elementary and therefore omitted. An application of Lemma 2 to the matrix representation of $M^N_{Q_{A,b}}$ yields

$$\begin{aligned} \frac{d}{db_k} \|R^{Q_{A,b}}_{\alpha,N} f\|^2 &= \frac{d}{db_k} ((M^N_{Q_{A,b}} + \alpha I)^{-1} M^N_{Q_{A,b}} (M^N_{Q_{A,b}} + \alpha I)^{-1} f, f) \\ &= ((-Y_k Z + Y_k - (Y_k Z)^T) f, f) \end{aligned}$$

with the abbreviations

$$X := M^N_{Q_{A,b}} + \alpha I,\ Y_k := X^{-1}(\frac{d}{db_k} M^N_{Q_{A,b}})X^{-1} \text{ and } Z := M^N_{Q_{A,b}} X^{-1},$$

where

$$[\frac{d}{db_k} M^N_{Q_{A,b}}]_{ij} = \int_{Q^k_{A,b}} \nabla u^i_0 \cdot \nabla u^j_0 \, d\xi$$

by Proposition 3. Thus we obtain a formula for $\nabla_b \| R^{Q_{A,b}}_{\alpha,N} f \|^2$ that is not only more precise than a numerical approximation by finite differences, but also much cheaper to compute, because the area of integration is just a lower-dimensional surface.

6 A First Numerical Simulation

We test our numerical algorithm on a simple 2d example, where all quantities are known explicitly and the algorithm can be observed under controlled conditions. Let $\Omega = B_1(0)$ be the unit circle and let $\sigma_0 \equiv 1$. We consider a point inhomogeneity which leads to a difference potential (cf. [1])

$$w(x) = \frac{1}{\pi} \frac{\langle z^* - x, \eta \rangle}{\|z^* - x\|_2^2}, \quad x \in B_1(0),$$

that solves the partial differential equation

$$\Delta w = \eta \cdot \nabla \delta_{z^*} \text{ in } \Omega, \quad \sigma_0 \partial_\nu u_{|\partial\Omega} = 0,$$

where z^* is the location of the point inhomogeneity, and $\eta \in \mathbb{R}^2$, $\|\eta\|_2 = 1$ is a dipole orientation vector depending on the applied current pattern. Using a standard smoothing argument, it is easily checked (see, e.g., [4]) that for each open set O containing z^* there exists $F \in L^2(O)^2$ so that

$$\Delta w = \operatorname{div} F.$$

Hence the convex source support of the difference measurement $w|_{\partial\Omega}$ is the inhomogeneity location z^*. In our example we used $z^* = (\frac{3}{10}, \frac{3}{10})^T$ and $\eta = (1, 0)^T$.

In the following computations, it is convenient to switch between standard coordinates (x_1, x_2) and polar coordinates (r, ξ). Consider the basis

$$\varphi_{2j}(\xi) = \frac{1}{\sqrt{\pi}} \cos(j\xi), \quad \varphi_{2j+1}(\xi) = \frac{1}{\sqrt{\pi}} \sin(j\xi), \quad j \in \mathbb{N}_1,$$

of $L^2_\diamond(\partial\Omega)$. Since the Laplace operator satisfies

$$u_{x_1x_1} + u_{x_2x_2} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\xi\xi},$$

it is easy to see that the corresponding solutions of problem (7) are

$$u_0^{2j}(r,\xi) = \frac{1}{j\sqrt{\pi}}\cos(j\xi)r^j, \quad u_0^{2j+1}(r,\xi) = \frac{1}{j\sqrt{\pi}}\sin(j\xi)r^j, \quad j \in \mathbb{N}_1.$$

Since the gradient satisfies

$$u_{x_1} = \cos\xi \cdot u_r + \frac{1}{r}\sin\xi \cdot u_\xi, \quad u_{x_2} = \sin\xi \cdot u_r + \frac{1}{r}\cos\xi \cdot u_\xi,$$

we have explicit representations

$$\frac{d}{dx_1}u_0^{2j} = \frac{r^{j-1}}{\sqrt{\pi}}\cos((j+1)\xi), \quad \frac{d}{dx_2}u_0^{2j} = -\frac{r^{j-1}}{\sqrt{\pi}}\sin((j-1)\xi),$$
$$\frac{d}{dx_1}u_0^{2j+1} = \frac{r^{j-1}}{\sqrt{\pi}}\sin((j+1)\xi), \quad \frac{d}{dx_2}u_0^{2j+1} = \frac{r^{j-1}}{\sqrt{\pi}}\cos((j-1)\xi).$$

Now we fix the matrix $A = (a_1, \ldots, a_8)^T$ given by

$$a_k = (\cos(k\pi/4), \sin(k\pi/4))^T, \quad k = 1, \ldots, 8,$$

and solve optimization problem (4) approximately by applying Matlab's interior point method to problem (10) with initial value $b_0 = (\frac{4}{5}, \ldots, \frac{4}{5})^T$ and $N = 6$, computing values and gradients of the objective $b \mapsto \mathrm{vol}_2(Q_{A,b})$ and the constraint $b \mapsto \|R^{Q_{A,b}}_{\alpha,N} w_{|\partial\Omega}\|^2$ as in Sect. 5. The results and the computation times on an ordinary desktop computer are displayed in Fig. 1.

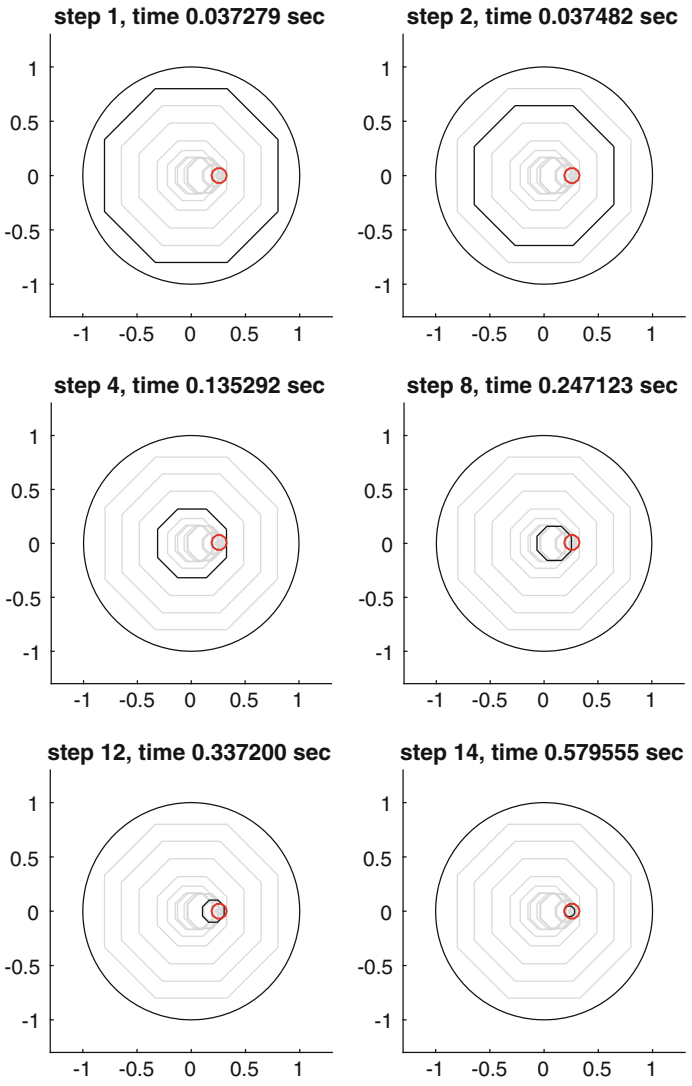

Fig. 1 Selected iterates of Matlab's interior point optimization tool applied to (10) with data specified in Sect. 6. The current iterate is highlighted in black. The position of the dipole is the center of the red circle

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