Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 5289–5294

Delay Variability Optimization Using Shockwave Theory at an Undersaturated Intersection

Reza Mohajerpoor * **Meead Saberi** * **Mohsen Ramezani** **

* *Institute of Transport Studies, Department of Civil Engineering, Monash University, Melbourne, Australia*
** *School of Civil Engineering, The University of Sydney, Australia*

Abstract: Signal optimization is one of the most crucial problems in the traffic flow theory. Delay at signalized intersections is the main component of travel time in urban transportation networks. This paper investigates an analytical approach based on the shockwave theory to estimate the delay of each vehicle joining the queue, and minimize the total delay and delay variability at an undersaturated intersection. The optimizations are carried out for an intersection with and without loss time and are formulated as convex programs. The global optimal cycle length and splits are attained to minimize the delay variability.

Keywords: Delay distribution; fundamental diagram; travel time reliability; convex optimization

1. INTRODUCTION

Signal optimization of isolated intersections has been studied for a long time (Wardrop, 1952; Webster, 1958; Gazis, 1964). The seminal work of Webster (1958) proposed an analytical formula for the optimal cycle length at an isolated undersaturated intersection. However, as demonstrated in Wagner et al. (2014), the formula is oversimplified and does not necessarily provide the optimal solution. Furthermore, the introduction of the *kinematic shockwave* theory (Lighthill and Whitham, 1955; Richards, 1956) provided alternative methods for modelling and analyzing the traffic flow of arterials and intersections (see e.g. Dion et al. (2004); Skabardonis and Geroliminis (2008); Liu et al. (2009); Cheng et al. (2011); Ramezani and Geroliminis (2015)). The delay estimation at intersections using the shockwave theory was first presented in Michalopoulos and Pisharody (1981), where the authors considered a variable density at the discharge phase of the intersection, which itself resulted in a rigorous formula. Michalopoulos et al. (1981) proposed a heuristic signal control algorithm for an over-saturated isolated intersection. Dion et al. (2004) presented a comprehensive literature review on the delay modelling approaches and classified them into deterministic queueing algorithms, shockwave based models, and microscopic simulation based approaches.

The literature on the optimization of traffic networks can be classified into studies that (i) focus on multiple interconnected intersections (Dinopoulou et al., 2006; Kosmatopoulos et al., 2007; Ramezani et al., 2016), (ii) investigate network-wide signal control e.g. (Diakaki et al., 2002; Geroliminis et al., 2013; Ramezani et al., 2015; Keyvan-Ekbatani et al., 2012), and (iii) optimize a single isolated intersection. Gazis (1964) propose an optimization approach based on the Pontryagin's minimum principle for an oversaturated isolated intersection, using a semi-graphical methodology and employing the queueing theory.

The *queueing theory* has been widely employed for the signal timing optimization, relying on various numerical optimization approaches (Haddad et al., 2010; Aboudolas et al., 2010; Ioslovich et al., 2011; Varaiya, 2013b,a; Haddad et al., 2014). Haddad et al. (2010) proposed a discrete-event max-plus approach to model the traffic flow of a two-way isolated undersaturated intersection and minimize a weighted sum of the red phases of the intersection as the optimization criterion. It assumes a *discrete* model and a *given fixed cycle-time* in the modelling and optimization stages. Later Haddad et al. (2014) extended the theory to consider the assumption of the lower and upper bounds of the green phase of each intersection. In addition, the above theory is applied in Ioslovich et al. (2011) to optimize an oversaturated intersection assuming a continuous model as in Gazis (1964). Moreover, Varaiya (2013b) proposed the concept of max-pressure and apply it on the *store and forward* queueing model to stabilize an arbitrary network of correlated intersections under uncertain demands and turning ratios.

Nevertheless, studies based on the store and forward queueing model do not provide full spatial and temporal characteristics of queuing dynamics. Specifically the spillback phenomenon cannot be fully captured. In addition, it is well-known that when the queue in an approach exceeds the detector's location, models based on the queuing theory face observability problems in practical cases (Liu et al., 2009; Cheng et al., 2011).

Accordingly, this paper investigates the traffic signal control problem at intersections using the shockwave theory. An analytical model is presented for estimating the delay of a vehicle joining the queue, and the distribution of

10.1016/j.ifacol.2017.08.622

intersection delay. Using the model, the paper formulates the optimal cycle time and the green phase allocations at an *undersaturated* intersection considering constant flow rates per cycle in two cases of with and without nominal loss times (LTs). The optimization of the signals is carried out by minimizing the following objective functions: (i) total delay and (ii) delay variability. The delay distribution at the intersection is established, and the variance of delay is introduced as a criterion for increasing the performance reliability of the signal control. Undersaturation and spillback avoidance are formulated to define the optimization constraints.

NOMENCLATURE

q_i^{a}	The arrival flow at Approach i; [veh/unit time]
q_i^{c}	The saturation flow (or capacity) at Approach i; [veh/unit time]
k_i^{a}	The arrival traffic density at Approach i; [veh/unit distance]
k_i^{c}	The saturation traffic density at Approach i; [veh/unit distance]
k_i^{j}	The jam density at Approach i; [veh/unit distance]
R_i, G_i	Respectively, the red phase and green phase of Approach i in the cycle; [unit time]
C	The cycle time; [unit time]
L_i, L	The loss time at Approach i, and the total loss time, respectively; [unit time]
t_i	The time of joining the queue at Approach i; [unit time]
$x_i(t_i)$	The position of the back of the queue in Approach i as a function of t_i; [unit length]
$D_i(t_i)$	Delay of the vehicle at Approach i joining the queue at t_i; [unit time]
x_i^{j} and t_i^{j}	The position and the time of queue clearance at Approach i, respectively
$\hat{i} \in \Omega$	The complement of i in the set Ω

2. PRELIMINARIES OF DELAY ESTIMATION

In this section the delay of every vehicle joining the queue at an intersection without LT is estimated using the shockwave theory. Consequently, the effect of drivers' reaction times is also amended in the estimation. It is emphasized that the formulas developed in this section are aimed at helping with the optimization and control of traffic at the intersection. Delay formulas have been extensively studied previously using various methods.

Variable $t_i \in [0, t_i^{\mathrm{j}}]$, $i = \{1, 2\}$ is defined as the time of joining the queue for vehicles at Approach i. The methodology of the paper is based on the time-space diagram (TSD) that contains the trajectory of every vehicle entering and exiting an intersection, and the fundamental diagram (FD) that relates the traffic flow to the traffic density based on the characteristics of the road. TSD and FD of an approach are related through the kinematic shockwaves. A shockwave is a wave created due to any abrupt change in the states of traffic flow (Lighthill and Whitham, 1955). Fig. 1 depicts TSDs, FDs, and shockwaves of the two approaches at a two-phase undersaturated intersection with constant inflows.

To derive analytical formulation of delays we assume: (A1) the transition of every individual vehicle between the free flow speed to zero occurs with an *infinite acceleration*; (A2) the average arrival and discharge flows and speeds of each approach are constant and known; (A3) a triangular FD is considered (Fig. 1(b)); (A4) the intersection is two-phase; and (A5) the LT of each approach at the intersection is known and constant. Although only two phases are considered, this does not fail the generality of the paper, as it is well-understood that the multiple-phase problem is an extension of the two-phase model (Webster, 1958; Haddad et al., 2010; Varaiya, 2013a).

The shockwaves are bold solid lines depicted in Fig. 1 and the trajectories of vehicles are shown using thin directed lines. The slope of the shockwave from free flow to jam density conditions in Approach $i = \{1, 2\}$ is equal to the slope of J_iA_i in its FD. Analogously, the transition from jam state to the saturation flow can be described through a shockwave with the slope of the line J_iC_i in the FD. This relation is helpful in deriving the formulas of the paper.

In addition, calculating the points $(t_i^{\mathrm{j}}, x_i^{\mathrm{j}})$, $i = \{1, 2\}$, provides an estimation of the back of the queue, which is essential in analyzing the existence of any spillback for non-isolated intersections, and thus devising an appropriate signal setting to avoid spillback.

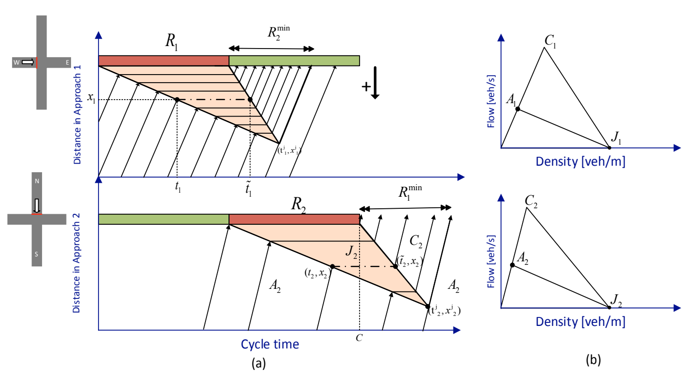

Fig. 1. (a) The time-space, and (b) the FDs of each approach at a two-phase isolated intersection.

Since the slopes of the arrival and discharge shockwaves can be obtained from the FD, the queue clearance point $(t_i^{\mathrm{j}}, x_i^{\mathrm{j}})$, $i = \{1, 2\}$, can be estimated as:

$$t_i^{\mathrm{j}} = \beta_i R_i, \tag{1}$$

$$x_i^{\mathrm{j}} = \frac{q_i^{\mathrm{a}}}{k_i^{\mathrm{j}} - k_i^{\mathrm{a}}} t_i^{\mathrm{j}}, \tag{2}$$

where

$$\beta_i \triangleq \frac{k_i^{\mathrm{c}} \left(k_i^{\mathrm{j}} - k_i^{\mathrm{a}}\right)}{k_i^{\mathrm{j}} \left(k_i^{\mathrm{c}} - k_i^{\mathrm{a}}\right)} > 1.$$

Since the vehicle that arrives to the queue at time t_i starts moving at time $\tilde{t}_i$ at the same location (see Fig. 1(a)), the delay of the vehicle is

$$D_i(t_i) = \tilde{t}_i - t_i \qquad \forall t_i \in [0, t_i^{\mathrm{j}}]. \tag{3}$$

Hence, the position of the back of the queue at time t_i is formulated as

$$x_i(t_i) = \frac{q_i^{\mathrm{a}}}{k_i^{\mathrm{j}} - k_i^{\mathrm{a}}} t_i$$
$$= \frac{q_i^{\mathrm{c}}}{k_i^{\mathrm{j}} - k_i^{\mathrm{c}}} (\tilde{t}_i - R_i). \quad (4)$$

Therefore,

$$\tilde{t}_i = \alpha_i t_i + R_i, \qquad \forall t_i \in [0, t_i^{\mathrm{j}}], \quad (5)$$

where

$$\alpha_i \triangleq \frac{k_i^{\mathrm{a}}(k_i^{\mathrm{j}} - k_i^{\mathrm{c}})}{k_i^{\mathrm{c}}(k_i^{\mathrm{j}} - k_i^{\mathrm{a}})} < 1 \qquad i = \{1, 2\}.$$

Note that $1 - \alpha_i = 1/\beta_i$. Finally, the closed-form formulation of delay can be written as

$$D_i(t_i) = (\alpha_i - 1)t_i + R_i \qquad \forall t_i \in [0, t_i^{\mathrm{j}}]. \quad (6a)$$

$$D_i(x_i) = (\alpha_i - 1)\frac{k_i^{\mathrm{j}} - k_i^{\mathrm{a}}}{q_i^{\mathrm{a}}} x_i + R_i \qquad \forall x_i \in [0, x_i^{\mathrm{j}}]. \quad (6b)$$

Considering constant LTs, L_i, $i = \{1, 2\}$, which models drivers reaction times, does *not* significantly change the derivation of delays. The constant LTs shift the discharge shockwaves, thus all of the formulas are still valid after replacing R_i with $\bar{R}_i \triangleq R_i + L_i$.

3. SIGNAL OPTIMIZATION

In this section, the optimization problem is formulated based on two objective functions with the constraint of spillback avoidance: minimization of (i) the total delay and (ii) the variance of delay distribution at the intersection.

3.1 Formulating the Constraints

Given the red phase R_i, it can be obtained that the minimum green phase of Approach i must satisfy the following inequality:

$$G_i^{\min} = R_{\hat{i}}^{\min} \geq t_i^{\mathrm{j}} - R_i + \frac{k_i^{\mathrm{a}}}{q_i^{\mathrm{a}}} x_i^{\mathrm{j}} \qquad i = \{1, 2\}, \quad (7)$$

where $\hat{i} \in \{1, 2\}$ is the complement of i in the set $\{1, 2\}$. After substituting (1) and (2) into (7), it is obtained that

$$R_{\hat{i}} - \eta_i R_i \geq 0, \quad (8)$$

where

$$\eta_i \triangleq \frac{k_i^{\mathrm{a}}}{k_i^{\mathrm{c}} - k_i^{\mathrm{a}}} \qquad i = \{1, 2\}.$$

Taking driver reaction times into account and recalling that $G_i + L_i = R_{\hat{i}}$, (8) is modified to

$$R_{\hat{i}} - \eta_i (R_i + L_i) - L_i \geq 0. \quad (9)$$

In addition, it can be shown using the queueing theory that the necessary and sufficient condition for keeping an intersection undersaturated is (Gazis, 1964)

$$\sum_{i=1,2} \frac{q_i^{\mathrm{a}}}{q_i^{\mathrm{c}}} + \frac{L}{C} \leq 1. \quad (10)$$

Since increasing the cycle time can potentially increase the chance of spillback, a holistic optimization is essential to avoid spillback. That is in Approach i, the position of the back of the queue at the clearance time, i.e. x_i^{j}, should be less than or equal to the link length of the approach, i.e. Δ_i. This constraint can be expressed as the following inequality:

$$R_i + L_i - k_i^{\mathrm{j}} \left(\frac{1}{q_i^{\mathrm{a}}} - \frac{1}{q_i^{\mathrm{c}}} \right) \Delta_i \leq 0 \quad i = \{1, 2\}. \quad (11)$$

3.2 Minimizing the Total Delay

Intersection without Loss Time One of the most crucial criteria for evaluating the performance of a signal control method is the total delay. Theoretically, under the assumption of *no* LT in a homogeneous undersaturated intersection satisfying Assumptions (A1)-(A4), the red phase of each approach can be ideally equal to zero. This indicates that there is no need for a signal to operate, and the intersection is no longer a bottleneck. Mathematically, this conclusion comes from Assumption (A1), where an infinite acceleration for every individual vehicle is realized. Nevertheless, a minimum green phase at each approach, represented as $G_i^{\min}$, should be allocated for pedestrians and for vehicles to safely pass the intersection. Note that $G_i^{\min} = R_{\hat{i}}^{\min}$.

Considering the pre-defined infimum values of the red phases, the feasibility region of the optimization problem can be depicted as in Fig. 2 (a-c). These figures demostrate the constraints, i.e. (8), (11), and the minimum red phases, of the optimization problem.

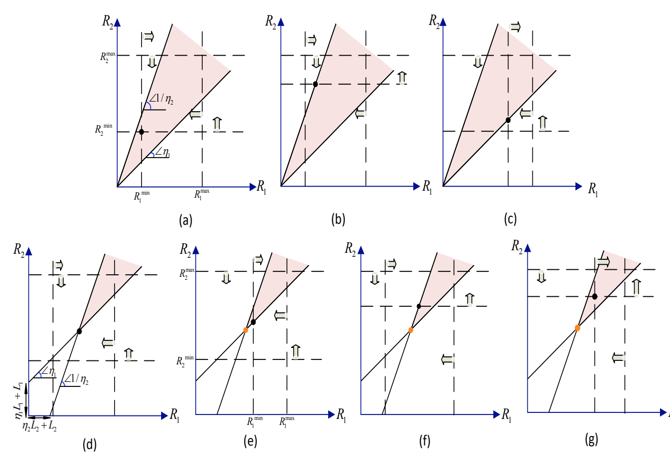

Fig. 2. The feasibility regions and possible optimal solution outcomes of the dynamic cycle length control scenario: (a-c) intersection without loss time and (d-g) intersection with constant loss times. The black dot point in each figure indicates the optimal signal setting. Note that $R_i^{\max} = k_i^{\mathrm{j}} \left(\frac{1}{q_i^{\mathrm{a}}} - \frac{1}{q_i^{\mathrm{c}}} \right) \Delta_i - L_i$.

Total delay is the integration of the delay function in the space direction, i.e.:

$$\begin{aligned} D_T &= \textstyle\sum_{i=1,2} \int_0^{x_i^{\mathrm{j}}} k_i^{\mathrm{j}} D_i(x_i) \mathrm{d}x_i \\ &= \textstyle\sum_{i=1,2} \gamma_i R_i^2, \end{aligned} \quad (12)$$

where $\gamma_i \triangleq 0.5\beta_i \frac{q_i^{\mathrm{a}}}{k_i^{\mathrm{j}} - k_i^{\mathrm{a}}} k_i^{\mathrm{j}}$. It follows that the average delay at the intersection can be calculated as

$$\bar{D}_T = \frac{D_T}{C(q_1^{\mathrm{a}} + q_2^{\mathrm{a}})}. \quad (13)$$

Hence, the global optimal cycle-length and signal split that minimize the total delay and ensure the spillback avoidance and undersaturation property of the intersection can be obtained from the following optimization problem ($i, \hat{i} \in \{1, 2\}$):

$$\begin{aligned} \underset{R_1,R_2}{\text{minimize}} \quad & D_T(R_1, R_2) \\ & \eta_i R_i - R_{\hat{i}} \leq 0 \\ & -R_i + R_i^{\min} \leq 0 \\ & R_i - k_i^{\text{j}} \left(\tfrac{1}{q_i^{\text{a}}} - \tfrac{1}{q_i^{\text{c}}} \right) \Delta_i \leq 0 \end{aligned} \tag{14}$$

D_T is a convex function of R_1 and R_2, and the constraints are linear, thus the optimization problem (14) is a convex optimization problem, which can be solved using any numerical convex optimization program. However, it can be shown using the Lagrange optimization method that the optimal solution lies on the border of the feasibility set, demonstrated in Fig. 2 (a-c). In other words, the optimal solution is attained from two non-zero values of the Lagrange multipliers which correspond to two constraints that can also be identified graphically (Fig. 2). Hence, there are three possible outcomes depending on the location of the point $D^\star \triangleq (R_1^{\min}, R_2^{\min})$ in the solution space:

(1) $D^\star$ is the optimal solution if it lies between the lines $-R_2 + \eta_1 R_1 = 0$ and $\eta_2 R_2 - R_1 = 0$ (the bold point in Fig. 2(a)).
(2) $(\eta_2 R_2^{\min}, R_2^{\min})$ is the solution if the point $D^\star$ lies on the left of the line $-R_1 + \eta_2 R_2 = 0$ (the bold point in Fig. 2(b)).
(3) $(R_1^{\min}, \eta_1 R_1^{\min})$ is the solution if the point $D^\star$ lies on the right of the line $\eta_1 R_1 - R_2 = 0$ (the bold point in Fig. 2(c)).

Considering LTs, four outcomes are possible that are depicted in Fig. 2 (d-g).

3.3 Minimizing the Delay Variability

The distribution of delay provides crucial information about the reliability of the signal control. A traffic signal cotrol that minimizes this objective provides candid green times to all approaches. As such, vehicles on minor approaches will experience more equitable delays compared to the case when minimizing the total delay at the intersection.

Theorem 1. *The following statements hold for a two-phase undersaturated isolated intersection wherein Assumptions (A1-A4) are satisfied:*

(a) The delay probability density function (PDF) of Approach i, $i = \{1, 2\}$, can be expressed as

$$f_{D_i}(d) = \frac{\tilde{n}_i}{C q_i^{\text{a}}} \delta(d) + \theta_i \left(H(d) - H(d - R_i) \right), \tag{15}$$

where $\theta_i = \frac{\beta_i k_i^{\text{j}}}{C(k_i^{\text{j}} - k_i^{\text{a}})}$, $d \geq 0$ is the delay experienced by a vehicle, $\delta(\cdot)$ is the Dirac delta function, and $\tilde{n}_i$ is the approximate number of undelayed vehicles in Approach i:

$$\tilde{n}_i = C q_i^{\text{a}} - q_i^{\text{a}} \nu_i \beta_i R_i, \tag{16}$$

where $\nu_i \triangleq \frac{k_i^{\text{j}}}{k_i^{\text{j}} - k_i^{\text{a}}}$.

(b) Without loss of generality, if Approach i is the dominant approach, i.e. $R_{\hat{i}} \geq R_i$, then for every $d \geq 0$ the delay PDF of the whole intersection can be expressed as

$$\begin{aligned} f_D(d) =& \zeta_1 \delta(d) + \zeta_2 \left(H(d) - H(d - R_i) \right) \\ &+ \zeta_3 \left(H(d - R_i) - H(d - R_{\hat{i}}) \right), \end{aligned} \tag{17}$$

where

$$\zeta_1 = \frac{\tilde{n}_1 + \tilde{n}_2}{C(q_1^{\text{a}} + q_2^{\text{a}})},$$

$$\zeta_2 = \sum_{i=1}^{2} \frac{\beta_i \nu_i q_i^{\text{a}}}{C(q_1^{\text{a}} + q_2^{\text{a}})},$$

$$\zeta_3 = \frac{\beta_{\hat{i}} \nu_{\hat{i}} q_{\hat{i}}^{\text{a}}}{C(q_1^{\text{a}} + q_2^{\text{a}})}.$$

(c) The expected value of delay is equal to

$$\bar{D}_\mu = \mathrm{E}[f_D] = 0.5 \zeta_2 R_i^2 + 0.5 \zeta_3 \left(R_{\hat{i}}^2 - R_i^2 \right). \tag{18}$$

(d) The variance of delay at the intersection assuming $R_{\hat{i}} \geq R_i$ is

$$\begin{aligned} D_{\text{var}} &= \zeta_1 \bar{D}_\mu^2 + \tfrac{1}{3} \zeta_2 \left(R_i - \bar{D}_\mu \right)^3 + \tfrac{1}{3} \zeta_2 \bar{D}_\mu^3 \\ &+ \tfrac{1}{3} \zeta_3 \left(R_{\hat{i}} - \bar{D}_\mu \right)^3 - \tfrac{1}{3} \zeta_3 \left(R_i - \bar{D}_\mu \right)^3. \end{aligned} \tag{19}$$

Considering constant loss times due to drivers reaction times, the formulas should be amended by replacing R_i with $R_i + L_i$, $i = \{1, 2\}$.

Proof. (a) A number of vehicles $\tilde{n}_i \approx C q_i^{\text{a}} - k_i^{\text{j}} x_i^{\text{j}}$ do not come to stop at Approach i. Every other vehicle joins the queue and experiences a delay between 0 and R_i (see Fig. 1(a)). Hence, the delay PDF of each approach can be approximated as a Dirac function at $d = 0$ and an *impulse* function of length R_i for $d > 0$, as formulated in (15) and depicted in Figs. 3(b-c). Moreover, the coefficient of the Dirac function should represent the probability that a vehicle at Approach i does not stop, which depends on $\tilde{n}_i$. To calculate θ_i the fact that the integration of the PDF function must be 1 is used.

(b) The delay PDF of the whole intersection can be obtained from the superposition of the delay PDFs of the two approaches, which graphically results in the shape depicted in Fig. 3(d) and can be formulated as (17). To obtain ζ_1 the total number of vehicles and non-stopped vehicles in a cycle can be approximated from $C(q_1^{\text{a}} + q_2^{\text{a}})$ and $\tilde{n}_1 + \tilde{n}_2$, respectively. Now, if we assume $R_{\hat{i}} \geq R_i$ the possibility that a vehicle experience a delay value between 0 and R_i is approximately (Fig. 3(a)):

$$\Pr[0 < d \leq R_i] \approx \frac{q_i^{\text{a}} \nu_i t_i^{\text{j}} + q_{\hat{i}}^{\text{a}} \nu_{\hat{i}} (t_{\hat{i}}^{\text{j}} - \Delta t_{h1})}{C(q_1^{\text{a}} + q_2^{\text{a}})}.$$

Furthermore, the probability of vehicle delays between R_i and $R_{\hat{i}}$ can be approximated as:

$$\Pr[R_i < d \leq R_{\hat{i}}] \approx \frac{q_{\hat{i}}^{\text{a}} \nu_{\hat{i}} \Delta t_{h1}}{C(q_1^{\text{a}} + q_2^{\text{a}})}.$$

We estimate Δt_{h1} through h. To obtain h, we define the speeds of the arrival and departure shockwaves of Approach $i = \{1, 2\}$ as u_i and w_i, respectively. From the FDs the absolute values of the speeds of the shockwaves are

$$|u_i| = \frac{q_i^{\text{a}}}{k_i^{\text{j}} - k_i^{\text{a}}}, \quad |w_i| = \frac{q_i^{\text{c}}}{k_i^{\text{j}} - k_i^{\text{c}}}.$$

Thereafter, from Fig. 3(a) it can be written that $h/\Delta t_{h1} = |u_{\hat{i}}|$, $h/\Delta t_{h2} = |w_{\hat{i}}|$, $R_{\hat{i}} + \Delta t_{h2} = R_i + \Delta t_{h1}$, and

$$R_{\hat{i}} - R_i = h\left(\frac{1}{|u_{\hat{i}}|} - \frac{1}{|w_{\hat{i}}|}\right).$$

Accordingly, Δt_{h1} is obtained as

$$\Delta t_{h1} = \beta_{\hat{i}}(R_{\hat{i}} - R_i).$$

Now, recalling $\zeta_2 R_i = \Pr[0 < d \le R_i]$ and $\zeta_3(R_{\hat{i}} - R_i) = \Pr[R_i < d \le R_{\hat{i}}]$, the expressions of ζ_i, $i = \{2, 3\}$ can be obtained as demonstrated in the theorem.

(c,d) These parts can be proved following the definitions of the expected value and variance of PDF.

□

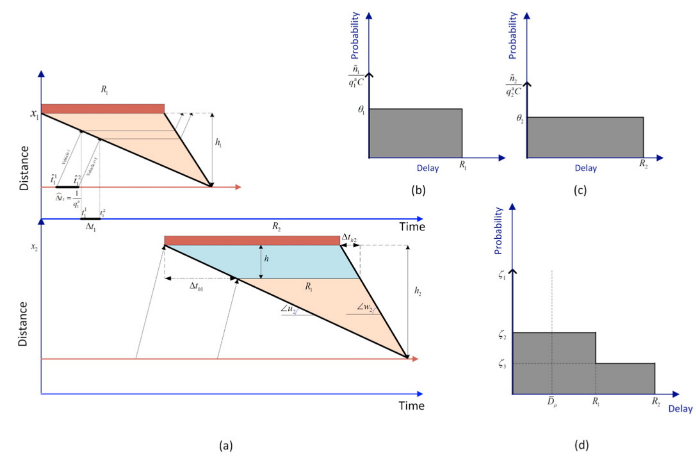

Fig. 3. (a) The description of the notations used in the proof of Theorem 1 on the TSD; (b)-(d) The PDFs of delay in Approaches 1, 2, and the whole intersection, respectively.

According to their definitions, ζ_1, ζ_2, ζ_3, and $\bar{D}_T$ are nonlinear (with respect to the optimization variables, R_1 and R_2). Hence, D_{var} is also a nonlinear function of R_i, $i = \{1, 2\}$. Furthermore, investigating the convexity of the variance function by plotting its shape based on some sample traffic states data affirms that D_{var} is a *non-convex* function of R_i, thus, the infimum of the objective function is *not* necessarily on the border of the feasibility set. This point makes the analytical calculation of the optimal solution cumbersome. Hence, the utilization of numerical approaches is preferred. Nevertheless, it can be shown after carrying out comprehensive mathematical manipulations that given $R_{\hat{i}} \ge R_i$, $D_{\mathrm{var}}(R_1, R_2)$ is strictly increasing with respect to $R_{\hat{i}}$ and it is a convex function of $R_{\hat{i}}$. Hence, the optimal cycle length and red phase allocations of the intersection based on the minimization of the variance function is on the line $R_{\hat{i}} = R_{\hat{i}}^{\min}$ of the (R_1,R_2) plane.

Signal Optimization Algorithm

I. Examine Inequality (10). If it is realized, there exists a global optimal solution and one may proceed to the next step. Otherwise, the intersection is over-saturated and the algorithm terminates with no solution.

II. Calculate η_1 and η_2 according to their definitions and identify the category of the optimization: (1) if $\eta_{\hat{i}} \ge 1$, $i, \hat{i} \in \{1, 2\}$, proceed to Step III; and (2) if $\eta_1 < 1$ and $\eta_2 < 1$, jump to Step IV.

III. It can be assumed that $R_{\hat{i}} \le R_i$. If $\eta_{\hat{i}} R_{\hat{i}}^{\min} \ge R_i^{\min}$, then the global optimal solution is $(R_{\hat{i}}^{\min}, \eta_{\hat{i}} R_{\hat{i}}^{\min})$ (the order may change depending on the value of $\hat{i}$). The optimal point is demonstrated as the blue bold points in Fig. 4(a,b). Otherwise, the following optimization problem should be solved using a numerical nonlinear non-convex optimization approach such as the method of Lagrange multipliers:

$$\begin{aligned} \underset{R_i = R_i^{\min}, R_{\hat{i}}}{\text{minimize}} \quad & D_{\mathrm{var}}(R_{\hat{i}}) \\ & \eta_{\hat{i}} R_{\hat{i}} - R_i^{\min} \le 0, \\ & -R_{\hat{i}} + R_{\hat{i}}^{\min} \le 0, \\ & -R_{\hat{i}} + \eta_i R_i^{\min} \le 0, \\ & R_{\hat{i}} - k_{\hat{i}}^{\mathrm{j}} \left(\frac{1}{q_{\hat{i}}^{\mathrm{a}}} - \frac{1}{q_{\hat{i}}^{\mathrm{c}}}\right) \Delta_{\hat{i}} \le 0, \\ & R_i^{\min} - k_i^{\mathrm{j}} \left(\frac{1}{q_i^{\mathrm{a}}} - \frac{1}{q_i^{\mathrm{c}}}\right) \Delta_i \le 0. \end{aligned} \tag{20}$$

The solution lies on the feasible closed set with $R_i = R_i^{\min}$, as demonstrated in the upper Figs. 4(a,b).

IV. Let us assume that $R_i^{\min} \le R_{\hat{i}}^{\min}$. The optimal solution can be found after solving the following problem (see Fig. 4(c)):

$$\begin{aligned} \underset{R_i, R_{\hat{i}} = R_{\hat{i}}^{\min}}{\text{minimize}} \quad & D_{\mathrm{var}}(R_i) \\ & \eta_{\hat{i}} R_{\hat{i}}^{\min} - R_i \le 0, \\ & -R_i + R_i^{\min} \le 0, \\ & R_i - R_{\hat{i}}^{\min} \le 0, \\ & R_i - k_i^{\mathrm{j}} \left(\frac{1}{q_i^{\mathrm{a}}} - \frac{1}{q_i^{\mathrm{c}}}\right) \Delta_i \le 0, \\ & R_{\hat{i}}^{\min} - k_{\hat{i}}^{\mathrm{j}} \left(\frac{1}{q_{\hat{i}}^{\mathrm{a}}} - \frac{1}{q_{\hat{i}}^{\mathrm{c}}}\right) \Delta_{\hat{i}} \le 0. \end{aligned} \tag{21}$$

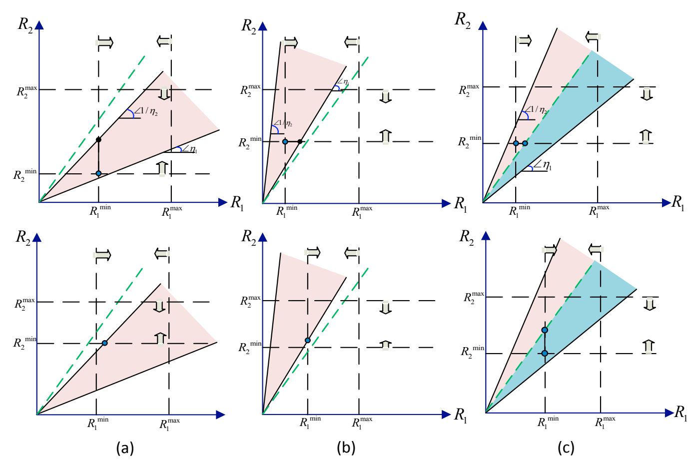

Fig. 4. The feasibility area (shaded) and the solution domains (bold solid lines) of the delay variability optimization problem assuming no LT at the intersection: (a,b) Case (1); (c) Case (2). The green dashed line represent $R_1 - R_2 = 0$; and $R_i^{\max} = k_i^{\mathrm{j}} \left(\frac{1}{q_i^{\mathrm{a}}} - \frac{1}{q_i^{\mathrm{c}}}\right) \Delta_i$.

4. SUMMARY AND FUTURE WORK

An analytical framework based on the shockwave theory has been presented to formulate the delay of each vehicle joining the queue, total delay, back of the queue, and delay variability at a two-phase undersaturated intersection.

The analytical global optimal cycle length and signal splits at the intersection for minimizing various objectives comprising the total delay and the variance of delay have been achieved assuming/ignoring constant loss times. The effect of loss times on the properties of the feasibility domain and the optimal signal settings are demonstrated. Future work will focus on addressing multiple phases at the intersection, and treating stochastic arrival flows. Moreover, considering variable upstream flow rates, as well as employing the proposed model for optimizing the signal settings at oversaturated intersections are open future directions.

REFERENCES

Aboudolas, K., Papageorgiou, M., Kouvelas, A., and Kosmatopoulos, E. (2010). A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks. *Transportation Research Part C: Emerging Technologies*, 18(5), 680 – 694.

Cheng, Y., Qin, X., Jin, J., Ran, B., and Anderson, J. (2011). Cycle-by-cycle queue length estimation for signalized intersections using sampled trajectory data. *Transportation Research Record: Journal of the Transportation Research Board*, (2257), 87–94.

Diakaki, C., Papageorgiou, M., and Aboudolas, K. (2002). A multivariable regulator approach to traffic-responsive network-wide signal control. *Control Engineering Practice*, 10(2), 183 – 195.

Dinopoulou, V., Diakaki, C., and Papageorgiou, M. (2006). Applications of the urban traffic control strategy tuc. *European Journal of Operational Research*, 175(3), 1652–1665.

Dion, F., Rakha, H., and Kang, Y.S. (2004). Comparison of delay estimates at under-saturated and over-saturated pre-timed signalized intersections. *Transportation Research Part B: Methodological*, 38(2), 99–122.

Gazis, D.C. (1964). Optimum control of a system of oversaturated intersections. *Operations Research*, 12(6), 815–831.

Geroliminis, N., Haddad, J., and Ramezani, M. (2013). Optimal perimeter control for two urban regions with macroscopic fundamental diagrams: A model predictive approach. *IEEE Transactions on Intelligent Transportation Systems*, 14(1), 348–359.

Haddad, J., Schutter, B.D., Mahalel, D., Ioslovich, I., and Gutman, P.O. (2010). Optimal steady-state control for isolated traffic intersections. *IEEE Transactions on Automatic Control*, 55(11), 2612–2617.

Haddad, J., Mahalel, D., Ioslovich, I., and Gutman, P.O. (2014). Constrained optimal steady-state control for isolated traffic intersections. *Control Theory and Technology*, 12(1), 84–94.

Ioslovich, I., Haddad, J., Gutman, P.O., and Mahalel, D. (2011). Optimal traffic control synthesis for an isolated intersection. *Control Engineering Practice*, 19(8), 900–911.

Keyvan-Ekbatani, M., Kouvelas, A., Papamichail, I., and Papageorgiou, M. (2012). Exploiting the fundamental diagram of urban networks for feedback-based gating. *Transportation Research Part B: Methodological*, 46(10), 1393 – 1403.

Kosmatopoulos, E.B., Papageorgiou, M., Vakouli, A., and Kouvelas, A. (2007). Adaptive fine-tuning of nonlinear control systems with application to the urban traffic control strategy tuc. *IEEE Transactions on Control Systems Technology*, 15(6), 991–1002.

Lighthill, M.J. and Whitham, G.B. (1955). On kinematic waves. ii. a theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 229(1178), 317–345.

Liu, H.X., Wu, X., Ma, W., and Hu, H. (2009). Real-time queue length estimation for congested signalized intersections. *Transportation Research Part C: Emerging Technologies*, 17(4), 412–427.

Mahmassani, H.S., Saberi, M., and Zockaie, A. (2013). Urban network gridlock: Theory, characteristics, and dynamics. *Transportation Research Part C: Emerging Technologies*, 36, 480–497.

Michalopoulos, P.G. and Pisharody, V.B. (1981). Derivation of delays based on improved macroscopic traffic models. *Transportation Research Part B: Methodological*, 15(5), 299–317.

Michalopoulos, P.G., Stephanopoulos, G., and Stephanopoulos, G. (1981). An application of shock wave theory to traffic signal control. *Transportation Research Part B: Methodological*, 15(1), 35–51.

Ramezani, M., de Lamberterie, N., Skabardonis, A., and Geroliminis, N. (2016). A link partitioning approach for real-time control of queue spillbacks on congested arterials. *Transportmetrica B: Transport Dynamics*, 1–14.

Ramezani, M. and Geroliminis, N. (2015). Queue profile estimation in congested urban networks with probe data. *Computer-Aided Civil and Infrastructure Engineering*, 30(6), 414–432.

Ramezani, M., Haddad, J., and Geroliminis, N. (2015). Dynamics of heterogeneity in urban networks: aggregated traffic modeling and hierarchical control. *Transportation Research Part B: Methodological*, 74, 1–19.

Richards, P.I. (1956). Shock waves on the highway. *Operations Research*, 4(1), 42–51.

Skabardonis, A. and Geroliminis, N. (2008). Real-time monitoring and control on signalized arterials. *Journal of Intelligent Transportation Systems*, 12(2), 64–74.

Varaiya, P. (2013a). Max pressure control of a network of signalized intersections. *Transportation Research Part C: Emerging Technologies*, 36, 177–195.

Varaiya, P. (2013b). *The Max-Pressure Controller for Arbitrary Networks of Signalized Intersections*, 27–66. Springer New York, New York, NY.

Wagner, P., Gartner, N., Oertel, R., and Lu, T. (2014). Websters delay formula revisited. In *Transportation Research Board 93nd Annual Meeting*. Washington, DC, USA.

Wardrop, J.G. (1952). Road paper, some theoretical aspects of road traffic research. *Proceedings of the institution of civil engineers*, 1(3), 325–362.

Webster, F.V. (1958). Traffic signal settings. Report.