

Concentration-Compactness and Finite-Time Singularities for Chen’s Flow

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Abstract. Chen’s flow is a fourth-order curvature flow motivated by the spectral decomposition of immersions, a program classically pushed by B.-Y. Chen since the 1970s. In curvature flow terms the flow sits at the critical level of scaling together with the most popular extrinsic fourth-order curvature flow, the Willmore and surface diffusion flows. Unlike them however the famous Chen conjecture indicates that there should be no stationary nonminimal data, and so in particular the flow should drive all closed submanifolds to singularities. We investigate this idea, proving that (1) closed data becomes extinct in finite time in all dimensions and for any codimension; (2) singularities are characterised by concentration of curvature in L^n for intrinsic dimension $n \in \{2, 4\}$ and any codimension (a Lifespan Theorem); and (3) for $n = 2$ and in any codimension, there exists an explicit ε_2 such that if the L^2 norm of the tracefree curvature is initially smaller than ε_2 , the flow remains smooth until it shrinks to a point, and that the blowup of that point is an embedded smooth round sphere.

1. Introduction

Suppose $f : M^n \rightarrow \mathbb{R}^N$, $N > n$ is a smooth isometric immersion. We assume that M^n is closed and complete. Denote by \vec{H} the mean curvature vector of f . Then

$$(\Delta f)(p) = \vec{H}(p)$$

for all $p \in M^n$, where Δ here refers to the rough Laplacian. The rough Laplacian is that induced by the connection on the pullback bundle $f^*(T\mathbb{R}^{n+1})$. Applying the operator again yields

$$(\Delta^2 f)(p) = (\Delta \vec{H})(p).$$

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If $\Delta^2 f \equiv 0$, we call f *biharmonic*. Chen's conjecture is the statement that $\Delta \vec{H} \equiv 0$ implies $\vec{H} \equiv 0$. This conjecture is motivated by Chen's work in the spectral decomposition of immersed submanifolds. There has been much activity on the conjecture (see as a sample the recent papers [2, 8, 19, 20, 26, 27, 28, 29, 30, 34, 38] and Chen's recent survey [6]), but still it remains open.

In this paper we study the heat flow for Δ^2 : this is a one-parameter family of smooth isometric immersions $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ satisfying $f(p, 0) = f_0(p)$ for a given smooth isometric immersion $f_0 : M^n \rightarrow \mathbb{R}^N$ and

$$(CF) \quad (\partial_t f)(p, t) = -(\Delta^2 f)(p, t),$$

for all $(p, t) \in M^n \times (0, T)$. We call (CF) *Chen's flow* and f_0 the initial data. Since Δ^2 is a fourth-order quasilinear elliptic operator, local existence and uniqueness for (CF) is standard. Details can be found in [1, Chapter 3]. See also [9, Chapter 5], [33] and [18].

THEOREM 1. *Let $f_0 : M^n \rightarrow \mathbb{R}^N$ be a smooth closed isometrically immersed submanifold. There exists a $T \in (0, \infty]$ and unique one-parameter family of smooth closed isometric immersions $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ such that (CF) is satisfied and T is maximal.*

Note that *maximal* above means that there does not exist another family $\hat{f} : M^n \times [0, \hat{T}) \rightarrow \mathbb{R}^N$ of smooth closed isometrically immersed hypersurfaces satisfying (CF), $\hat{f}(p, 0) = f_0(p)$ with $\hat{T} > T$.

A simple consequence of the argument used by Jiang [10] is that there are no closed biharmonic submanifolds of Euclidean space. Therefore it is natural to expect that the flow may only exist for at most finite time, that is, $T < \infty$. The following result gives a precise estimate, sharp for $n \in \{2, 3, 4\}$.

THEOREM 2. *Chen's flow $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ with smooth, closed initial data $f_0 : M^n \rightarrow \mathbb{R}^N$ has finite maximal time of existence, with the explicit estimate*

$$(1) \quad T \leq \frac{\mu(f_0)^{\frac{4}{n}}}{C_n},$$

where for $n \in \{2, 3, 4\}$ we have $C_n = 4\omega_n^{\frac{4}{n}}n^2$, and for $n > 4$ we have $C_n = \frac{\omega_n^{\frac{4}{n}}}{n^2 4^{4n+3}}$. Here ω_n denotes the area of the unit n -sphere. Furthermore, if equality is achieved in (1), then $\mu(f_t) \searrow 0$ as $t \nearrow T$.

REMARK 1. Round spheres are driven to points under Chen's flow with $T = \frac{r_0^4}{4n^2}$, where r_0 is the initial radius. This shows that the estimate (1) is sharp in dimensions 2, 3 and 4. We expect that the same estimate holds in higher dimensions.

Given Theorem 2, it is natural to ask for a classification of finite-time singularities. For higher-order curvature flow such as Chen's flow, such classifications are very difficult. For example, a classification of singular geometries remains well open for the two most popular extrinsic fourth-order curvature flow, that is, the Willmore flow and the surface diffusion flow (see for example [14, 15, 16, 21, 24, 35, 36, 37, 39]).

For both the surface diffusion and Willmore flows, the general principle of concentration or compactness from the classical theory of harmonic map heat flow remains valid. We are able to obtain a similar result here: We present the following characterisation of finite-time singularities, also called a concentration-compactness alternative or lifespan theorem.

THEOREM 3. *Let $n \in \{2, 4\}$. There exist constants $\varepsilon_1 > 0$ and $c < \infty$ depending only on n and N with the following property. Let $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ be a Chen flow with smooth initial data.*

(Case 1: $n = 2$.) *Let ρ be chosen such that*

$$(2) \quad \int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^N.$$

Then the maximal time T of smooth existence satisfies

$$T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq c\varepsilon_1 \quad \text{for all } t \in \left[0, \frac{1}{c} \rho^4\right].$$

(Case 2: $n = 4$.) Let ρ be chosen such that

$$(3) \quad \int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^N.$$

Then the maximal time T of smooth existence satisfies

$$T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$(4) \quad \int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu \leq c\varepsilon_1 \quad \text{for all } t \in \left[0, \frac{1}{c} \rho^4\right].$$

REMARK 2. Our proof applies to a general class of flows, including the Willmore flow and the surface diffusion flow. This is new for the Willmore flow and the surface diffusion flow in four dimensions (the two dimensional case for the Willmore flow is the main result of [15], and a corresponding theorem surface diffusion flow is contained in [36]). In three dimensions a lifespan theorem for the surface diffusion flow is known [36], however the constants (ε_1, c) there for $n = 3$ depend on the measure of the initial data. Here, new estimates enable our constants to be universal.

The main result of [15] and the lifespan theorems from [21, 23, 35, 36, 37] (assuming the external force vanishes identically) are generalised by our work here. See Theorem 20 for a precise statement.

The concentration phenomenon that Theorem 3 guarantees can be seen as follows. If $\rho(t)$ denotes the largest radius such that either of the concentration conditions ((2) or (3)) holds at time t , then $\rho(t) \leq \sqrt[4]{c(T-t)}$ and so at least ε_1 of the curvature (or its derivative if $n = 4$) concentrates in a ball $f^{-1}(B_{\rho(T)}(x))$. That is,

$$\begin{cases} (n = 2) & \lim_{t \rightarrow T} \int_{f^{-1}(B_{\rho(t)}(x))} |A|^2 d\mu \geq \varepsilon_1, \\ (n = 4) & \lim_{t \rightarrow T} \int_{f^{-1}(B_{\rho(t)}(x))} |A|^4 + |\nabla A|^2 d\mu \geq \varepsilon_1, \end{cases}$$

where $x = x(t)$ is understood to be the centre of a ball where the integral above is maximised. In either case, this implies that a blowup of such a singularity will be nontrivial.

Although Theorem 3 yields a characterisation of finite time singularities as space-time concentrations of curvature, it does not give any information at all about the asymptotic geometry of such a singularity. One of the simplest observations in this direction is that for spherical initial data with radius r_0 , the flow shrinks homothetically to a point with maximal time

$$T = \frac{r_0^4}{4n^2}.$$

As the evolution is homothetic, parabolic rescaling about the space-time singularity reveals a standard round sphere. This asymptotic behaviour is called *shrinking to a round point*.

One may therefore hope that this behaviour holds in a neighbourhood of a sphere. This is our final result of the paper, proved using blowup analysis.

THEOREM 4. *There exists an absolute constant $\varepsilon_2 > 0$ depending only on N such that if $f : M^2 \times [0, T) \rightarrow \mathbb{R}^N$ is Chen's flow satisfying*

$$(5) \quad \int_M |A^\circ|^2 d\mu \Big|_{t=0} \leq \varepsilon_2 < 8\pi$$

then $T < \infty$, and $f(M^2, t)$ shrinks to a round point as $t \rightarrow T$.

This paper is organised as follows. In Section 2 we describe our notation, some fundamental identities, and the Chen flow in the normal bundle. Section 3 gives evolution equations and the proof of Theorem 2. Our analysis throughout the paper relies on control obtained via localised integral estimates. The key tools that facilitate this are the Michael-Simon Sobolev inequality [25] and the divergence theorem. Section 4 contains the consequences of these that we need here, and proofs of all new statements. One especially long proof is delayed to the Appendix. Integral estimates valid along the flow are also proved in Section 4, including control on the local growth of the L^n norm of A . The section is concluded with a proof of the lifespan theorem. Section 5 is concerned with global analysis for the flow, and contains a proof of the monotonicity result for the L^2 norm of A , and blowup analysis, ending in the proof of Theorem 4.

2. Notation and the Normal Flow

Let us first collect various general formulae from the differential geometry of submanifolds which we need for later analysis. We use notation similar to that of Kuwert-Schätzle [14, 15, 16], Hamilton [11] and Huisken [12, 13]. We have as our principal object of study a smooth isometric immersion $f : M^n \rightarrow \mathbb{R}^N$ of a Riemannian manifold $(M^n, g) = (M^n, f^* \delta^{\mathbb{R}^N})$ into \mathbb{R}^N .

The induced metric has components

$$(6) \quad g_{ij} = \langle \partial_i f, \partial_j f \rangle,$$

where ∂ denotes the regular partial derivative and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. Integration on Σ is performed with respect to the induced area element

$$(7) \quad d\mu = \sqrt{\det g} \, d\mathcal{L}^n,$$

where $d\mathcal{L}^n$ is the standard Hausdorff measure on \mathbb{R}^n .

The *second fundamental form* A is a symmetric $(0, 2)$ tensor field in the normal bundle of f with components

$$(8) \quad A_{ij} = (\partial_{ij}^2 f)^\perp$$

There are two invariants of A relevant to our work here: the first is the trace with respect to the metric

$$\vec{H} = \text{trace}_g A = g^{ij} A_{ij}$$

called the *mean curvature vector*, and the second the *tracefree second fundamental form* defined by

$$A_{ij}^o = A_{ij} - \frac{1}{n} \vec{H} g_{ij}.$$

We define the *Gauss curvature* to be

$$K = \frac{1}{2} (|\vec{H}|^2 - |A|^2).$$

From (8) and the smoothness of f we see that the second fundamental form is symmetric; less obvious but equally important is the symmetry of the first covariant derivatives of A :

$$\nabla_i A_{jk} = \nabla_j A_{ik} = \nabla_k A_{ij};$$

these are the Codazzi equations. In the case here of high codimension they follow with ∇ the connection induced in the normal bundle along f from the fact that the ambient space has constant curvature.

One basic consequence of the Codazzi equations which we shall make use of is that the gradient of the mean curvature is completely controlled by a contraction of the $(0, 3)$ tensor ∇A^o . To see this, first note that

$$\nabla_j A_j^i = \nabla_i \vec{H} = \nabla_i \left((A^o)_j^i + \frac{1}{n} g_j^i \vec{H} \right),$$

then factorise to find

$$(9) \quad \nabla_j \vec{H} = 2 \nabla_i (A^o)_j^i =: \frac{n}{n-1} (\nabla^* A^o)_j.$$

This in fact shows that all derivatives of A are controlled by derivatives of A^o . For a (p, q) tensor field T , let us denote by $\nabla_{(n)} T$ the tensor field with components $\nabla_{i_1 \dots i_n} T_{j_1 \dots j_q}^{k_1 \dots k_p} = \nabla_{i_1} \dots \nabla_{i_n} T_{j_1 \dots j_q}^{k_1 \dots k_p}$. In our notation, the i_n -th covariant derivative is applied first. Since

$$\nabla_{(k)} A = \left(\nabla_{(k)} A^o + \frac{1}{n} g \nabla_{(k)} \vec{H} \right) = \left(\nabla_{(k)} A^o + \frac{1}{n-1} g \nabla_{(k-1)} \nabla^* A^o \right),$$

we have

$$(10) \quad |\nabla_{(k)} A|^2 \leq \frac{2n-1}{n-1} |\nabla_{(k)} A^o|^2.$$

The fundamental relations between components of the Riemann curvature tensor R_{ijkl} , the Ricci tensor R_{ij} and scalar curvature R are given by Gauss' equation

$$R_{ijkl} = \langle A_{ik}, A_{jl} \rangle - \langle A_{il}, A_{jk} \rangle,$$

with contractions

$$g^{jl} R_{ijkl} = R_{ik} = \left\langle \vec{H}, A_{ik} \right\rangle - \left\langle A_i^j, A_{jk} \right\rangle, \text{ and}$$

$$g^{ik} R_{ik} = R = |H|^2 - |A|^2.$$

We will need to interchange covariant derivatives. For a $(0, m)$ -tensor T normal along f we have

$$(11) \quad \nabla_{ij}T = \nabla_{ji}T + R_{ij}^\perp T$$

where

$$R_{ij}^\perp T = A_{ki} \langle A_{kj}, T \rangle - A_{kj} \langle A_{ki}, T \rangle = A_{ki}^o \langle A_{kj}^o, T \rangle - A_{kj}^o \langle A_{ki}^o, T \rangle .$$

Note that $\langle R_{ij}^\perp T, T \rangle = 0$

We also use for normal tensor fields T and S the notation $T * S$ to denote a linear combination of new tensors, each formed by contracting pairs of indices from T and S by the metric g with multiplication by a universal constant. The resultant tensor will have the same type as the other quantities in the expression it appears. We denote polynomials in the iterated normal derivatives of T by

$$P_j^i(T) = \sum_{k_1 + \dots + k_j = i} c_{ij} \nabla_{(k_1)} T * \dots * \nabla_{(k_j)} T,$$

where the constants $c_{ij} \in \mathbb{R}$ are absolute. As is common for the $*$ -notation, we slightly abuse these constants when certain subterms do not appear in our P -style terms. For example

$$|\nabla A|^2 = \langle \nabla A, \nabla A \rangle = 1 \cdot (\nabla_{(1)} A * \nabla_{(1)} A) + 0 \cdot (A * \nabla_{(2)} A) = P_2^2(A).$$

Using the Codazzi equation with the interchange of covariant derivative formula given above, we obtain Simons' identity [32]:

$$(12) \quad \Delta A = \nabla_{(2)} H + A * A * A.$$

The interchange of covariant derivatives formula for mixed tensor fields T is simple to state in $*$ -notation:

$$(13) \quad \nabla_{ij}T = \nabla_{ji}T + T * A * A.$$

Let $\{\partial_1 f, \dots, \partial_n f\}$ be an orthonormal basis for $T_p M$ and $\{\nu_1, \dots, \nu_{N-n}\}$ be an orthonormal basis for $N_p M$ with Christoffel symbols in the normal bundle

vanishing at p , that is, $\bar{\Gamma}(p) = 0$. We call such a frame for $T_pM \otimes N_pM$ a *normal frame*. Then

$$(14) \quad \begin{aligned} \partial_i \nu_\alpha &= \langle \partial_i \nu_\alpha, \partial_k f \rangle \partial_k f + \langle \partial_i \nu_\alpha, \nu_\beta \rangle \nu_\beta \\ &= - \left\langle A_i^k, \nu_\alpha \right\rangle \partial_k f + \bar{\Gamma}_{i\alpha}^\beta \nu_\beta = - \left\langle A_i^k, \nu_\alpha \right\rangle \partial_k f \end{aligned}$$

so that

$$(15) \quad \begin{aligned} \Delta \nu_\alpha &= -g^{ij} \left(\left\langle \partial_j A_i^k, \nu_\alpha \right\rangle \partial_k f + \left\langle A_i^k, \partial_j \nu_\alpha \right\rangle \partial_k f + \left\langle A_i^k, \nu_\alpha \right\rangle \partial_j \partial_k f \right) \\ &= - \left\langle \nabla \vec{H}, \nu_\alpha \right\rangle - \left\langle A^{ij}, \nu_\alpha \right\rangle A_{ij}. \end{aligned}$$

In most of our integral estimates, we include a function $\gamma : M^n \rightarrow \mathbb{R}$ in the integrand. Eventually, this will be specialised to a smooth cutoff function on the preimage of balls on \mathbb{R}^{n+1} via the immersion f . For now however, let us only assume that $\gamma = \tilde{\gamma} \circ f$, where

$$(17) \quad 0 \leq \tilde{\gamma} \leq 1, \quad \text{and} \quad \|\tilde{\gamma}\|_{C^2(\mathbb{R}^{n+1})} \leq c_\gamma < \infty.$$

Using the chain rule, this implies $D\gamma = (D\tilde{\gamma} \circ f)Df$ and then $D^2\gamma = (D^2\tilde{\gamma} \circ f)(Df, Df) + (D\tilde{\gamma} \circ f)D^2f(\cdot, \cdot)$. A routine calculation shows that there exists a constant $c_\gamma = c_\gamma(c_\gamma) \in \mathbb{R}$ such that

$$(18) \quad \begin{aligned} |\nabla \gamma| &\leq c c_\gamma, \quad |\nabla_{(2)} \gamma| \leq c c_\gamma (c_\gamma + |A|), \quad \text{and} \\ |\nabla_{(3)} \gamma| &\leq c c_\gamma (c_\gamma^2 + c_\gamma |A| + |A|^2 + |\nabla A|). \end{aligned}$$

When we write ‘‘for a function $\gamma : M^n \rightarrow \mathbb{R}$ as in (17)’’ we mean a function $\gamma : M^n \rightarrow \mathbb{R}$ as above, satisfying all conditions labeled (17), which additionally achieves the values zero and one in at least two points on M^n .

We note that if $\tilde{\gamma}$ is a cutoff function on a ball in \mathbb{R}^{n+1} of radius ρ , then we may choose $c_\gamma = \frac{c}{\rho}$ where c is a universal constant and we have used that $c_{\tilde{\gamma}} = c_{\tilde{\gamma}}(\rho)$.

2.1. The normal flow

Chen's flow has tangential and normal components. We calculate in a normal frame using (15):

$$\begin{aligned} \Delta^2 f &= \Delta \vec{H} = \Delta \left(\left\langle \vec{H}, \nu_\alpha \right\rangle \nu_\alpha \right) = \Delta (H_\alpha \nu_\alpha) \\ &= (\Delta H_\alpha) \nu_\alpha + H_\alpha \Delta \nu_\alpha + 2 \langle \nabla H_\alpha, \nabla \nu_\alpha \rangle \\ &= \Delta H_\alpha \nu_\alpha - H_\alpha A_{ij} \langle A^{ij}, \nu_\alpha \rangle + \left(2 \langle \nabla H_\alpha, \nabla \nu_\alpha \rangle - H_\alpha \left\langle \nabla \vec{H}, \nu_\alpha \right\rangle \right). \end{aligned}$$

To see that the bracketed term is tangential, we compute using (14):

$$\begin{aligned}
& \left\langle 2 \langle \nabla H_\alpha, \nabla \nu_\alpha \rangle - H_\alpha \langle \nabla \vec{H}, \nu_\alpha \rangle, \nu_\beta \right\rangle \\
&= \left\langle -2g^{ij} \partial_i H_\alpha \langle A_j^k, \nu_\alpha \rangle \partial_k f - H_\alpha g^{ik} \langle \partial_i \vec{H}, \nu_\alpha \rangle \partial_k f, \nu_\beta \right\rangle \\
&= \left(-2g^{ij} \partial_i H_\alpha \langle A_j^k, \nu_\alpha \rangle - H_\alpha g^{ik} \langle \partial_i \vec{H}, \nu_\alpha \rangle \right) \langle \partial_k f, \nu_\beta \rangle \\
&= 0.
\end{aligned}$$

It is a standard result that for closed curvature flow tangential motion acts in the diffeomorphism group of M^n , which is tantamount to a reparametrisation at each time (see for example [1, Chapter 3]). Therefore Chen's flow is equivalent to the purely normal flow:

$$(\text{NCF}) \quad (\partial_t f)(p, t) = -(\Delta^2 f)^\perp = -(\Delta H_\alpha \nu_\alpha - H_\alpha A_{ij} \langle A^{ij}, \nu_\alpha \rangle) = -\mathbf{F},$$

with initial conditions $f(\cdot, 0) = f_0$. For simplicity we conduct our analysis with this formulation.

Note that we may express the velocity \mathbf{F} in a coordinate invariant manner as

$$\mathbf{F} = \Delta^\perp \vec{H} - Q(A) \vec{H}$$

where Q is a normal endomorphism of NM acting on a section ϕ by

$$Q(A)\phi = A_{ij} \langle A^{ij}, \phi \rangle.$$

The same endomorphism arises in the study of the Willmore flow in high codimension, see for example (2.4) in [15].

3. Finite-Time Singularities and Evolution Equations

The following evolution equations hold (see Lemma 2.2 in [15]):

LEMMA 5. *For $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ evolving by $\partial_t f = -\mathbf{F}$ the following equations hold:*

$$\begin{aligned}
\partial_t g_{ij} &= 2 \langle \mathbf{F}, A_{ij} \rangle, & \partial_t d\mu &= \langle \vec{H}, \mathbf{F} \rangle d\mu, & \partial_t g^{ij} &= -2 \langle \mathbf{F}, A^{ij} \rangle, \\
\partial_t^\perp A_{ij} &= -\nabla_{ij} \mathbf{F} + A_{ik} \langle A_j^k, \mathbf{F} \rangle,
\end{aligned}$$

where $\partial_t^\perp \phi = (\partial_t \phi)^\perp$.

Using the P -notation introduced in the previous section we write the evolution of the second fundamental form as

$$\partial_t^\perp A_{ij} = -\nabla_{ij} \Delta^\perp \vec{H} + (P_3^2 + P_5^0)(A).$$

Interchanging covariant derivatives and applying (12) then gives the following lemma:

LEMMA 6. *For $f : M^n \times [0, T] \rightarrow \mathbb{R}^N$ evolving by (NCF) the following equation holds:*

$$\partial_t^\perp A_{ij} = -(\Delta^\perp)^2 A_{ij} + (P_3^2 + P_5^0)(A).$$

LEMMA 7. *For $f : M^n \times [0, T] \rightarrow \mathbb{R}^N$ evolving by (NCF) the following equation holds:*

$$\partial_t \nabla_{(k)} A_{ij} = -\Delta^2 \nabla_{(k)} A + (P_3^{k+2} + P_5^k)(A).$$

Note that this is exactly the same structure that arises in the Willmore flow. Therefore the $n = 2$ case of the lifespan theorem can be proved using the methods of [15]. For $n = 3$, the work in [36] can be adapted along the lines of [37, 21]. For $n = 4$, different arguments are required.

We now state the evolution of curvature quantities along the flow. The proof is standard, and can be adapted from [15].

LEMMA 8. *Let $f : M^n \times [0, T] \rightarrow \mathbb{R}^N$ be a solution of (NCF) and γ be as in (γ). Suppose $s \geq 2k + 4$. For each $\delta > 0$ there exists a constant $c \in (0, \infty)$ depending only on s, n, N and δ such that the following estimate holds:*

$$\begin{aligned} & \frac{d}{dt} \int_M |\nabla_{(k)} A|^2 \gamma^s d\mu + (2 - \delta) \int_M |\nabla_{(k+2)} A|^2 \gamma^s d\mu \\ & \leq c(c_\gamma)^{2k+4} \int_M |A|^2 \gamma^{s-2k-4} d\mu \\ & \quad + c \int_M \nabla_{(k)} A * \left(P_3^{k+2}(A) + P_5^k(A) \right) \gamma^s d\mu. \end{aligned}$$

Area is monotone under the flow:

LEMMA 9. *Let $n \in \{2, 3, 4\}$. For $f : M^n \times [0, T] \rightarrow \mathbb{R}^N$ evolving by (NCF) we have*

$$\mu(f_t)^{\frac{4}{n}} \leq \mu(f_0)^{\frac{4}{n}} - C_n t$$

where for $n \in \{2, 3, 4\}$ we have $C_n = 4\omega_n^{\frac{4}{n}} n^2$, and for $n > 4$ we have $C_n = \frac{\omega_n^{\frac{4}{n}}}{n^2 4^{4n+3}}$. Here ω_n denotes the area of the unit n -sphere.

PROOF. Differentiating,

$$\begin{aligned} \frac{d}{dt} \mu(f_t) &= \frac{d}{dt} \int_M d\mu \\ &= \int_M \langle \vec{H}, \Delta^\perp \vec{H} \rangle - \langle Q(A) \vec{H}, \vec{H} \rangle d\mu. \end{aligned}$$

Using the estimate $\langle Q(A) \vec{H}, \vec{H} \rangle \geq \frac{1}{n} |\vec{H}|^4$ and the divergence theorem we estimate

$$\frac{d}{dt} \mu(f_t) \leq - \int_M |\nabla \vec{H}|^2 d\mu - \frac{1}{n} \int_M |\vec{H}|^4 d\mu \leq - \frac{1}{n} \int_M |\vec{H}|^4 d\mu.$$

Now we use the inequality

$$(16) \quad \int_M |\vec{H}|^4 d\mu \geq \hat{C}_n \mu(f_t)^{\frac{n-4}{n}},$$

to estimate

$$\frac{n}{4} \left(\mu(f_t)^{\frac{4}{n}} \right)' \leq - \frac{\hat{C}_n}{n}.$$

This implies

$$\mu(f_t)^{\frac{4}{n}} \leq \mu(f_0)^{\frac{4}{n}} - 4 \frac{\hat{C}_n}{n^2} t$$

as required. The inequality (16) follows for $n = 2, 3, 4$ from the fundamental sharp estimate

$$\int_M |\vec{H}|^n d\mu \geq \omega_n n^n$$

of Chen [5]. For $n = 4$ (16) is immediate, whereas for $n = 2, 3$ we first use Hölder's inequality

$$\int_M |\vec{H}|^n d\mu \leq \left(\int_M |\vec{H}|^4 d\mu \right)^{\frac{n}{4}} \mu(f_t)^{\frac{4-n}{4}}$$

and then rearrange, to obtain

$$\int_M |\vec{H}|^4 d\mu \geq \left(\int_M |\vec{H}|^n d\mu \right)^{\frac{4}{n}} \mu(f_t)^{\frac{n-4}{n}} \geq \omega_n^{\frac{4}{n}} n^4 \mu(f_t)^{\frac{n-4}{n}},$$

that is, the estimate (16) with $\hat{C}_n = \omega_n^{\frac{4}{n}} n^4$. For $n > 4$, this argument does not work and we must lose some sharpness in the constant. In this case, we use Theorem 28.4.1 of [4] to estimate $\|H\|_1$ from below in terms of the area scaled appropriately. Such an estimate follows directly from the Michael-Simon Sobolev inequality (see Theorem 2.1 in [25], stated in Theorem 10 below) by an approximation argument:

$$\mu(f_t)^{\frac{n-1}{n}} \leq \frac{4^{n+1}}{\omega_n^{1/n}} \int_M |\vec{H}| d\mu.$$

Using Hölder's inequality we find

$$\int_M |\vec{H}| d\mu \leq \left(\int_M |\vec{H}|^4 d\mu \right)^{\frac{1}{4}} \mu(f_t)^{\frac{3}{4}}$$

and so

$$\int_M |\vec{H}|^4 d\mu \geq \frac{\omega_n^{\frac{4}{n}}}{4^{4n+4}} \mu(f_t)^{\frac{n-4}{n}}.$$

This establishes (16) with $\hat{C}_n = \frac{\omega_n^{\frac{4}{n}}}{4^{4n+4}}$. \square

PROOF OF THEOREM 2. Assume that the flow remains smooth with $T > \frac{\mu(f_0)^{\frac{4}{n}}}{C_n}$. Then, applying Lemma 9 with

$$t = \frac{\mu(f_0)^{\frac{4}{n}}}{C_n}$$

shows that $\mu(f_t) = 0$, contradicting the assumption that the flow remains smooth for $t \in [0, T)$. Therefore either the family f shrink to a point, in which case $T \leq \frac{\mu(f_0)}{C_n}$, or there is a loss of regularity beforehand. In either case we have the estimate (1) as required. \square

4. Integral Estimates with Small Concentration of Curvature

The argument for $n = 3$ and $n = 4$ is by necessity different to that for $n = 2$. This is due to the important role played by the Michael-Simon Sobolev inequality.

THEOREM 10 (Theorem 2.1 in [25]). *Let $f : M^n \rightarrow \mathbb{R}^N$ be a smooth immersed submanifold. Then for any $u \in C_c^1(M)$ we have*

$$\left(\int_M |u|^{n/(n-1)} d\mu \right)^{(n-1)/n} \leq \frac{4^{n+1}}{\omega_n^{1/n}} \int_M |\nabla u| + |u| |\vec{H}| d\mu.$$

Notice the exponent on the left. Our eventual goal for this section is to prove local L^∞ estimates for all derivatives of curvature under a hypothesis that the local concentration of curvature is small. Our main tool to convert L^q bounds to L^∞ bounds is the following theorem, which is an n -dimensional analogue of Theorem 5.6 from [15]. The proof is contained in Appendix A of [37].

THEOREM 11. *Let $f : M^n \rightarrow \mathbb{R}^N$ be a smooth immersed submanifold. For $u \in C_c^1(M)$, $n < p \leq \infty$, $0 \leq \beta \leq \infty$ and $0 < \alpha < 1$ where $\frac{1}{\alpha} = (\frac{1}{n} - \frac{1}{p})\beta + 1$ we have*

$$(17) \quad \|u\|_\infty \leq c \|u\|_\beta^{1-\alpha} (\|\nabla u\|_p + \|\vec{H}u\|_p)^\alpha,$$

where $c = c(p, n, N, \beta)$.

The proof follows ideas from [17] and [15]. Due to the exponent in the Michael-Simon Sobolev inequality (which is itself an isoperimetric obstruction), it is not possible to decrease the lower bound on p , even at the expense of other parameters in the inequality.

For $n = 3$, it is possible to use $p = 4$ in Theorem 11. This means that estimates on the same quantities as in the $n = 2$ case may be used. For $n = 4$ we are not able to use $p = 4$ in Theorem 11. We thus need to estimate new quantities in this case.

LEMMA 12. *Let γ be as in (γ) . Then:*

(i) For an immersed surface $f : M^2 \rightarrow \mathbb{R}^N$, $s \geq 4$, we have

$$\int_M (|A|^2 |\nabla A|^2 + |A|^6) \gamma^s d\mu \leq c \int_{[\gamma>0]} |A|^2 d\mu \int_M (|\nabla_{(2)} A|^2 + |A|^6) \gamma^s d\mu + c(c_\gamma)^4 \left(\int_{[\gamma>0]} |A|^2 d\mu \right)^2,$$

where $c = c(s, N)$ is an absolute constant.

(ii) For an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 2$, we have

$$\begin{aligned} & \int_M (|\nabla A|^2 |A|^2 + |A|^6) \gamma^s d\mu \\ & \leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + c(\|A\|_{4, [\gamma>0]}^{\frac{4}{3}} + \|A\|_{4, [\gamma>0]}^4) \int_M |A|^6 \gamma^s d\mu \\ & \quad + (c_\gamma)^2 \|A\|_{4, [\gamma>0]}^4, \end{aligned}$$

and for $s \geq 4$ we have

$$\begin{aligned} & \int_M (|\nabla A|^2 |A|^3 + |A|^7) \gamma^s d\mu \\ & \leq (c\|A\|_{3, [\gamma>0]}^2 + \theta) \int_M (|\nabla_{(2)} A|^2 |A| + |\nabla A|^2 |A|^3 + |A|^7) \gamma^s d\mu \\ & \quad + (c_\gamma)^4 \|A\|_{3, [\gamma>0]}^3, \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(iii) For an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 8$, we have

$$\begin{aligned} & \int_M (|A|^2 |\nabla_{(2)} A|^2 + |A|^4 |\nabla A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu \\ & + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\ & \leq (\theta + c\|A\|_{4, [\gamma>0]}^{\frac{4}{3}}) \\ & \quad \times \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu \\ & \quad + c\|A\|_{4, [\gamma>0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\ & \quad + c\|A\|_{4, [\gamma>0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \end{aligned}$$

$$+ c(c_\gamma)^4 \left(1 + \|A\|_{4, [\gamma > 0]}^4 + [(c_\gamma)^4 \mu_\gamma(f)]^6 \right) \|A\|_{4, [\gamma > 0]}^3,$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(iv) for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 16$, we have

$$\begin{aligned} & \int_M (|\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2 \\ & \quad + |\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\ & + \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \leq (\theta + c \|A\|_{4, [\gamma > 0]}^4) \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla_{(2)} A|^2 |A|^4 \\ & \quad + |\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\ & \quad + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\ & \quad + c(\theta + \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}}) \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \quad + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^2 (1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}}) (1 + \|A\|_{4, [\gamma > 0]}^2). \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(v) for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 4$, we have

$$(18) \quad \begin{aligned} & \int_M |A|^8 \gamma^s d\mu \\ & \leq c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M (|\nabla A|^4 + |A|^8) \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^{\frac{16}{3}} \end{aligned}$$

and

$$(19) \quad \begin{aligned} & \int_M |\nabla A|^4 \gamma^s d\mu \\ & \leq c \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^4 \end{aligned}$$

where $c = c(s, N)$ is an absolute constant.

We postpone the proof of Lemma 12 to the Appendix. Under an appropriate smallness condition, many terms can be absorbed, yielding the following Corollary.

COROLLARY 13. *Let γ be as in (γ) . There exists an $\varepsilon > 0$ depending only on n , s , and N such that if*

$$\int_{[\gamma>0]} |A|^n d\mu \leq \varepsilon \leq 1$$

we have

(i) *for an immersed surface $f : M^2 \rightarrow \mathbb{R}^N$, $s \geq 4$:*

$$\int_M (|A|^2 |\nabla A|^2 + |A|^6) \gamma^s d\mu \leq c\varepsilon \left(\int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + (c_\gamma)^4 \right)$$

where $c = c(s, N)$ is an absolute constant.

(ii) *for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 2$, we have*

$$\int_M (|\nabla A|^2 |A|^2 + |A|^6) \gamma^s d\mu \leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + (c_\gamma)^2 \|A\|_{4, [\gamma>0]}^4,$$

and for $s \geq 4$ we have

$$\begin{aligned} & \int_M (|\nabla A|^2 |A|^3 + |A|^7) \gamma^s d\mu \\ & \leq (c \|A\|_{3, [\gamma>0]}^2 + \theta) \int_M |\nabla_{(2)} A|^2 |A| \gamma^s d\mu + (c_\gamma)^4 \|A\|_{3, [\gamma>0]}^3, \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(iii) *for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 8$, we have*

$$\begin{aligned} & \int_M (|A|^2 |\nabla_{(2)} A|^2 + |A|^4 |\nabla A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu \\ & \leq (\theta + c\varepsilon^{\frac{1}{3}}) \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\ & \quad + c(c_\gamma)^4 \left(1 + \varepsilon + [(c_\gamma)^4 \mu_\gamma(f)]^6 \right) \|A\|_{4, [\gamma>0]}^3, \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(iv) for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 16$, we have

$$\begin{aligned} & \int_M (|\nabla_{(3)}A|^2|A|^2 + |\nabla_{(2)}A|^2|A|^4 + |\nabla_{(2)}A|^2|\nabla A|^2 \\ & \quad + |\nabla A|^2|A|^6 + |A|^{10})\gamma^s d\mu \\ & \leq (\theta + c\|A\|_{4, [\gamma>0]}^4) \int_M |\nabla_{(4)}A|^2\gamma^s d\mu \\ & \quad + c(c_\gamma)^6\|A\|_{4, [\gamma>0]}^2(1 + [(c_\gamma)^4\mu_\gamma(f)]^{\frac{1}{2}})(1 + \|A\|_{4, [\gamma>0]}^2). \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant.

(v) for an immersion $f : M^4 \rightarrow \mathbb{R}^N$, $s \geq 4$, we have

$$\int_M |A|^8 \gamma^s d\mu \leq c\|A\|_{4, [\gamma>0]}^{\frac{4}{3}} \int_M |\nabla A|^4 \gamma^s d\mu + c(c_\gamma)^4\|A\|_{4, [\gamma>0]}^{\frac{16}{3}}$$

and

$$\int_M |\nabla A|^4 \gamma^s d\mu \leq c \int_M |\nabla_{(2)}A|^2|A|^2 \gamma^s d\mu + c(c_\gamma)^4\|A\|_{4, [\gamma>0]}^4$$

where $c = c(s, N)$ is an absolute constant.

Next we give a local refinement of Theorem 11.

PROPOSITION 14. *Suppose γ is as in (γ) . For any tensor T normal along $f : M^n \rightarrow \mathbb{R}^N$, if $n = 2$, we have*

$$(20) \quad \begin{aligned} & \|T\|_{\infty, [\gamma=1]}^4 \\ & \leq c\|T\|_{2, [\gamma>0]}^2 (\|\nabla_{(2)}T\|_{2, [\gamma>0]}^2 + (c_\gamma)^4\|T\|_{2, [\gamma>0]}^2 + \|TA^2\|_{2, [\gamma>0]}^2), \end{aligned}$$

and if $n = 4$, then we have

$$(21) \quad \begin{aligned} \|T\|_{\infty, [\gamma=1]}^3 & \leq c\|T\|_{2, [\gamma>0]} \left(\|\nabla_{(3)}T\|_{2, [\gamma>0]}^2 + \|TA^3\|_{2, [\gamma>0]}^2 \right. \\ & \quad + (c_\gamma)^2\|A\nabla T\|_{2, [\gamma>0]}^2 + (c_\gamma)^2\|T\nabla A\|_{2, [\gamma>0]}^2 \\ & \quad \left. + (c_\gamma)^4\|\nabla T\|_{2, [\gamma>0]}^2 + (c_\gamma)^6\|T\|_{2, [\gamma>0]}^2 \right), \end{aligned}$$

where $c = c(n, N)$.

Assume $T = A$. There exists an $\varepsilon_0 = \varepsilon_0(n, N)$ such that if

$$\|A\|_{n, [\gamma > 0]}^n \leq \varepsilon_0$$

we have for $n = 2$:

$$\|A\|_{\infty, [\gamma=1]}^4 \leq c\varepsilon_0 (\|\nabla_{(2)} A\|_{2, [\gamma > 0]}^2 + \varepsilon_0 (c_\gamma)^4),$$

and for $n = 4$:

$$(22) \quad \|A\|_{\infty, [\gamma=1]}^3 \leq c \|A\|_{2, [\gamma > 0]} \left(\|\nabla_{(3)} A\|_{2, [\gamma > 0]}^2 + (c_\gamma)^4 (1 + \|A\|_{4, [\gamma > 0]}^4 + (c_\gamma)^4 \mu_\gamma(f)) \right),$$

with $c = c(n, N, \varepsilon_0)$.

PROOF. The proof proceeds in two parts: first we deal with the case where $n = 2$. Then we prove the statements for $n = 4$. In each part we will estimate an arbitrary tensor field S , and then we will localise the estimate for S by using a γ function. Precisely, we specialise the estimate for S to $S = T\gamma^2$ in the first part and $S = T\gamma^4$ in the second, taking care to factor out the quantity $\|T\|_{2, [\gamma > 0]}^2$ to conclude our desired inequality. Note that for $n = 2$ the result is in [15] (except here we keep track of c_γ , and in the relevant result from [15] the constant c depends on γ).

Here, and until we deal with the case $n = 4$, we leave n as a free parameter. This is because the proof below works for both $n = 2$ and $n = 3$. Therefore, let us take $p = 4$, $\beta = 2$ in Theorem 11 to obtain

$$(23) \quad \|S\|_\infty \leq c \|S\|_2^{\frac{4-n}{n+4}} (\|\nabla S\|_4 + \|S \vec{H}\|_4)^{\frac{2n}{n+4}}.$$

We now use integration by parts and the Hölder inequality to derive

$$(24) \quad \begin{aligned} \|\nabla S\|_4^4 &\leq \int_M S * (\nabla_{(2)} S |\nabla S|^2 + 2\nabla S * \nabla S * \nabla_{(2)} S) d\mu \\ &\leq c \|S\|_\infty \|\nabla S\|_4^2 \|\nabla_{(2)} S\|_2, \text{ so} \\ \|\nabla S\|_4 &\leq c \|S\|_\infty^{\frac{1}{2}} \|\nabla_{(2)} S\|_2^{\frac{1}{2}}. \end{aligned}$$

Combine inequality (24) with (23) and use Jensen's inequality to obtain

$$(25) \quad \|S\|_\infty \leq c \|S\|_2^{\frac{4-n}{n+4}} \left[(\|S\|_\infty^{\frac{1}{2}} \|\nabla_{(2)} S\|_2^{\frac{1}{2}})^{\frac{2n}{n+4}} + \|S \vec{H}\|_4^{\frac{2n}{n+4}} \right].$$

Using Hölder's inequality we estimate

$$\|S \vec{H}\|_4^{\frac{2n}{n+4}} \leq \left(\|S^2\|_\infty^{\frac{1}{4}} \|S^{\frac{1}{2}} \vec{H}\|_4 \right)^{\frac{2n}{n+4}} \leq \|S\|_\infty^{\frac{n}{n+4}} \|S^{\frac{1}{2}} \vec{H}\|_4^{\frac{2n}{n+4}},$$

and combining this with (25) above we conclude

$$(26) \quad \begin{aligned} \|S\|_\infty^4 &= \left(\|S\|_\infty^{1-\frac{n}{n+4}} \right)^{n+4} \\ &\leq \left(c \|S\|_2^{\frac{4-n}{n+4}} (\|\nabla_{(2)} S\|_2^{\frac{n}{n+4}} + \|S^{\frac{1}{2}} \vec{H}\|_4^{\frac{2n}{n+4}}) \right)^{n+4} \\ &\leq c \|S\|_2^{4-n} (\|\nabla_{(2)} S\|_2^n + \|S |\vec{H}|^2\|_2^n). \end{aligned}$$

We now turn our attention to localising the estimate for S . As mentioned earlier, for this purpose we set $S = T\gamma^2$. We first evaluate and estimate the second derivative term $\|\nabla_{(2)} S\|_2^2$:

$$(27) \quad \begin{aligned} \|\nabla_{(2)} S\|_2^2 &= \int_M |\nabla_{(2)}(T\gamma^2)|^2 d\mu \\ &\leq c \left(\int_M |\nabla_{(2)} T|^2 \gamma^4 d\mu + \int_M |\nabla T|^2 |\nabla \gamma^2|^2 d\mu + \int_M |T|^2 |\nabla_{(2)} \gamma^2|^2 d\mu \right) \\ &\leq c \left(\int_M |\nabla_{(2)} T|^2 \gamma^4 d\mu + \int_M |\nabla T|^2 |\nabla \gamma|^2 \gamma^2 d\mu \right. \\ &\quad \left. + \int_M |T|^2 [|\nabla_{(2)} \gamma|^2 + |\nabla \gamma|^2]^2 d\mu \right) \\ &\leq c \left(\int_M |\nabla_{(2)} T|^2 \gamma^4 d\mu + (c_\gamma)^2 \int_M |\nabla T|^2 \gamma^2 d\mu \right. \\ &\quad \left. + (c_\gamma)^2 \int_M |T|^2 |A|^2 \gamma^2 d\mu + (c_\gamma)^4 \int_{[\gamma>0]} |T|^2 d\mu \right). \end{aligned}$$

We interpolate the first derivative term:

$$(c_\gamma)^2 \int_M |\nabla T|^2 \gamma^2 d\mu \leq (c_\gamma)^2 \int_M |T| |\nabla_{(2)} T| \gamma^2 d\mu + c(c_\gamma)^3 \int_M |T| |\nabla T| \gamma d\mu$$

$$\begin{aligned} &\leq \frac{1}{2}(c_\gamma)^2 \int_M |\nabla T|^2 \gamma^2 d\mu + c \int_M |\nabla_{(2)} T|^2 \gamma^4 d\mu \\ &\quad + c(c_\gamma)^4 \int_{[\gamma>0]} |T|^2 d\mu \end{aligned}$$

and thus

$$(c_\gamma)^2 \int_M |\nabla T|^2 \gamma^2 d\mu \leq c \int_M |\nabla_{(2)} T|^2 \gamma^4 d\mu + c(c_\gamma)^4 \int_{[\gamma>0]} |T|^2 d\mu.$$

Inserting this result into (27), and estimating

$$c(c_\gamma)^2 \int_M |T|^2 |A|^2 \gamma^2 d\mu \leq c \int_M |T|^2 |A|^4 \gamma^4 d\mu + c(c_\gamma)^4 \int_{[\gamma>0]} |T|^2 d\mu$$

we obtain

$$\begin{aligned} \|\nabla_{(2)} S\|_2^2 &\leq c \int_{[\gamma>0]} |\nabla_{(2)} T|^2 d\mu \\ (28) \quad &\quad + c \int_M |T|^2 |A|^4 \gamma^4 d\mu + c(c_\gamma)^4 \int_{[\gamma>0]} |T|^2 d\mu. \end{aligned}$$

Combining this with our estimate for $\|S\|_\infty$ earlier, inequality (26), gives

$$\begin{aligned} \|S\|_\infty^4 &\leq c \|S\|_2^{4-n} (\|\nabla_{(2)} T\|_{2,[\gamma>0]}^n + (c_\gamma)^{2n} \|T\|_{2,[\gamma>0]}^n \\ &\quad + \|S\|_2^2 \|H^2\|_2^2 + \|TA^2\|_2^2) \\ (29) \quad &\leq c \|T\|_{2,[\gamma>0]}^{4-n} (\|\nabla_{(2)} T\|_{2,[\gamma>0]}^n + (c_\gamma)^{2n} \|T\|_{2,[\gamma>0]}^n + \|TA^2\|_{2,[\gamma>0]}^n). \end{aligned}$$

Estimating $\|T\|_{\infty,[\gamma=1]}^4 \leq \|S\|_\infty^4$ proves (20).

Now set $T = A$ in (20).

For $n = 2$, Lemma 12 (i) implies

$$\int_M |A|^6 \gamma^4 d\mu \leq c \|A\|_{2,[\gamma>0]}^2 (\|\nabla_{(2)} A\|_{2,[\gamma>0]}^2 + \|A\gamma^{\frac{2}{3}}\|_6^6) + c(c_\gamma)^4 \|A\|_{2,[\gamma>0]}^4,$$

and absorbing on the left we obtain

$$\int_M |A|^6 \gamma^4 d\mu \leq c \|A\|_{2,[\gamma>0]}^2 (\|\nabla_{(2)} A\|_{2,[\gamma>0]}^2 + (c_\gamma)^4 \|A\|_{2,[\gamma>0]}^2).$$

Inserting this into (29) gives the second statement for $n = 2$.

For the $n = 4$ inequalities, we proceed similarly. We first claim

$$(30) \quad \|S\|_\infty \leq c \|S\|_2^{\frac{1}{3}} (\|\nabla_{(3)} S\|_2^{\frac{2}{3}} + \|S^{\frac{1}{3}} \vec{H}\|_6^2).$$

In order to prove (30), we need some auxilliary estimates. First, we calculate

$$\begin{aligned} \int_M |\nabla S|^6 &\leq c \int_M |S| |\nabla S|^4 |\nabla_{(2)} S| d\mu \\ &\leq c \|S\|_\infty \|\nabla S\|_6^4 \left(\int_M |\nabla_{(2)} S|^3 d\mu \right)^{\frac{1}{3}} \end{aligned}$$

so

$$(31) \quad \|\nabla S\|_6^6 \leq c \|S\|_\infty^3 \|\nabla_{(2)} S\|_3^3.$$

We also need

$$\begin{aligned} \int_M |\nabla_{(2)} S|^3 d\mu &\leq c \int_M |\nabla S| |\nabla_{(2)} S| |\nabla_{(3)} S| d\mu \\ &\leq \frac{1}{2} \|\nabla_{(2)} S\|_3^3 + c \int_M |\nabla S|^{\frac{3}{2}} |\nabla_{(3)} S|^{\frac{3}{2}} d\mu \end{aligned}$$

so

$$\begin{aligned} \int_M |\nabla_{(2)} S|^3 d\mu &\leq c \int_M |\nabla S|^{\frac{3}{2}} |\nabla_{(3)} S|^{\frac{3}{2}} d\mu \\ &\leq \frac{1}{2c \|S\|_\infty^3} \int_M |\nabla S|^6 d\mu + c \|S\|_\infty \int_M |\nabla_{(3)} S|^2 d\mu. \end{aligned}$$

(Note that if $\|S\|_\infty = 0$ then the estimate is trivially true, and so we assume this is not the case.) Combining with (31) and absorbing we find

$$\|\nabla S\|_6^6 \leq c \|S\|_\infty^4 \|\nabla_{(3)} S\|_2^2.$$

Now applying Theorem 11 yields

$$\begin{aligned} \|S\|_\infty &\leq c \|S\|_\beta^{1-\alpha} (\|\nabla S\|_6 + \|\vec{H} S\|_6)^\alpha \\ &\leq c \|S\|_\beta^{1-\alpha} (\|S\|_\infty^{\frac{2}{3}} \|\nabla_{(3)} S\|_2^{\frac{1}{3}} + \|S\|_\infty^{\frac{2}{3}} \|S^{\frac{1}{3}} \vec{H}\|_6)^\alpha \end{aligned}$$

so

$$\|S\|_\infty \leq c \|S\|_\beta^{\frac{3-3\alpha}{3-2\alpha}} (\|\nabla_{(3)}S\|_2^{\frac{1}{3}} + \|S^{\frac{1}{3}}\vec{H}\|_6)^{\frac{3\alpha}{3-2\alpha}}$$

where $\alpha^{-1} = 1 + (\frac{1}{n} - \frac{1}{6})\beta$. Since $n = 4$, $\alpha^{-1} = \frac{12+\beta}{12}$, and

$$\frac{3\alpha}{3-2\alpha} = \frac{12}{12+\beta} \frac{3}{3-2\frac{12}{12+\beta}} = \frac{36}{36+3\beta-24} = \frac{12}{4+\beta}$$

so in particular if $\beta = 2$ then $3\alpha/(3-2\alpha) = 2$ or $\alpha = 6/7$. We also note that $\frac{3-3\alpha}{3-2\alpha} = \frac{1}{3}$. This proves the estimate (30).

Now we set $S = T\gamma^4$ and calculate

$$\begin{aligned} & \int_M |\nabla_{(3)}(T\gamma^4)|^2 d\mu \\ & \leq c \int_M |\nabla_{(3)}T|^2 \gamma^8 d\mu + c(c_\gamma)^2 \int_M |\nabla_{(2)}T|^2 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |\nabla T|^2 ((c_\gamma^2 + |A|^2)\gamma^2 + c_\gamma^2)\gamma^4 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |T|^2 ((c_\gamma^4 + c_\gamma^2|A|^2 + |A|^4 + |\nabla A|^2)\gamma^4 \\ & \quad \quad + (c_\gamma^2(c_\gamma^2 + |A|^2))\gamma^2 + c_\gamma^4)\gamma^2 d\mu \\ & \leq c \int_M |\nabla_{(3)}T|^2 \gamma^8 d\mu + c(c_\gamma)^2 \int_M |\nabla_{(2)}T|^2 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |\nabla T|^2 |A|^2 \gamma^6 d\mu + c(c_\gamma)^2 \int_M |T|^2 |\nabla A|^2 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |T|^2 |A|^4 \gamma^6 d\mu + c(c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu \\ & \quad + c(c_\gamma)^4 \int_M |T|^2 |A|^2 \gamma^4 d\mu + c(c_\gamma)^6 \int_M |T|^2 \gamma^2 d\mu \\ & \leq c \int_M |\nabla_{(3)}T|^2 \gamma^8 d\mu + c(c_\gamma)^2 \int_M |\nabla_{(2)}T|^2 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |\nabla T|^2 |A|^2 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^2 \int_M |T|^2 |\nabla A|^2 \gamma^6 d\mu + c(c_\gamma)^2 \int_M |T|^2 |A|^4 \gamma^6 d\mu \\ & \quad + c(c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu + c(c_\gamma)^6 \int_M |T|^2 \gamma^2 d\mu. \end{aligned}$$

Note that

$$\begin{aligned}
& (c_\gamma)^2 \int_M |\nabla_{(2)} T|^2 \gamma^6 d\mu \\
& \leq c(c_\gamma)^2 \int_M |\nabla T| |\nabla_{(3)} T| \gamma^6 d\mu + c(c_\gamma)^3 \int_M |\nabla T| |\nabla_{(2)} T| \gamma^5 d\mu \\
& \leq \frac{1}{2}(c_\gamma)^2 \int_M |\nabla_{(2)} T|^2 \gamma^6 d\mu + c(c_\gamma)^2 \int_M |\nabla T| |\nabla_{(3)} T| \gamma^6 d\mu \\
& \quad + c(c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu
\end{aligned}$$

so that

$$(c_\gamma)^2 \int_M |\nabla_{(2)} T|^2 \gamma^6 d\mu \leq \int_M |\nabla_{(3)} T|^2 \gamma^8 d\mu + (c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu.$$

This refines the above to

$$\begin{aligned}
& \int_M |\nabla_{(3)}(T\gamma^4)|^2 d\mu \\
& \leq c \int_M |\nabla_{(3)} T|^2 \gamma^8 d\mu + c(c_\gamma)^2 \int_M |\nabla T|^2 |A|^2 \gamma^6 d\mu \\
& \quad + c(c_\gamma)^2 \int_M |T|^2 |\nabla A|^2 \gamma^6 d\mu + c(c_\gamma)^2 \int_M |T|^2 |A|^4 \gamma^6 d\mu \\
& \quad + c(c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu + c(c_\gamma)^6 \int_M |T|^2 \gamma^2 d\mu.
\end{aligned}$$

Combining this with (30) and then cubing everything yields

$$\begin{aligned}
\|T\gamma^4\|_\infty^3 & \leq c\|T\gamma^4\|_2 \left(\int_M |\nabla_{(3)} T|^2 \gamma^8 d\mu + \int_M |T|^2 |A|^6 \gamma^8 d\mu \right. \\
& \quad + (c_\gamma)^2 \int_M |\nabla T|^2 |A|^2 \gamma^6 d\mu + (c_\gamma)^2 \int_M |T|^2 |\nabla A|^2 \gamma^6 d\mu \\
& \quad + (c_\gamma)^2 \int_M |T|^2 |A|^4 \gamma^6 d\mu + (c_\gamma)^4 \int_M |\nabla T|^2 \gamma^4 d\mu \\
& \quad \left. + (c_\gamma)^6 \int_M |T|^2 \gamma^2 d\mu \right).
\end{aligned}$$

Using the definition of γ we have

$$\begin{aligned}
\|T\|_{\infty, [\gamma=1]}^3 & \leq c\|T\|_{2, [\gamma>0]} \left(\|\nabla_{(3)} T\|_{2, [\gamma>0]}^2 + \|T A^3\|_{2, [\gamma>0]}^2 \right. \\
& \quad \left. + (c_\gamma)^2 \|A \nabla T\|_{2, [\gamma>0]}^2 + (c_\gamma)^2 \|T \nabla A\|_{2, [\gamma>0]}^2 \right)
\end{aligned}$$

$$+ (c_\gamma)^4 \|\nabla T\|_{2, [\gamma > 0]}^2 + (c_\gamma)^6 \|T\|_{2, [\gamma > 0]}^2 \Big).$$

Note that we interpolated one term.

In the particular case where $T = A$ we find

$$\begin{aligned} \|A\|_{\infty, [\gamma=1]}^3 &\leq c \|A\|_{2, [\gamma > 0]} \left(\int_M |\nabla_{(3)} A|^2 \gamma^8 d\mu + \int_M (|\nabla A|^4 + |A|^8) \gamma^8 d\mu \right. \\ &\quad \left. + (c_\gamma)^4 (1 + \|A\|_{4, [\gamma > 0]}^4 + (c_\gamma)^4 \mu_\gamma(f) + (c_\gamma)^2 \|A\|_{2, [\gamma > 0]}^2) \right). \end{aligned}$$

When $\|A\|_{4, [\gamma > 0]}^4$ is small, we may use Corollary 13 (iii) to absorb the second integral on the right, and conclude

$$\begin{aligned} \|A\|_{\infty, [\gamma=1]}^3 &\leq c \|A\|_{2, [\gamma > 0]} \left(\|\nabla_{(3)} A\|_{2, [\gamma > 0]}^2 \right. \\ &\quad \left. + (c_\gamma)^4 (1 + \|A\|_{4, [\gamma > 0]}^4 + (c_\gamma)^4 \mu_\gamma(f) + (c_\gamma)^2 \|A\|_{2, [\gamma > 0]}^2) \right). \quad \square \end{aligned}$$

The Lifespan Theorem is proved using an alternative that relies on being able to, in a weak sense, preserve the assumption

$$\int_{[\gamma > 0]} |A|^n d\mu < \varepsilon_0$$

at later times. A key difficulty is that the flow lives naturally in the L^2 heirarchy, and so does not directly control the L^n norm of curvature. This in turn introduces difficulties in obtaining pointwise control of curvature. For $n = 2$ this does not cause any issue. For $n = 4$ the same Sobolev inequalities can not apply. Nevertheless we are able to use those proved above to obtain pointwise control in this case as well.

We begin with the L^2 -control.

PROPOSITION 15. *Let $n \in \{2, 4\}$. Suppose $f : M^n \times [0, T^*] \rightarrow \mathbb{R}^N$ evolves by (NCF) and γ is a cutoff function as in (γ) . Then there is a universal $\varepsilon_0 = \varepsilon_0(N)$ such that if*

$$(32) \quad \varepsilon = \sup_{[0, T^*]} \int_{[\gamma > 0]} |A|^n d\mu \leq \varepsilon_0$$

then for any $t \in [0, T^*]$ we have

$$\begin{aligned} & \int_{[\gamma=1]} |A|^2 d\mu + \int_0^t \int_{[\gamma=1]} (|\nabla_{(2)}A|^2 + |A|^2|\nabla A|^2 + |A|^6) d\mu d\tau \\ & \leq \int_{[\gamma>0]} |A|^2 d\mu \Big|_{t=0} + ct(c_\gamma)^{6-n} \left(1 + (n-2)(4-n)[(c_\gamma)^3\mu(f_0)]^{\frac{1}{3}} \right. \\ & \qquad \qquad \qquad \left. + (n-2)(n-3)[(c_\gamma)^4\mu(f_0)]^{\frac{1}{2}} \right) \varepsilon^{\frac{2}{n}}, \end{aligned}$$

where $c = c(n, N)$.

PROOF. The idea of the proof is to integrate Lemma 8, and then use the multiplicative Sobolev inequality Lemma 12. This will introduce a multiplicative factor of $\|A\|_{n, [\gamma>0]}$ in front of several integrals, which we can then absorb on the left. The proof for $n = 2$ is the same as that in [15]. Therefore here we give only the proof for $n = 4$.

Setting $k = 0$ and $s = 4$ in Lemma 8 we have

$$\begin{aligned} & \frac{d}{dt} \int_M |A|^2 \gamma^4 d\mu + (2 - \theta) \int_M |\nabla_{(2)}A|^2 \gamma^4 d\mu \\ & \leq c(c_\gamma)^4 \int_{[\gamma>0]} |A|^2 d\mu + c \int_M ([P_3^2(A) + P_5^0(A)] * A) \gamma^4 d\mu. \end{aligned}$$

We estimate the P -style terms:

$$\begin{aligned} & \int_M ([P_3^2(A) + P_5^0(A)] * A) \gamma^4 d\mu \\ & \leq c \int_M \left([|A|^2 \cdot |\nabla_{(2)}A| + |\nabla A|^2 |A| + |A|^5] |A| \right) \gamma^4 d\mu \\ & \leq c \int_M [|A|^3 |\nabla_{(2)}A| + |\nabla A|^2 |A|^2 + |A|^6] \gamma^4 d\mu \\ & \leq \theta_0 \int_M |\nabla_{(2)}A|^2 \gamma^4 d\mu + c \int_M (|A|^6 + |\nabla A|^2 |A|^2) \gamma^4 d\mu. \end{aligned}$$

We use Corollary 13 (ii) to estimate the second integral and obtain

$$(33) \quad \begin{aligned} & \int_M ([P_3^2(A) + P_5^0(A)] * A) \gamma^4 d\mu \\ & \leq \theta \int_M |\nabla_{(2)}A|^2 \gamma^s d\mu + (c_\gamma)^2 \|A\|_{4, [\gamma>0]}^4, \end{aligned}$$

We add the integrals $\int_M |A|^6 \gamma^4 d\mu$ and $\int_M |\nabla A|^2 |A|^2 \gamma^4 d\mu$ to the estimate of Lemma 8 (with $k = 0$, $s = 4$) and find

$$\begin{aligned} & \frac{d}{dt} \int_M |A|^2 \gamma^4 d\mu + (2 - \theta) \int_M (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu \\ & \leq c(c_\gamma)^4 \int_{[\gamma > 0]} |A|^2 d\mu + c \int_M (|A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu \\ & \quad + c \int_M ([P_3^2(A) + P_5^0(A)] * A) \gamma^4 d\mu \\ & \leq c(c_\gamma)^4 \int_{[\gamma > 0]} |A|^2 d\mu + \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + (c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4, \end{aligned}$$

which upon absorbing and choosing θ small yields

$$\begin{aligned} & \frac{d}{dt} \int_M |A|^2 \gamma^4 d\mu + \int_M (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu \\ & \leq c(c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^2 \left((c_\gamma)^2 \mu_\gamma(f_t)^{\frac{1}{2}} + \|A\|_{4, [\gamma > 0]}^2 \right). \end{aligned}$$

Integrating, we have

$$\begin{aligned} & \int_{[\gamma=1]} |A|^2 d\mu + \int_0^t \int_{[\gamma=1]} (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^6) d\mu d\tau \\ & \leq \int_{[\gamma > 0]} |A|^2 d\mu \Big|_{t=0} + ct(c_\gamma)^{6-n} \left(1 + (n-2)(4-n)[(c_\gamma)^3 \mu(f_0)]^{\frac{1}{3}} \right. \\ & \quad \left. + (n-2)(n-3)[(c_\gamma)^4 \mu(f_0)]^{\frac{1}{2}} \right) \varepsilon^{\frac{2}{n}}, \end{aligned}$$

where we have incorporated the three cases into one statement, and used $\varepsilon \leq 1$, $\mu_\gamma(f_t) \leq \mu(f_t) \leq \mu(f_0)$, $[\gamma = 1] \subset [\gamma > 0]$ and $0 \leq \gamma \leq 1$. \square

REMARK 3. It is possible to proceed as in [24] and prove a bound directly for $\mu_\gamma(f_t)$ in terms of $\mu_\gamma(f_0)$, under the smallness hypothesis (32). However this yields a bound exponential in time, which is quickly worse than the simple but uniform in time bound used above. It is an interesting open question on how to control the area locally uniformly in time without resorting to this crude estimate. In order to overcome this issue we prove the following estimate for the scale-invariant $\|\nabla A\|_{2, \gamma^s}^2 + \|A\|_{4, \gamma^s}^4$ directly.

PROPOSITION 16. *Suppose $f : M^4 \times [0, T^*] \rightarrow \mathbb{R}^N$ evolves by (NCF) and γ is a cutoff function as in (γ) . Then there is a universal $\varepsilon_0 = \varepsilon_0(N)$ such that if*

$$\varepsilon = \sup_{[0, T^*]} \int_{[\gamma > 0]} |A|^4 + |\nabla A|^2 d\mu \leq \varepsilon_0$$

then for any $t \in [0, T^*]$ we have

$$\begin{aligned} & \int_{[\gamma=1]} |A|^4 + |\nabla A|^2 d\mu \\ & + \int_0^t \int_{[\gamma=1]} (|\nabla_{(3)} A|^2 + |\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^2 |A|^4 + |\nabla A|^4 + |A|^8) d\mu d\tau \\ & \leq \int_{[\gamma > 0]} |A|^4 + |\nabla A|^2 d\mu \Big|_{t=0} + ct(c_\gamma)^4 \varepsilon_0, \end{aligned}$$

where c depends only on c_γ and N .

PROOF. Let us first calculate

$$\begin{aligned} \frac{d}{dt} \int_M |A|^4 \gamma^s d\mu &= 4 \int_M |A|^2 \langle A, A_t \rangle \gamma^s d\mu + \int_M |A|^4 \langle \vec{H}, \mathbf{F} \rangle \gamma^s d\mu \\ & \quad + s \int_M |A|^4 \gamma_t \gamma^{s-1} d\mu \\ &= 4 \int_M |A|^2 \langle A, -(\Delta^\perp)^2 A + (P_3^2 + P_5^0)(A) \rangle \gamma^s d\mu \\ & \quad + \int_M |A|^4 \langle \vec{H}, (P_1^2 + P_3^0)(A) \rangle \gamma^s d\mu \\ & \quad + s \int_M |A|^4 \gamma_t \gamma^{s-1} d\mu. \end{aligned}$$

Observe that

$$\begin{aligned} & 4 \int_M |A|^2 \langle A, -(\Delta^\perp)^2 A + (P_3^2 + P_5^0)(A) \rangle \gamma^s d\mu \\ & \quad + \int_M |A|^4 \langle \vec{H}, (P_1^2 + P_3^0)(A) \rangle \gamma^s d\mu \\ &= -4 \int_M |A|^2 \langle A, \nabla^p \nabla^q \nabla_q \nabla_p A \rangle \gamma^s d\mu + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu \end{aligned}$$

$$\begin{aligned}
 &= -4 \int_M \langle \nabla_{(2)}(A|A|^2), \nabla_{(2)}A \rangle \gamma^s d\mu \\
 &\quad + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad - 4s \int_M \langle \nabla^p(A|A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu \\
 &= -4 \int_M \langle \nabla^q(\nabla^p A |A|^2 + 2A \langle A, \nabla^p A \rangle), \nabla_{qp} A \rangle \gamma^s d\mu \\
 &\quad + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad - 4s \int_M \langle \nabla^p(A|A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu \\
 &= -4 \int_M \langle \nabla^{qp} A |A|^2 + 2\nabla^p A \langle A, \nabla^q A \rangle + 2\nabla^q A \langle A, \nabla^p A \rangle \\
 &\quad \quad + 2A \langle \nabla^p A, \nabla^q A \rangle + 2A \langle A, \nabla^{qp} A \rangle, \nabla_{qp} A \rangle \gamma^s d\mu \\
 &\quad + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad - 4s \int_M \langle \nabla^p(A|A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu \\
 &= -4 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu - 8 \int_M |\langle A, \nabla_{(2)} A \rangle|^2 \gamma^s d\mu \\
 &\quad - 4 \int_M \langle 2\nabla^p A \langle A, \nabla^q A \rangle + 2\nabla^q A \langle A, \nabla^p A \rangle \\
 &\quad \quad + 2A \langle \nabla^p A, \nabla^q A \rangle, \nabla_{qp} A \rangle \gamma^s d\mu \\
 &\quad + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
 &\quad - 4s \int_M \langle \nabla^p(A|A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu
 \end{aligned}$$

$$\begin{aligned}
& + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu \\
= & -4 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu - 8 \int_M |\langle A, \nabla_{(2)} A \rangle|^2 \gamma^s d\mu \\
& + \int_M (A * \nabla A * \nabla A * \nabla_{(2)} A) \gamma^s d\mu \\
& + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
& - 4s \int_M \langle \nabla^p (A |A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
& + \int_M P_3^0(A) * (P_3^2 + P_5^0)(A) \gamma^s d\mu
\end{aligned}$$

Using $\gamma = \tilde{\gamma} \circ f$, we combine this with the evolution of $\|A\|_{4, \gamma^{s/4}}^4$ and estimate to find

$$\begin{aligned}
& \frac{d}{dt} \int_M |A|^4 \gamma^s d\mu \\
& \leq -4 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu - 8 \int_M |\langle A, \nabla_{(2)} A \rangle|^2 \gamma^s d\mu \\
& \quad + 4s \int_M |A|^2 \langle A \nabla^p \gamma, \nabla^q \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
& \quad - 4s \int_M \langle \nabla^p (A |A|^2) \nabla^q \gamma, \nabla_q \nabla_p A \rangle \gamma^{s-1} d\mu \\
& \quad + \int_M (A * \nabla A * \nabla A * \nabla_{(2)} A + P_3^0(A) * P_3^2(A) + P_8^0(A)) \gamma^s d\mu \\
& \quad + s \int_M |A|^4 \gamma_t \gamma^{s-1} d\mu \\
& \leq -4 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
& \quad + \int_M (A * \nabla A * \nabla A * \nabla_{(2)} A + P_3^0(A) * P_3^2(A) + P_8^0(A)) \gamma^s d\mu \\
& \quad + c(c_\gamma) \int_M (|A|^4 |\nabla_{(2)} A| + |A|^3 |\nabla_{(3)} A| \\
& \quad \quad + |A|^2 |\nabla A| |\nabla_{(2)} A| + |A|^7) \gamma^{s-1} d\mu \\
& \leq (-4 + \delta_1 + \delta_5) \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + \delta_3 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu
\end{aligned}$$

$$\begin{aligned}
 & + (\delta_2 + \delta_4 + \delta_7) \int_M |A|^8 \gamma^s d\mu + \delta_6 \int_M |\nabla A|^4 \gamma^s d\mu \\
 & + \int_M (A * \nabla A * \nabla A * \nabla_{(2)} A + P_3^0(A) * P_3^2(A) + P_8^0(A)) \gamma^s d\mu \\
 & + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

In the above, we used the estimates (c varies from line to line, is fixed depending on δ_i , s , $\tilde{\gamma}$ to be chosen)

$$\begin{aligned}
 & c(c_\gamma) \int_M |A|^4 |\nabla_{(2)} A| \gamma^{s-1} d\mu \\
 & \leq \delta_1 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_M |A|^6 \gamma^{s-2} d\mu \\
 & \leq \delta_1 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + \delta_2 \int_M |A|^8 \gamma^s d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu,
 \end{aligned}$$

$$\begin{aligned}
 & c(c_\gamma) \int_M |A|^3 |\nabla_{(3)} A| \gamma^{s-1} d\mu \\
 & \leq \delta_3 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_M |A|^6 \gamma^{s-2} d\mu \\
 & \leq \delta_3 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + \delta_4 \int_M |A|^8 \gamma^s d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu,
 \end{aligned}$$

$$\begin{aligned}
 & c(c_\gamma) \int_M |A|^2 |\nabla A| |\nabla_{(2)} A| \gamma^{s-1} d\mu \\
 & \leq \delta_5 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c(c_\gamma)^2 \int_M |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu \\
 & \leq \delta_5 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + \delta_6 \int_M |\nabla A|^4 \gamma^s d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu,
 \end{aligned}$$

$$\begin{aligned}
 c(c_\gamma) \int_M |A|^7 \gamma^{s-1} d\mu & \leq \frac{\delta_7}{2} \int_M |A|^8 \gamma^s d\mu + c(c_\gamma)^2 \int_M |A|^6 \gamma^{s-2} d\mu \\
 & \leq \delta_7 \int_M |A|^8 \gamma^s d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

Now let us deal with the P -style terms by estimating

$$\begin{aligned} & \int_M (A * \nabla A * \nabla A * \nabla_{(2)} A + P_3^0(A) * P_3^2(A) + P_8^0(A)) \gamma^s d\mu \\ & \leq c \int_M (|A| |\nabla A|^2 |\nabla_{(2)} A| + |A|^3 (|A|^2 |\nabla_{(2)} A| + |A| |\nabla A|^2) + |A|^8) \gamma^s d\mu \\ & \leq \delta_8 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu + c \int_M (|\nabla A|^4 + |A|^8) \gamma^s d\mu. \end{aligned}$$

Combining, we find

$$\begin{aligned} (34) \quad \frac{d}{dt} \int_M |A|^4 \gamma^s d\mu & \leq (-4 + \delta_1 + \delta_5 + \delta_8) \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\ & \quad + \delta_3 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\ & \quad + (c + \delta_2 + \delta_4 + \delta_7) \int_M |A|^8 \gamma^s d\mu \\ & \quad + (c + \delta_6) \int_M |\nabla A|^4 \gamma^s d\mu \\ & \quad + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu. \end{aligned}$$

Now we turn to the next term, $\int_M |\nabla A|^2 \gamma^s d\mu$. Since this is an L^2 -norm, the evolution is standard. Unfortunately, the typical approach with Lemma 8 interpolates between $\nabla_{(3)} A$ and A in L^2 , whereas we wish to go down instead to A in L^4 . So we calculate

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla A|^2 \gamma^s d\mu & = 2 \int_M \langle \nabla A, -\nabla^p \Delta \nabla_p \nabla A + (P_3^3 + P_5^1)(A) \rangle \gamma^s d\mu \\ & \quad + \int_M |\nabla A|^2 \langle \vec{H}, \mathbf{F} \rangle \gamma^s d\mu + s \int_M |\nabla A|^2 \gamma_t \gamma^{s-1} d\mu \\ & = -2 \int_M \langle \nabla A, \nabla^p \Delta \nabla_p \nabla A \rangle \gamma^s d\mu \\ & \quad + \int_M (\nabla A * (P_3^3(A) + P_5^1(A))) \gamma^s d\mu \\ & \quad + s \int_M |\nabla A|^2 \gamma_t \gamma^{s-1} d\mu. \end{aligned}$$

For the first two terms, we find

$$-2 \int_M \langle \nabla A, \nabla^p \Delta \nabla_p \nabla A \rangle \gamma^s d\mu + \int_M (\nabla A * (P_3^3(A) + P_5^1(A))) \gamma^s d\mu$$

$$\begin{aligned}
 &= -2 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + 2s \int_M \langle \nabla^r A \nabla^q \gamma, \Delta \nabla_{qr} A \rangle \gamma^{s-1} d\mu \\
 &\quad - 2s \int_M \langle \nabla^{qr} A \nabla^p \gamma, \nabla_{pqr} A \rangle \gamma^{s-1} d\mu \\
 &\quad + \int_M (\nabla A * (P_3^3(A) + P_5^1(A))) \gamma^s d\mu \\
 &= -2 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu - 2s \int_M \langle \nabla^{qr} A \nabla^p \gamma, \nabla_{pqr} A \rangle \gamma^{s-1} d\mu \\
 &\quad - 2s \int_M \langle \gamma \nabla^{pr} A \nabla^q \gamma + \gamma \nabla^r A \nabla^{pq} \gamma \\
 &\quad\quad + (s-1) \nabla^r A \nabla^p \gamma \nabla^q \gamma, \nabla_{pqr} A \rangle \gamma^{s-2} d\mu \\
 &\quad + \int_M (\nabla A * (P_3^3(A) + P_5^1(A))) \gamma^s d\mu.
 \end{aligned}$$

Note that

$$\begin{aligned}
 &- 2s(s-1) \int_M \langle \nabla^r A \nabla^p \gamma \nabla^q \gamma, \nabla_{pqr} A \rangle \gamma^{s-2} d\mu \\
 &\quad \leq c(c_\gamma)^2 \int_M (\nabla_{(3)} A * \nabla A) \gamma^{s-2} d\mu.
 \end{aligned}$$

Classifying and estimating terms in this way, using also $\gamma = \tilde{\gamma} \circ f$, we combine this with the evolution of $\|\nabla A\|_{2, \gamma^{s/2}}^2$ to find (note that the c here depends on s and N)

$$\begin{aligned}
 (35) \quad &\frac{d}{dt} \int_M |\nabla A|^2 \gamma^s d\mu \leq -2 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\
 &\quad + \int_M (\nabla A * (P_3^3(A) + P_5^1(A))) \gamma^s d\mu \\
 &\quad + c(c_\gamma)^2 \int_M (\nabla_{(3)} A * \nabla A) \gamma^{s-2} d\mu \\
 &\quad + c(c_\gamma) \int_M (\nabla_{(3)} A * (\nabla_{(2)} A + A * \nabla A)) \gamma^{s-1} d\mu \\
 &\quad + c(c_\gamma) \int_M |\nabla A|^2 (|\nabla_{(2)} A| + |A|^3) \gamma^{s-1} d\mu.
 \end{aligned}$$

For the P -style terms we estimate (here c depends additionally on $\delta_9, \delta_{10}, \delta_{11}$)

$$\int_M ([P_3^3(A) + P_5^1(A)] * \nabla A) \gamma^s d\mu$$

$$\begin{aligned}
&\leq c \int_M \left([|A|^2 |\nabla_{(3)} A| + |\nabla_{(2)} A| |\nabla A| |A| \right. \\
&\quad \left. + |\nabla A|^3 + |A|^4 |\nabla A| \right) |\nabla A| \gamma^s d\mu \\
&\leq c \int_M \left(|A|^2 |\nabla A| |\nabla_{(3)} A| + |\nabla_{(2)} A| |\nabla A|^2 |A| \right. \\
&\quad \left. + |\nabla A|^4 + |A|^4 |\nabla A|^2 \right) \gamma^s d\mu \\
&\leq \delta_9 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + \delta_{10} \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
&\quad + c \int_M \left(|\nabla A|^4 + |A|^4 |\nabla A|^2 \right) \gamma^s d\mu \\
&\leq \delta_9 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + \delta_{10} \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
&\quad + \delta_{11} \int_M |A|^8 \gamma^s d\mu + c \int_M |\nabla A|^4 \gamma^s d\mu.
\end{aligned}$$

We additionally observe the estimate

$$\begin{aligned}
&c(c_\gamma)^2 \int_M (\nabla_{(3)} A * \nabla A) \gamma^{s-2} d\mu \\
&\quad + c(c_\gamma) \int_M (\nabla_{(3)} A * (\nabla_{(2)} A + A * \nabla A)) \gamma^{s-1} d\mu \\
&\quad + c(c_\gamma) \int_M |\nabla A|^2 (|\nabla_{(2)} A| + |A|^3) \gamma^{s-1} d\mu \\
(36) \quad &\leq \delta_{12} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + \delta_{13} \int_M |A|^8 \gamma^s d\mu \\
&\quad + c \int_M |\nabla A|^4 \gamma^s d\mu + c(c_\gamma)^2 \int_M |\nabla_{(2)} A|^2 \gamma^{s-2} d\mu \\
&\quad + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
\end{aligned}$$

Since

$$\begin{aligned}
&c(c_\gamma)^2 \int_M |\nabla_{(2)} A|^2 \gamma^{s-2} d\mu \\
&\leq \delta_{14} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu
\end{aligned}$$

we refine (36) to

$$\begin{aligned}
 (37) \quad & c(c_\gamma) \int_M (\nabla_{(3)}A * (\nabla_{(2)}A + (c_\gamma)\nabla A + A * \nabla A))\gamma^{s-1} d\mu \\
 & + c(c_\gamma) \int_M |\nabla A|^2 (|\nabla_{(2)}A| + |A|^3)\gamma^{s-1} d\mu \\
 & \leq (\delta_{12} + \delta_{14}) \int_M |\nabla_{(3)}A|^2 \gamma^s d\mu + \delta_{13} \int_M |A|^8 \gamma^s d\mu \\
 & + c \int_M |\nabla A|^4 \gamma^s d\mu \\
 & + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

Combining (36) and (37) with (35) we find

$$\begin{aligned}
 (38) \quad & \frac{d}{dt} \int_M |\nabla A|^2 \gamma^s d\mu \\
 & \leq -(2 - \delta_9 - \delta_{12} - \delta_{14}) \int_M |\nabla_{(3)}A|^2 \gamma^s d\mu \\
 & + \delta_{10} \int_M |\nabla_{(2)}A|^2 |A|^2 \gamma^s d\mu \\
 & + (\delta_{11} + \delta_{13}) \int_M |A|^8 \gamma^s d\mu + c \int_M |\nabla A|^4 \gamma^s d\mu \\
 & + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

Taking the final estimates (34) and (38) together, we obtain

$$\begin{aligned}
 (39) \quad & \frac{d}{dt} \int_M (|A|^4 + |\nabla A|^2)\gamma^s d\mu \\
 & \leq -(2 - \delta_3 - \delta_9 - \delta_{12} - \delta_{14}) \int_M |\nabla_{(3)}A|^2 \gamma^s d\mu \\
 & - (4 - \delta_1 - \delta_5 - \delta_8 - \delta_{10}) \int_M |\nabla_{(2)}A|^2 |A|^2 \gamma^s d\mu \\
 & + (c + \delta_2 + \delta_4 + \delta_7 + \delta_{11} + \delta_{13}) \int_M |A|^8 \gamma^s d\mu \\
 & + (c + \delta_6) \int_M |\nabla A|^4 \gamma^s d\mu \\
 & + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

With appropriate choices for δ_i we find

$$\begin{aligned}
 (40) \quad & \frac{d}{dt} \int_M (|A|^4 + |\nabla A|^2) \gamma^s d\mu \\
 & \leq -\frac{3}{2} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu - 3 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
 & \quad + c \int_M |A|^8 \gamma^s d\mu + c \int_M |\nabla A|^4 \gamma^s d\mu \\
 & \quad + c(c_\gamma)^4 \int_M |\nabla A|^2 \gamma^{s-4} d\mu + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

We deal with each of the integrals with a large coefficient in turn. By the Michael-Simon Sobolev inequality we estimate

$$\begin{aligned}
 \int_M |\nabla A|^4 \gamma^s d\mu & \leq c \left(\int_M |\nabla A|^2 |\nabla_{(2)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \quad + c \left(\int_M |A| |\nabla A|^3 \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \quad + c \left((c_\gamma) \int_M |\nabla A|^3 \gamma^{\frac{3s-4}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \leq c \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\
 & \quad + c \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{1}{3}} \int_M |\nabla A|^4 \gamma^s d\mu \\
 & \quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_{[\gamma>0]} |\nabla A|^2 d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{2}{3}} \\
 & \leq c \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \quad + \left(\frac{1}{2} + c\varepsilon_0^{\frac{1}{3}} \right) \int_M |\nabla A|^4 \gamma^s d\mu \\
 & \quad + c(c_\gamma)^4 \left(\int_{[\gamma>0]} |\nabla A|^2 d\mu \right)^2
 \end{aligned}$$

Now observe the intermediate estimate

$$\left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 \leq c \left(\int_M |\nabla_{(3)} A| |\nabla A| \gamma^{\frac{s}{2}} d\mu \right)^2$$

$$\begin{aligned}
 & + c \left((c_\gamma) \int_M |\nabla_{(2)} A| |\nabla A| \gamma^{\frac{s-2}{2}} d\mu \right)^2 \\
 \leq & \frac{1}{2} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 + c\varepsilon_0 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\
 & + c \left((c_\gamma)^2 \int_M |\nabla A|^2 \gamma^{\frac{s-4}{2}} d\mu \right)^2,
 \end{aligned}$$

that is,

$$\begin{aligned}
 \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{s}{2}} d\mu \right)^2 & \leq c\varepsilon_0 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\
 & + c(c_\gamma)^4 \left(\int_M |\nabla A|^2 \gamma^{\frac{s-4}{2}} d\mu \right)^2.
 \end{aligned}$$

Combining this with the above yields

$$\begin{aligned}
 (41) \quad \int_M |\nabla A|^4 \gamma^s d\mu & \leq c\varepsilon_0 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + \left(\frac{1}{2} + c\varepsilon_0^{\frac{1}{3}} \right) \int_M |\nabla A|^4 \gamma^s d\mu \\
 & + c(c_\gamma)^4 \left(\int_{[\gamma>0]} |\nabla A|^2 d\mu \right)^2.
 \end{aligned}$$

A similar argument applies to the integral $\int_M |A|^8 \gamma^s d\mu$ (see the derivation of (90) for details), yielding:

$$\begin{aligned}
 (42) \quad \int_M |A|^8 \gamma^s d\mu & \leq c \|A\|_{4, [\gamma>0]}^{\frac{4}{3}} \int_M (|\nabla A|^4 + |A|^8) \gamma^s d\mu \\
 & + c(c_\gamma)^4 \|A\|_{4, [\gamma>0]}^{\frac{16}{3}}.
 \end{aligned}$$

Combining (41), (42) and taking a sufficiently small ε_0 such that the left hand side absorbs, we have

$$(43) \quad \int_M (|A|^8 + |\nabla A|^4) \gamma^s d\mu \leq c\varepsilon_0 \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + c(c_\gamma)^4 \varepsilon_0.$$

Combining (43) with (40) above and choosing ε_0 again if necessary, we finally arrive at the estimate

$$\frac{d}{dt} \int_M (|A|^4 + |\nabla A|^2) \gamma^s d\mu$$

$$\begin{aligned}
& + \int_M \left(|\nabla_{(3)} A|^2 + |\nabla_{(2)} A|^2 |A|^2 + |\nabla A|^4 + |A|^4 |\nabla A|^2 + |A|^8 \right) \gamma^s d\mu \\
& \leq c(c_\gamma)^4 \varepsilon_0.
\end{aligned}$$

Integrating the above finishes the proof. \square

PROPOSITION 17. *Suppose $f : M^4 \times [0, T^*] \rightarrow \mathbb{R}^N$ evolves by (NCF) and γ is a cutoff function as in (γ) . Then there is an $\varepsilon_0 = \varepsilon_0(N)$ such that if*

$$(44) \quad \varepsilon = \sup_{[0, T^*]} \int_{[\gamma > 0]} |A|^4 d\mu \leq \varepsilon_0$$

then for any $t \in [0, T^*]$ we have

$$\begin{aligned}
(45) \quad & \int_{[\gamma=1]} |\nabla_{(2)} A|^2 d\mu + \int_0^t \int_{[\gamma=1]} \left(|\nabla_{(4)} A|^2 + |\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |A|^4 \right. \\
& \quad \left. + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla A|^4 |A|^2 \right. \\
& \quad \left. + |\nabla A|^2 |A|^6 + |A|^{10} \right) d\mu d\tau \\
& \leq \int_{[\gamma > 0]} |\nabla_{(2)} A|^2 d\mu \Big|_{t=0} + ct(c_\gamma)^6 \varepsilon^{\frac{1}{2}} \left(1 + [(c_\gamma)^4 \mu(f_0)]^{\frac{1}{2}} \right) (1 + \varepsilon^{\frac{1}{2}}),
\end{aligned}$$

where $c = c(N)$.

PROOF. Lemma 8 with $k = 2$, $s = 16$ gives

$$\begin{aligned}
& \frac{d}{dt} \int_M |\nabla_{(2)} A|^2 \gamma^{16} d\mu + (2 - \theta) \int_M |\nabla_{(4)} A|^2 \gamma^{16} d\mu \\
& \leq c(c_\gamma)^8 \int_{[\gamma > 0]} |A|^2 d\mu + c \int_M \left([P_3^4(A) + P_5^2(A)] * \nabla_{(2)} A \right) \gamma^{16} d\mu.
\end{aligned}$$

We estimate the P -style terms as follows:

$$\begin{aligned}
& c \int_M \left([P_3^4(A) + P_5^2(A)] * \nabla_{(2)} A \right) \gamma^{16} d\mu \\
& \leq \theta \int_M |\nabla_{(4)} A|^2 \gamma^{16} d\mu + c \int_M |\nabla_{(2)} A|^2 |A|^4 \gamma^{16} d\mu \\
& \quad + c \int_M \nabla_{(2)} A * (\nabla_{(3)} A * \nabla A * A + \nabla_{(2)} A * \nabla A * \nabla A) \gamma^{16} d\mu
\end{aligned}$$

$$\begin{aligned}
 & + c \int_M \nabla_{(2)} A * (\nabla_{(2)} A * A * A * A * A \\
 & \qquad \qquad \qquad + \nabla A * \nabla A * A * A * A) \gamma^{16} d\mu \\
 & \leq \theta \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(3)} A|^2 |A|^2 + |\nabla A|^4 |A|^2) \gamma^{16} d\mu \\
 (46) \quad & + c \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2) \gamma^{16} d\mu.
 \end{aligned}$$

The equality

$$\begin{aligned}
 \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu & = - \int_M \langle \Delta A, A \rangle |\nabla A|^2 |A|^2 \gamma^{16} d\mu \\
 & \quad - 2 \int_M \langle \nabla_p A, A \rangle \langle \nabla^p \nabla A, \nabla A \rangle |A|^2 \gamma^{16} d\mu \\
 & \quad - 2 \int_M \langle \nabla_p A, A \rangle |\nabla A|^2 \langle \nabla^p A, A \rangle \gamma^{16} d\mu \\
 & \quad - 16 \int_M \langle \nabla_p A, A \nabla^p \gamma \rangle |\nabla A|^2 |A|^2 \gamma^{15} d\mu \\
 & = - \int_M (\nabla_{(2)} A * A) * (\nabla A * \nabla A * A * A) \gamma^{16} d\mu \\
 & \quad - \frac{1}{2} \int_M |\nabla |A|^2|^2 |\nabla A|^2 \gamma^{16} d\mu \\
 & \quad - 16 \int_M \langle \nabla_p A, A \nabla^p \gamma \rangle |\nabla A|^2 |A|^2 \gamma^{15} d\mu
 \end{aligned}$$

implies the estimate

$$\begin{aligned}
 & \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + \frac{1}{2} \int_M |\nabla |A|^2|^2 |\nabla A|^2 \gamma^{16} d\mu \\
 & = - \int_M (\nabla_{(2)} A * A) * (\nabla A * \nabla A * A * A) \gamma^{16} d\mu \\
 & \quad - 16 \int_M \langle \nabla_p A, A \nabla^p \gamma \rangle |\nabla A|^2 |A|^2 \gamma^{15} d\mu \\
 & \leq \frac{1}{4} \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + c \int_M |\nabla_{(2)} A|^2 |A|^4 \gamma^{16} d\mu \\
 (47) \quad & - 16 \int_M \langle \nabla_p A, A \nabla^p \gamma \rangle |\nabla A|^2 |A|^2 \gamma^{15} d\mu.
 \end{aligned}$$

To deal with the last term we use Young's inequality twice (on the first line with exponents 4 and $\frac{4}{3}$, for the second with exponents 3 and $\frac{3}{2}$) to estimate

$$\begin{aligned}
& -16 \int_M \langle \nabla_p A, A \nabla^p \gamma \rangle |\nabla A|^2 |A|^2 \gamma^{15} d\mu \\
& \leq \frac{1}{4} \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + c(c_\gamma)^4 \int_M |A|^6 \gamma^{12} d\mu \\
& \leq \frac{1}{4} \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + c \int_M |A|^{10} \gamma^{16} d\mu \\
(48) \quad & + c(c_\gamma)^6 \int_M |A|^4 \gamma^{10} d\mu.
\end{aligned}$$

Combining (47) and (48) we find

$$\begin{aligned}
& \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + \frac{1}{2} \int_M |\nabla |A|^2|^2 |\nabla A|^2 \gamma^{16} d\mu \\
& \leq \frac{1}{2} \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + c \int_M (|\nabla_{(2)} A|^2 |A|^4 + |A|^{10}) \gamma^{16} d\mu \\
& \quad + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^4,
\end{aligned}$$

which after absorbing yields

$$\begin{aligned}
& \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu + \int_M |\nabla |A|^2|^2 |\nabla A|^2 \gamma^{16} d\mu \\
(49) \quad & \leq c \int_M (|\nabla_{(2)} A|^2 |A|^4 + |A|^{10}) \gamma^{16} d\mu + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^4.
\end{aligned}$$

Combining (49) with (46), we find

$$\begin{aligned}
& c \int_M ([P_3^4(A) + P_5^2(A)] * \nabla_{(2)} A) \gamma^{16} d\mu + c \int_M |\nabla A|^4 |A|^2 \gamma^{16} d\mu \\
& \leq \theta \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(3)} A|^2 |A|^2) \gamma^{16} d\mu \\
& \quad + c \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2 + |A|^{10}) \gamma^{16} d\mu \\
(50) \quad & + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^4.
\end{aligned}$$

Now we require the multiplicative Sobolev inequality in (iv) of Corollary 13. This is particularly useful for estimating the right hand side of (50):

$$\int_M (|\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2)$$

$$\begin{aligned}
 & + |\nabla A|^2 |A|^6 + |A|^{10} \gamma^{16} d\mu \\
 \leq & (\theta + c \|A\|_{4, [\gamma > 0]}^4) \int_M |\nabla_{(4)} A|^2 \gamma^{16} d\mu \\
 & + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^2 (1 + [(c_\gamma)^4 \mu_\gamma(f_t)]^{\frac{1}{2}}) (1 + \|A\|_{4, [\gamma > 0]}^2).
 \end{aligned}$$

where $\theta \in (0, 1)$ and $c = c(s, \theta, N)$ is an absolute constant. Applying this and absorbing, we find

$$\begin{aligned}
 & \frac{d}{dt} \int_M |\nabla_{(2)} A|^2 \gamma^{16} d\mu + \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |A|^4 \\
 & \quad + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla A|^4 |A|^2 + |\nabla A|^2 |A|^6 + |A|^{10}) \gamma^{16} d\mu d\tau \\
 & \leq c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f_t)]^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} + c(c_\gamma)^6 \varepsilon^{\frac{1}{2}} (1 + [(c_\gamma)^4 \mu_\gamma(f_t)]^{\frac{1}{2}}) (1 + \varepsilon^{\frac{1}{2}}) \\
 & \leq c(c_\gamma)^6 \varepsilon^{\frac{1}{2}} (1 + [(c_\gamma)^4 \mu_\gamma(f_t)]^{\frac{1}{2}}) (1 + \varepsilon^{\frac{1}{2}}).
 \end{aligned}$$

Integrating finishes the proof. \square

For L^∞ control we use the following estimated form of Lemma 8. The proof of Proposition 18 carries over essentially unchanged from [15]. The $n = 2$ case of Proposition 19 is very similar to [15] for $n = 2$. Therefore we focus only on the case $n = 4$ in the proof below.

PROPOSITION 18. *Suppose $f : M^n \times [0, T^*] \rightarrow \mathbb{R}^N$ evolves by (NCF) and γ is a cutoff function as in (γ) . Then, for $s \geq 2k + 4$ the following estimate holds:*

$$\begin{aligned}
 (51) \quad & \frac{d}{dt} \int_M |\nabla_{(k)} A|^2 \gamma^s d\mu + \int_M |\nabla_{(k+2)} A|^2 \gamma^s d\mu \\
 & \leq c \|A\|_{\infty, [\gamma > 0]}^4 \int_M |\nabla_{(k)} A|^2 \gamma^s d\mu \\
 & \quad + c(c_\gamma)^{2k} \|A\|_{2, [\gamma > 0]}^2 (1 + \|A\|_{\infty, [\gamma > 0]}^4)
 \end{aligned}$$

where $c = c(N)$.

PROPOSITION 19. *Let $n \in \{2, 4\}$. Suppose $f : M^n \times [0, T^*] \rightarrow \mathbb{R}^N$ evolves by (NCF) and γ is a cutoff function as in (γ) . Then there is an $\varepsilon_0 = \varepsilon_0(n, N)$ such that if*

$$(52) \quad \sup_{[0, T^*]} \int_{[\gamma > 0]} |A|^n d\mu \leq \varepsilon_0,$$

we can conclude

$$(53) \quad \begin{aligned} & \|\nabla^{(k)}A\|_{\infty, [\gamma=1]}^2 \\ & \leq (c_\gamma)^{2k+2} c(k, T^*, [(c_\gamma)^4 \mu(f_0)], N, \alpha_0(0), \dots, \alpha_0(k+3)), \end{aligned}$$

where $\alpha_0(j) = \mu(f_0)^{\frac{j-1}{2}} \|\nabla^{(j)}A\|_{2, [\gamma>0]}^2|_{t=0}$.

PROOF. The idea is to use our previous estimates and then integrate. We fix γ and consider nested cutoff functions $\gamma_{\sigma, \tau}$. Define for $0 \leq \sigma < \tau \leq 1$ functions $\gamma_{\sigma, \tau} = \psi_{\sigma, \tau} \circ \gamma$ satisfying $\gamma_{\sigma, \tau} = 0$ for $\gamma \leq \sigma$ and $\gamma_{\sigma, \tau} = 1$ for $\gamma \geq \tau$. The function $\psi_{\sigma, \tau}$ is chosen such that $\gamma_{\sigma, \tau}$ satisfies inequalities (γ) , with the estimate

$$c_{\gamma_{\sigma, \tau}} = \|\nabla \psi_{\sigma, \tau}\|_{\infty} \cdot c_\gamma.$$

Note that $\|\nabla \psi_{\sigma, \tau}\|_{\infty}$ depends only on σ and τ , so that when they are fixed we have $c_{\gamma_{\sigma, \tau}} \leq c c_\gamma$. We use this below.

As noted above, we present the proof for $n = 4$ only and refer to [15] for $n = 2$. We first estimate

$$\begin{aligned} & (c_\gamma)^{\frac{2}{3}} \left(\int_M |\nabla_{(3)}A|^2 \gamma^s d\mu \right)^{\frac{4}{3}} \\ & \leq c(c_\gamma)^{\frac{2}{3}} \left(\int_M |\nabla_{(4)}A| |\nabla_{(2)}A| \gamma^s d\mu \right)^{\frac{4}{3}} \\ & \quad + c(c_\gamma)^2 \left(\int_M |\nabla_{(3)}A| |\nabla_{(2)}A| \gamma^{s-1} d\mu \right)^{\frac{4}{3}} \\ & \leq c(c_\gamma)^{\frac{2}{3}} \left(\int_M |\nabla_{(4)}A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)}A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \\ & \quad + c(c_\gamma)^2 \left(\int_M |\nabla_{(3)}A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)}A|^2 \gamma^{s-2} d\mu \right)^{\frac{2}{3}} \\ & \leq \frac{1}{2} (c_\gamma)^{\frac{2}{3}} \left(\int_M |\nabla_{(3)}A|^2 \gamma^s d\mu \right)^{\frac{4}{3}} + c \int_M |\nabla_{(4)}A|^2 \gamma^s d\mu \\ & \quad + c(c_\gamma)^2 \left(\int_M |\nabla_{(2)}A|^2 \gamma^s d\mu \right)^2 + c(c_\gamma)^{\frac{10}{3}} \left(\int_M |\nabla_{(2)}A|^2 \gamma^{s-2} d\mu \right)^{\frac{4}{3}} \end{aligned}$$

which upon absorption yields

$$\begin{aligned}
 (c_\gamma)^{\frac{2}{3}} \left(\int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \right)^{\frac{4}{3}} &\leq c \int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \\
 &\quad + c(c_\gamma)^2 \left(\int_M |\nabla_{(2)} A|^2 \gamma^s d\mu \right)^2 \\
 (54) \quad &\quad + c(c_\gamma)^{\frac{10}{3}} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{s-2} d\mu \right)^{\frac{4}{3}}.
 \end{aligned}$$

We apply the estimate (54) with $\gamma = \gamma_{\frac{1}{2}, \frac{3}{4}}$ to find

$$\begin{aligned}
 (c_\gamma)^{\frac{2}{3}} \|\nabla_{(3)} A\|_{2, [\gamma \geq \frac{3}{4}]}^{\frac{8}{3}} &\leq c \|\nabla_{(4)} A\|_{2, [\gamma \geq \frac{1}{2}]}^2 + c(c_\gamma)^2 \|\nabla_{(2)} A\|_{2, [\gamma \geq \frac{1}{2}]}^4 \\
 (55) \quad &\quad + c(c_\gamma)^{\frac{10}{3}} \|\nabla_{(2)} A\|_{2, [\gamma \geq \frac{1}{2}]}^{\frac{8}{3}}.
 \end{aligned}$$

Taking ε_0 as in (44), we can apply the estimate (45) of Proposition 17 for $\gamma = \gamma_{\frac{1}{4}, \frac{1}{2}}$. In particular we have the estimate

$$(56) \quad \|\nabla_{(2)} A\|_{2, [\gamma \geq \frac{1}{2}]}^2 \leq \alpha_0(2) + cT^*(c_\gamma)^6 \varepsilon^{\frac{1}{2}} (1 + [(c_\gamma)^4 \mu(f_0)]^{\frac{1}{2}}) (1 + \varepsilon^{\frac{1}{2}}).$$

Combining (55) with (56) we have

$$\|\nabla_{(3)} A\|_{2, [\gamma \geq \frac{3}{4}]}^{\frac{8}{3}} \leq c \|\nabla_{(4)} A\|_{2, [\gamma \geq \frac{1}{2}]}^2 + c$$

where c depends on T^* , $\alpha_0(2)$, $[(c_\gamma)^4 \mu(f_0)]$ and N as in (53). We have also used $\varepsilon \leq 1$. From now until the rest of this proof all constants c (that may vary from line to line) shall depend on these quantities. Later in the proof c may additionally depend on $\alpha_0(k)$; when this occurs it will be explicitly stated.

From Proposition 14 we find, using $\gamma_{\frac{3}{4}, \frac{7}{8}}$ instead of γ ,

$$\begin{aligned}
 &\int_0^t \|A\|_{\infty, [\gamma \geq \frac{7}{8}]}^4 d\tau \\
 &\leq c \int_0^t \|A\|_{2, [\gamma \geq \frac{3}{4}]}^{\frac{4}{3}} \left(\|\nabla_{(3)} A\|_{2, [\gamma \geq \frac{3}{4}]}^{\frac{8}{3}} \right. \\
 &\quad \left. + (c_\gamma)^{\frac{16}{3}} (1 + \|A\|_{4, [\gamma \geq \frac{3}{4}]}^{\frac{16}{3}} + [(c_\gamma)^4 \mu_\gamma(f_t)]^{\frac{4}{3}}) \right) d\tau
 \end{aligned}$$

$$\begin{aligned} &\leq c(\mu(f_0)\varepsilon_0)^{\frac{1}{3}} \int_0^t \left(\|\nabla_{(4)}A\|_{2, [\gamma \geq \frac{3}{4}]}^2 + c \right) d\tau \\ &\leq c\varepsilon_0^{\frac{1}{3}} \int_0^t \|\nabla_{(4)}A\|_{2, [\gamma \geq \frac{3}{4}]}^2 d\tau + c\varepsilon_0^{\frac{1}{3}}. \end{aligned}$$

Now from Proposition 17 with $\gamma = \gamma_{\frac{1}{2}, \frac{3}{4}}$ we have the estimate

$$\int_0^t \|\nabla_{(4)}A\|_{2, [\gamma \geq \frac{3}{4}]}^2 d\tau \leq \alpha_0(2) + cT^*(c_\gamma)^6 \varepsilon^{\frac{1}{2}} (1 + [(c_\gamma)^4 \mu(f_0)]^{\frac{1}{2}}) (1 + \varepsilon^{\frac{1}{2}}) \leq c.$$

In the above we used $\varepsilon \leq 1$. This implies

$$(57) \quad \int_0^t \|A\|_{\infty, [\gamma \geq \frac{7}{8}]}^4 d\tau \leq c\varepsilon^{\frac{1}{3}}.$$

Now, integrating Proposition 18 with $\gamma = \gamma_{\frac{7}{8}, \frac{15}{16}}$ yields an inequality of the form

$$\alpha(t) \leq \beta(t) + \int_c^t \lambda(\tau)\alpha(\tau)d\tau,$$

where

$$\begin{aligned} \alpha(t) &= \|\nabla_{(k)}A\|_{2, [\gamma \geq \frac{15}{16}]}^2, \\ \beta(t) &= \|\nabla_{(k)}A\|_{2, [\gamma \geq \frac{7}{8}]}^2 \Big|_{t=0} + c \int_0^t \left[\|A\|_{2, [\gamma \geq \frac{7}{8}]}^2 \left(1 + \|A\|_{\infty, [\gamma \geq \frac{7}{8}]}^4 \right) \right] d\tau, \end{aligned}$$

and

$$\lambda(t) = \|A\|_{\infty, [\gamma \geq \frac{7}{8}]}^4.$$

Noting that β and $\int \lambda d\tau$ are bounded as shown above, we can invoke Grönwall's inequality and conclude

$$(58) \quad \|\nabla_{(k)}A\|_{2, [\gamma \geq \frac{15}{16}]}^2 \leq \beta(t) + \int_0^t \beta(\tau)\lambda(\tau)e^{\int_\tau^t \lambda(\nu)d\nu} d\tau \leq c,$$

where now c depends additionally on $\alpha_0(k)$. Therefore using (22) with $\gamma_{\frac{15}{16}, \frac{31}{32}}$ we have

$$(59) \quad \|A\|_{\infty, [\gamma \geq \frac{31}{32}]} \leq c\varepsilon_0^{\frac{1}{2}}.$$

Finally, using (21) with $T = \nabla_{(k)}A$ and $\gamma = \gamma_{\frac{31}{32},1}$ we obtain for any $l \in \mathbb{N}_0$

$$\begin{aligned}
 \|\nabla_{(l)}A\|_{\infty, [\gamma=1]}^3 &\leq c \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]} \left(\|\nabla_{(l+3)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \right. \\
 &\quad + \|A^3 \nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 + (c_\gamma)^2 \|A \nabla_{(l+1)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \\
 &\quad + (c_\gamma)^2 \|\nabla_{(l)}A \nabla A\|_{2, [\gamma \geq \frac{31}{32}]}^2 + (c_\gamma)^4 \|\nabla_{(l+1)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \\
 &\quad \left. + (c_\gamma)^6 \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \right) \\
 &\leq c \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]} \left(\|\nabla_{(l+3)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \right. \\
 &\quad + \|A\|_{\infty, [\gamma \geq \frac{31}{32}]}^6 \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \\
 &\quad + (c_\gamma)^2 \|A\|_{\infty, [\gamma \geq \frac{31}{32}]}^2 \|\nabla_{(l+1)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \\
 &\quad + (c_\gamma)^2 \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]} \|\nabla A\|_{2, [\gamma \geq \frac{31}{32}]} \\
 &\quad \left. + (c_\gamma)^4 \|\nabla_{(l+1)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 + (c_\gamma)^6 \|\nabla_{(l)}A\|_{2, [\gamma \geq \frac{31}{32}]}^2 \right).
 \end{aligned}$$

The estimate (58), applied for $k = 1, k, k + 1, k + 3$ then yields

$$\|\nabla_{(l)}A\|_{\infty, [\gamma=1]} \leq c.$$

Tracing through the dependence of the above c on (c_γ) and the scale-invariant $[(c_\gamma)^4 \mu(f_0)]$ reveals the structure of the constant given in (53). This completes the proof of the proposition. \square

PROOF OF THEOREM 3. The proof for $n = 2$ follows exactly as in [15]. It should be noted that the argument given in [37] results in a constant that depends on the measure of the initial immersion. That was natural in the setting of [37] where volume was a-priori along the flow possibly not controlled, depending on the given global force field. Here, we have no external forcing term, and so it is desirable to obtain the theorem with universal constants not depending on the initial data.

This improvement is possible due to the validity of Proposition 16. We make the definition

$$(60) \quad \eta(t) = \sup_{x \in \mathbb{R}^N} \int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu.$$

By covering $B_1(x) \subset \mathbb{R}^N$ with several translated copies of $B_{\frac{1}{2}}$ there is a constant c_η depending only on N such that

$$(61) \quad \eta(t) \leq c_\eta \sup_{x \in \mathbb{R}^N} \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^4 + |\nabla A|^2 d\mu.$$

By short time existence the function $\eta : [0, T) \rightarrow \mathbb{R}$ is continuous. We now define

$$(62) \quad t_0 = \sup\{0 \leq t \leq \min(T, \lambda) : \eta(\tau) \leq \delta \text{ for } 0 \leq \tau \leq t\},$$

where λ, δ are parameters to be specified later.

The proof continues in three steps.

$$(63) \quad t_0 = \min(T, \lambda),$$

$$(64) \quad t_0 = \lambda \implies \text{Lifespan Theorem,}$$

$$(65) \quad T \neq \infty \implies t_0 \neq T.$$

The three statements (63), (64), (65) together imply the Lifespan Theorem. The argument is as follows: first notice that by (63) $t_0 = \lambda$ or $t_0 = T$, and if $t_0 = \lambda$ then by (64) we have the Lifespan Theorem. Also notice that if $t_0 = \infty$ then $T = \infty$ and the Lifespan Theorem follows from estimate (67) below (used to prove statement (64)). Therefore the only remaining case where the Lifespan Theorem may fail to be true is when $t_0 = T < \infty$. But this is impossible by statement (65), so we are finished.

To prove step 1, suppose it is false. This means that $t_0 < \min(\lambda, T)$, so that on $[0, t_0)$ we have $\eta(t) \leq \delta$, and

$$(66) \quad \eta(t_0) = \delta.$$

Setting $\tilde{\gamma}$ to be a cutoff function that is identically one on $B_{\frac{\rho}{2}}(x)$ and zero outside $B_\rho(x)$, so that γ has the corresponding properties on the preimages of these balls under f , Proposition 16 implies

$$\begin{aligned} & \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^4 + |\nabla A|^2 d\mu \\ & \leq \int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu \Big|_{t=0} + c_0 t \rho^{-4} \varepsilon_1, \quad t \in [0, t_0). \end{aligned}$$

A covering argument implies

$$\eta(t) \leq c_\eta \sup_{x \in \mathbb{R}^N} \int_{f^{-1}(B_{\frac{\rho}{2}}(x))} |A|^4 + |\nabla A|^2 d\mu$$

so that

$$(67) \quad \int_{f^{-1}(B_1(x))} |A|^4 + |\nabla A|^2 d\mu < c_\eta \varepsilon_1 + c_\eta c_0 \lambda \rho^{-4} \varepsilon_1.$$

We choose $\delta = 3c_\eta \varepsilon_1$, and ε_1 small enough such that $\delta \leq \varepsilon_0$ where ε_0 is the smaller of those appearing in Proposition 16 and Proposition 19. Then, for $\lambda \leq \rho^4/c_0$, the above estimate implies

$$\eta(t) < 2c_\eta \varepsilon_1,$$

for all $t \in [0, t_0)$. Therefore (recall that $t_0 < T$) $\lim_{t \rightarrow t_0} \eta(t) \leq 2c_\eta \varepsilon_1$. This is a contradiction with (66).

This establishes step one (63). We have also proved the second step (64). Observe that if $t_0 = \lambda$ then by the definition (62) of t_0 ,

$$T \geq \lambda,$$

which is the lower bound for maximal time claimed by the lifespan theorem. The estimate (4) follows from (67). That is, we have proved if $t_0 = \lambda$, then the lifespan theorem holds, which is the second step.

We assume

$$t_0 = T \neq \infty;$$

since if $T = \infty$ then the lower bound on T holds automatically and again the previous estimates imply the a-priori control on $\|A\|_{4,f^{-1}(B_1(x))}^4 + \|\nabla A\|_{2,f^{-1}(B_1(x))}^2$. Note also that we can safely assume $T < \lambda$, since otherwise we can apply step two to conclude the Lifespan Theorem.

In this case, Proposition 19 implies that the flow exists smoothly up to and including time T . The proof of this claim follows exactly as in [15]. In particular, we have uniform control in C^∞ for the flow, allowing us to reapply short time existence and extend the flow. This contradicts the maximality of T , and finishes the proof. \square

All steps in the proof rely only on the flow having the form

$$\mathbf{F} = \Delta^\perp \vec{H} + P_3^0(A).$$

Both the surface diffusion flow and the Willmore flow have this form, in addition to Chen's flow. The work in this section extends results from [15, 37, 36] to the case where $n = 4$ for the flows considered there. We state a general version of the lifespan theorem here incorporating this.

THEOREM 20. *Let $n \in \{2, 4\}$. There exist constants $\varepsilon_1 > 0$ and $c < \infty$ depending only on n and N with the following property. Consider a curvature flow $f : M^n \times [0, T) \rightarrow \mathbb{R}^N$ with smooth initial data satisfying*

$$(\partial_t f)^\perp = -\mathbf{F}$$

where $\mathbf{F} = \Delta^\perp \vec{H} + P_3^0(A)$.

(Case 1: $n = 2$.) *Let ρ be chosen such that*

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^N.$$

Then the maximal time T of smooth existence satisfies

$$T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq c\varepsilon_1 \quad \text{for all } t \in \left[0, \frac{1}{c} \rho^4\right].$$

(Case 2: $n = 4$.) *Let ρ be chosen such that*

$$\int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu \Big|_{t=0} = \varepsilon(x) \leq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^N.$$

Then the maximal time T of smooth existence satisfies

$$T \geq \frac{1}{c} \rho^4,$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^4 + |\nabla A|^2 d\mu \leq c\varepsilon_1 \quad \text{for all } t \in \left[0, \frac{1}{c} \rho^4\right].$$

5. Global Analysis of the Flow

Now we move from a local condition on the concentration of curvature for the initial data, to a global condition on the tracefree second fundamental form. Unlike the estimates we have already discussed, we are now restricted to $n = 2$. We follow the same strategy as in [23], where asymptotic convergence to a round point is proved for a Willmore/Helfrich flow. The key difference here is in showing that the energy is monotone. This is where the restriction on dimension arises.

LEMMA 21. *Let $f : M^2 \times [0, T) \rightarrow \mathbb{R}^N$ be Chen's flow. There exists an absolute constant $\varepsilon_2 > 0$ such that if*

$$\int_M |A^\circ|^2 d\mu \leq \varepsilon_2$$

then

$$\begin{aligned} \frac{d}{dt} \int_M |A^\circ|^2 d\mu &\leq - \int_M |\Delta H|^2 d\mu - \frac{1}{2} \int_M |A^\circ|^2 H^4 d\mu \\ &\quad - \frac{17}{4} \int_M |\nabla A^\circ|^2 H^2 d\mu. \end{aligned}$$

PROOF. We first compute

$$\begin{aligned} \frac{d}{dt} \int_M |A^\circ|^2 d\mu &= \frac{d}{dt} \frac{1}{2} \int_M |\vec{H}|^2 d\mu \\ &= - \int_M \langle \Delta^\perp \vec{H} + Q(A^\circ) \vec{H}, \mathbf{F} \rangle d\mu \\ &= - \int_M \langle \Delta^\perp \vec{H} + Q(A^\circ) \vec{H}, \Delta^\perp \vec{H} - Q(A) \vec{H} \rangle d\mu \\ &= - \int_M |\Delta^\perp \vec{H}|^2 d\mu + \int_M \langle Q(A^\circ) \vec{H}, Q(A) \vec{H} \rangle d\mu \\ &\quad + \int_M \langle \Delta^\perp \vec{H}, Q(A) \vec{H} \rangle d\mu - \int_M \langle Q(A^\circ) \vec{H}, \Delta^\perp \vec{H} \rangle d\mu. \end{aligned}$$

Note that

$$\int_M \langle \Delta^\perp \vec{H}, Q(A) \vec{H} - Q(A^\circ) \vec{H} \rangle d\mu$$

$$\begin{aligned}
&= \frac{1}{2} \int_M \langle \Delta^\perp \vec{H}, |\vec{H}|^2 \vec{H} \rangle d\mu \\
&= -\frac{1}{2} \int_M |\nabla \vec{H}|^2 |\vec{H}|^2 d\mu - \int_M \left| \langle \nabla \vec{H}, \vec{H} \rangle \right|^2 d\mu \\
&= -\frac{3}{2} \int_M |\nabla \vec{H}|^2 |\vec{H}|^2 d\mu.
\end{aligned}$$

Therefore we find

$$\begin{aligned}
\frac{d}{dt} \int_M |A^\circ|^2 d\mu &\leq - \int_M |\Delta^\perp \vec{H}|^2 d\mu - \frac{3}{2} \int_M |\nabla \vec{H}|^2 |\vec{H}|^2 d\mu \\
&\quad + \int_M |A^\circ|^2 |A|^2 |\vec{H}|^2 d\mu.
\end{aligned}$$

We now use estimate [22][(14)] (see also [14][(17)]), valid analogously in high codimension, which reads

$$\begin{aligned}
(1 - \delta) \int_M |\vec{H}|^2 |\nabla A^\circ|^2 \gamma^4 d\mu &+ \left(\frac{1}{2} - 2\delta\right) \int_M |\vec{H}|^4 |A^\circ|^2 \gamma^4 d\mu \\
&\leq \left(\frac{1}{2} + 3\delta\right) \int_M |\vec{H}|^2 |\nabla \vec{H}|^2 \gamma^4 d\mu \\
&\quad + c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) \gamma^4 d\mu + c_\gamma^4 c \int_{[\gamma>0]} |A^\circ|^2 d\mu,
\end{aligned}$$

for $\delta > 0$, where c is a constant depending only on δ .

Rarranging this with $\gamma \equiv 1$ yields

$$\begin{aligned}
- \int_M |\vec{H}|^2 |\nabla \vec{H}|^2 d\mu &\leq -\frac{2 - 2\delta}{1 + 6\delta} \int_M |\vec{H}|^2 |\nabla A^\circ|^2 d\mu - \frac{1 - 4\delta}{1 + 6\delta} \int_M |\vec{H}|^4 |A^\circ|^2 d\mu \\
&\quad + c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu.
\end{aligned}$$

In order to absorb the bad term we need

$$\frac{3}{2} \frac{1 - 4\delta}{1 + 6\delta} > \frac{1}{2} \quad \iff \quad 1 - 4\delta > \frac{1}{3} + 2\delta \quad \iff \quad \frac{2}{3} > 6\delta.$$

This is satisfied for $\delta < \frac{1}{9}$, so let's pick $\delta = \frac{1}{18}$. This implies

$$-\frac{3}{2} \int_M |\vec{H}|^2 |\nabla \vec{H}|^2 d\mu \leq -\frac{17}{8} \int_M |\vec{H}|^2 |\nabla A^\circ|^2 d\mu - \frac{7}{8} \int_M |\vec{H}|^4 |A^\circ|^2 d\mu$$

$$+ c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu.$$

The evolution of $\|A^\circ\|_2^2$ can then be estimated by

$$\begin{aligned} \frac{d}{dt} \int_M |A^\circ|^2 d\mu &\leq - \int_M |\Delta^\perp \vec{H}|^2 d\mu + \int_M |A^\circ|^4 |\vec{H}|^2 d\mu + \frac{1}{2} \int_M |A^\circ|^2 |\vec{H}|^4 d\mu \\ &\quad - \frac{17}{8} \int_M |\nabla A^\circ|^2 |\vec{H}|^2 d\mu - \frac{7}{8} \int_M |A^\circ|^2 |\vec{H}|^4 d\mu \\ &\quad + c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu \\ &\leq - \int_M |\Delta^\perp \vec{H}|^2 d\mu + \int_M |A^\circ|^4 |\vec{H}|^2 d\mu - \frac{3}{8} \int_M |A^\circ|^2 |\vec{H}|^4 d\mu \\ &\quad - \frac{17}{8} \int_M |\nabla A^\circ|^2 |\vec{H}|^2 d\mu + c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu. \end{aligned}$$

Estimating $\int_M |A^\circ|^4 |\vec{H}|^2 d\mu \leq \frac{1}{8} \int_M |\vec{H}|^4 |A^\circ|^2 d\mu + c \int_M |A^\circ|^6 d\mu$, this becomes

$$\begin{aligned} \frac{d}{dt} \int_M |A^\circ|^2 d\mu &\leq - \int_M |\Delta^\perp \vec{H}|^2 d\mu - \frac{1}{4} \int_M |A^\circ|^2 |\vec{H}|^4 d\mu \\ &\quad - \frac{17}{8} \int_M |\nabla A^\circ|^2 |\vec{H}|^2 d\mu + c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu. \end{aligned}$$

Now we use the smallness assumption, so that the Sobolev inequalities

$$\begin{aligned} &c \int_M (|A^\circ|^6 + |A^\circ|^2 |\nabla A^\circ|^2) d\mu \\ &\leq c \|A^\circ\|_2^2 \int_M (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu \end{aligned}$$

and

$$\int_M (|\nabla_{(2)} A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu \leq c \int_M |\Delta^\perp \vec{H}|^2 d\mu$$

from [14] and [36] respectively, become valid. Combining these together we find that, for ε_2 sufficiently small,

$$\frac{d}{dt} \int_M |A^\circ|^2 d\mu \leq -\frac{1}{2} \int_M |\Delta \vec{H}|^2 d\mu - \frac{1}{4} \int_M |A^\circ|^2 |\vec{H}|^4 d\mu$$

$$-\frac{17}{8} \int_M |\nabla A^o|^2 \vec{H}^2 d\mu,$$

as required. \square

REMARK 4. The integral identity we use, as well as the relationship between A , A^o and \vec{H} , are only valid for $n = 2$.

Interior estimates for the flow follow using an argument analogous to [23, Theorem 3.11].

THEOREM 22. *Suppose $f : M^2 \times (0, \delta] \rightarrow \mathbb{R}^N$ flows by (NCF) and satisfies*

$$\sup_{0 < t \leq \delta} \int_{f^{-1}(B_{2\rho}(0))} |A|^2 d\mu \leq \varepsilon < \varepsilon_0,$$

where $\delta \leq c\rho^4$. Then for any $k \in \mathbb{N}_0$ and $t \in (0, \delta)$ we have

$$\begin{aligned} \|\nabla^{(k)} A\|_{2, f^{-1}(B_\rho(0))} &\leq c_k \sqrt{\varepsilon} t^{-\frac{k}{4}} \\ \|\nabla^{(k)} A\|_{\infty, f^{-1}(B_\rho(0))} &\leq c_k \sqrt{\varepsilon} t^{-\frac{k+1}{4}} \end{aligned}$$

where c_k is an absolute constant for each k .

We know by the Lifespan Theorem that for any sequence of radii $r_j \searrow 0$ there exists a sequence of times $t_j \nearrow T$ such that

$$t_j = \inf \left\{ t \geq 0 : \sup_{x \in \mathbb{R}^N} \int_{f^{-1}(B_{r_j}(x))} |A|^2 d\mu > \varepsilon_3 \right\} < T,$$

where $\varepsilon_3 = \varepsilon_1/c_1$ and ε_1, c_1 are the constants from the Lifespan Theorem. Curvature is quantised along $f(\cdot, t_j)$ so that

$$\int_{f^{-1}(B_{r_j}(x))} |A|^2 d\mu \Big|_{t=t_j} \leq \varepsilon_3 \text{ for any } x \in \mathbb{R}^N,$$

and

$$(68) \quad \int_{f^{-1}(B_{r_j}(x_j))} |A|^2 d\mu \Big|_{t=t_j} \geq \varepsilon_3 \text{ for some } x_j \in \mathbb{R}^N.$$

Consider the rescaled immersions

$$f_j : M^2 \times [-r_j^{-4}t_j, r_j^{-4}(T - t_j)] \rightarrow \mathbb{R}^N,$$

$$f_j(p, t) = \frac{1}{r_j} (f(p, t_j + r_j^4 t) - x_j).$$

The Lifespan Theorem implies $r_j^{-4}(T - t_j) \geq c_0$ for any j and also that

$$\sup_{x \in \mathbb{R}^N} \int_{f_j^{-1}(B_1(x))} |A|^2 d\mu \leq \varepsilon_0 \text{ for } 0 < t \leq c_0.$$

Interior estimates on parabolic cylinders $B_1(x) \times (t - 1, t]$ yields

$$\|\nabla_{(k)} A\|_{\infty, f_j} \leq c(k) \text{ for } -r_j^{-4}t_j + 1 \leq t \leq c_0.$$

The Willmore energy is bounded and so a local area bound may be obtained by Simon's estimate [31]. Therefore applying Kuwert-Schätzle's compactness theorem [14, Theorem 4.2] (see also [3, 7]) to the sequence $f_j = f_j(\cdot, 0) : M^2 \rightarrow \mathbb{R}^N$ we recover a limit immersion $\hat{f}_0 : \hat{M}^2 \rightarrow \mathbb{R}^N$, where $\hat{M}^2 \cong M^2$.

We also obtain the diffeomorphisms $\phi_j : \hat{M}^2(j) \rightarrow U_j \subset M^2$, such that the reparametrisation

$$f_j(\phi_j, \cdot) : \hat{M}^2(j) \times [0, c_0] \rightarrow \mathbb{R}^N$$

is a Chen flow with initial data

$$f_j(\phi_j, 0) = \hat{f}_0 + u_j : \hat{M}^2(j) \rightarrow \mathbb{R}^N.$$

We obtain the locally smooth convergence

$$(69) \quad f_j(\phi_j, \cdot) \rightarrow \hat{f},$$

where $\hat{f} : \hat{M}^2 \times [0, c_0] \rightarrow \mathbb{R}^N$ is a Chen flow with initial data \hat{f}_0 .

THEOREM 23. *Let $f : M^2 \times [0, T] \rightarrow \mathbb{R}^N$ be a Chen flow satisfying the smallness hypothesis. Then the blowup \hat{f} as constructed above satisfies $\|\mathbf{Q}\|_2^2 \equiv 0$, where $(\|A^\circ\|_2^2)' \leq -2\|\mathbf{Q}\|_2^2$.*

PROOF. The monotonicity calculation implies

$$\begin{aligned} 2 \int_0^{c_0} \int_{\tilde{M}^2(j)} |\mathbf{Q}(f_j(\phi_j, t))|^2 d\mu_{f_j(\phi_j, \cdot)} dt &= 2 \int_0^{c_0} \int_{U_j} |\mathbf{Q}_j|^2 d\mu_j dt \\ &\leq \int_{M^2} |A_j^o(0)|^2 d\mu_j - \int_{M^2} |A_j^o(c_0)|^2 d\mu_j \\ &= \int_{M^2} |A^o(t_j)|^2 d\mu - \int_{M^2} |A^o(t_j + r_j^4 c_0)|^2 d\mu, \end{aligned}$$

and this converges to zero as $j \rightarrow \infty$. \square

PROOF OF THEOREM 4. Theorem 23 implies

$$\int_{M^2} (|\Delta^\perp \vec{H}|^2 + |\nabla A^o|^2 |\vec{H}|^2 + |\vec{H}|^4 |A^o|^2) d\mu = 0$$

and so the blowup is a union of embedded spheres and planes. Ruling out disconnected components using [14, Lemma 4.3] and noting that by (68) we have $\|\tilde{A}\|_2^2 > 0$, we conclude that \tilde{f} is a round sphere.

As the sequence of radii was arbitrary and area is monotone, this shows that $\mu(f_t) \searrow 0$ and that f_t is asymptotic to a round point. \square

Appendix

PROOF OF LEMMA 12. Statement (i) is Lemma 4.2 in [15].

Let us now prove (ii). We estimate

$$\begin{aligned} \int_M |A|^6 \gamma^s d\mu &\leq c \left(\int_M |\nabla A| |A|^{\frac{7}{2}} \gamma^{\frac{3s}{2}} d\mu + \int_M |A|^{\frac{11}{2}} \gamma^{\frac{3s}{2}} d\mu \right. \\ &\quad \left. + (c_\gamma) \int_M |A|^{\frac{9}{2}} \gamma^{\frac{3s-2}{2}} d\mu \right)^{\frac{4}{3}} \\ &\leq c \left(\int_M |\nabla A|^2 |A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^5 \gamma^{2s} d\mu \right)^{\frac{2}{3}} \\ &\quad + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^6 \gamma^s d\mu \\ &\quad + c (c_\gamma)^{\frac{4}{3}} \left(\int_M |A|^{\frac{9}{2}} \gamma^{\frac{3s-2}{2}} d\mu \right)^{\frac{4}{3}} \end{aligned}$$

$$\begin{aligned}
 &\leq c \left(\int_M |\nabla A|^2 |A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^5 \gamma^{2s} d\mu \right)^{\frac{2}{3}} \\
 &\quad + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^6 \gamma^s d\mu \\
 &\quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_{[\gamma > 0]} |A|^4 d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^5 \gamma^{3s-2} d\mu \right)^{\frac{2}{3}} \\
 &\leq \theta \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu + c \left(\int_M |A|^5 \gamma^{2s} d\mu \right)^2 \\
 &\quad + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^6 \gamma^s d\mu + c(c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4 \\
 &\leq \theta \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu \\
 &\quad + c(\|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} + \|A\|_{4, [\gamma > 0]}^4) \int_M |A|^6 \gamma^s d\mu \\
 &\quad + c(c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4.
 \end{aligned}$$

Note that in the above we used $3s - 2 \geq 2s$. Now for the other term we estimate

$$\begin{aligned}
 \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu &= -2 \int_M (\nabla_i A_{jk})(A^{jk})(\nabla^i A_{lm})(A^{lm}) \gamma^s d\mu \\
 &\quad - \int_M \Delta A * A * A * A \gamma^s d\mu \\
 &\quad - \int_M \nabla A * A * A * A * D\gamma \gamma^{s-1} d\mu \\
 &\leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + c \int_M |A|^6 \gamma^s d\mu \\
 &\quad + (c_\gamma) \int_M |\nabla A| |A|^3 \gamma^{s-1} d\mu \\
 &\leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu \\
 &\quad + c \int_M |A|^6 \gamma^s d\mu \frac{1}{2} \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu \\
 &\quad + (c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4
 \end{aligned}$$

where we used $s \geq 2$. Absorbing yields

$$\begin{aligned} \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu &\leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu \\ &\quad + c \int_M |A|^6 \gamma^s d\mu + (c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4. \end{aligned}$$

We add these estimates together to obtain

$$\begin{aligned} \int_M (|\nabla A|^2 |A|^2 + |A|^6) \gamma^s d\mu &\leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu + c\theta \int_M |\nabla A|^2 |A|^2 \gamma^s d\mu \\ &\quad + c(\|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} + \|A\|_{4, [\gamma > 0]}^4) \int_M |A|^6 \gamma^s d\mu \\ &\quad + (c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4. \end{aligned}$$

Absorbing again for θ sufficiently small yields

$$\begin{aligned} \int_M (|\nabla A|^2 |A|^2 + |A|^6) \gamma^s d\mu &\leq \theta \int_M |\nabla_{(2)} A|^2 \gamma^s d\mu \\ &\quad + c(\|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} + \|A\|_{4, [\gamma > 0]}^4) \int_M |A|^6 \gamma^s d\mu \\ &\quad + (c_\gamma)^2 \|A\|_{4, [\gamma > 0]}^4. \end{aligned}$$

as required.

For the second estimate in (ii) we begin with

$$\begin{aligned} \int_M |\nabla A|^2 |A|^3 \gamma^s &\leq c \int_M |\nabla_{(2)} A| |A|^4 \gamma^s d\mu + c(c_\gamma) \int_M |\nabla A| |A|^4 \gamma^{s-1} d\mu \\ &\quad - \frac{3}{4} \int_M |\nabla |A|^2|^2 |A| \gamma^s d\mu \\ &\leq \frac{1}{2} \int_M |\nabla A|^2 |A|^3 \gamma^s + \theta \int_M |\nabla_{(2)} A|^2 |A| \gamma^s d\mu \\ &\quad + c \int_M |A|^7 \gamma^s d\mu + c(c_\gamma)^2 \int_M |A|^5 \gamma^{s-2} d\mu \\ &\leq \frac{1}{2} \int_M |\nabla A|^2 |A|^3 \gamma^s + \theta \int_M |\nabla_{(2)} A|^2 |A| \gamma^s d\mu \\ &\quad + c \int_M |A|^7 \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{3, [\gamma > 0]}^3 \end{aligned}$$

which upon absorbing yields

$$(70) \quad \int_M |\nabla A|^2 |A|^3 \gamma^s \leq \theta \int_M |\nabla_{(2)} A|^2 |A| \gamma^s d\mu + c \int_M |A|^7 \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{3, [\gamma > 0]}^3.$$

Note that in the estimate above we used $2s - 4 \geq s$. Now Michael-Simon yields

$$(71) \quad \begin{aligned} \int_M |A|^7 \gamma^s d\mu &\leq c \left(\int_M |\nabla A| |A|^{\frac{11}{3}} \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + c \left(\int_M |A|^{\frac{17}{3}} \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\ &\quad + c(c_\gamma)^{\frac{3}{2}} \left(\int_M |A|^{\frac{14}{3}} \gamma^{\frac{2s-3}{3}} d\mu \right)^{\frac{3}{2}} \\ &\leq c \|A\|_{3, [\gamma > 0]}^{\frac{3}{2}} \int_M |\nabla A|^{\frac{3}{2}} |A|^4 \gamma^s d\mu \\ &\quad + c \|A\|_{3, [\gamma > 0]}^{\frac{3}{2}} \int_M |A|^7 \gamma^s d\mu \\ &\quad + c(c_\gamma)^{\frac{3}{2}} \int_M |A|^{\frac{7}{2}} \gamma^{\frac{s-3}{2}} d\mu \left(\int_M |A|^7 \gamma^s d\mu \right)^{\frac{1}{2}} \\ &\leq c \|A\|_{3, [\gamma > 0]}^2 \int_M |\nabla A|^2 |A|^3 \gamma^s d\mu \\ &\quad + (c \|A\|_{3, [\gamma > 0]}^{\frac{3}{2}} + \theta) \int_M |A|^7 \gamma^s d\mu \\ &\quad + c(c_\gamma)^3 \left(\int_M |A|^{\frac{7}{2}} \gamma^{\frac{s-3}{2}} d\mu \right)^2. \end{aligned}$$

To deal with the last term we estimate

$$\begin{aligned} c(c_\gamma)^3 \left(\int_M |A|^{\frac{7}{2}} \gamma^{\frac{s-3}{2}} d\mu \right)^2 &\leq c(c_\gamma)^3 \|A\|_{3, [\gamma > 0]}^3 \int_M |A|^4 \gamma^{s-3} d\mu \\ &\leq c(c_\gamma)^3 \|A\|_{3, [\gamma > 0]}^{\frac{9}{2}} \left(\int_M |A|^5 \gamma^{2s-6} d\mu \right)^{\frac{1}{2}} \\ &\leq c(c_\gamma)^3 \|A\|_{3, [\gamma > 0]}^{\frac{21}{4}} \left(\int_M |A|^7 \gamma^{4s-12} d\mu \right)^{\frac{1}{4}} \\ &\leq \theta \int_M |A|^7 \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{3, [\gamma > 0]}^7 \end{aligned}$$

where we used $4s - 12 \geq s$. This allows us to improve estimate (71) to

$$\begin{aligned} \int_M |A|^7 \gamma^s d\mu &\leq c \|A\|_{3, [\gamma > 0]}^{\frac{3}{2}} \int_M |\nabla A|^{\frac{3}{2}} |A|^4, \gamma^s d\mu \\ &\quad + (c \|A\|_{3, [\gamma > 0]}^{\frac{3}{2}} + \theta) \int_M |A|^7 \gamma^s d\mu \\ &\quad + c(c_\gamma)^4 \|A\|_{3, [\gamma > 0]}^7. \end{aligned}$$

Combining the above with estimate (70), we have

$$\begin{aligned} &\int_M (|\nabla A|^2 |A|^3 + |A|^7) \gamma^s d\mu \\ &\leq (c \|A\|_{3, [\gamma > 0]}^2 + \theta) \int_M (|\nabla_{(2)} A|^2 |A| + |\nabla A|^2 |A|^3 + |A|^7) \gamma^s d\mu \\ &\quad + (c_\gamma)^4 \|A\|_{3, [\gamma > 0]}^3, \end{aligned}$$

as required.

For (iii) we begin by noting

$$\begin{aligned} &-2 \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \gamma^s d\mu \\ &= \int_M \langle A_{ij}, \Delta^\perp A^{ij} \rangle |\nabla A|^2 \gamma^s d\mu + \int_M |\nabla A|^4 \gamma^s d\mu \\ &\quad + s \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^q \gamma \gamma^{s-1} d\mu \end{aligned}$$

and

$$\begin{aligned} &-2 \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \gamma^s d\mu \\ &= \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu + \int_M |A|^2 \langle \nabla^q A_{kl}, \Delta^\perp \nabla_q A^{kl} \rangle \gamma^s d\mu \\ &\quad + s \int_M |A|^2 \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^p \gamma \gamma^{s-1} d\mu \end{aligned}$$

so that

$$(72) \quad \int_M |\nabla A|^4 \gamma^s d\mu = -2 \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \gamma^s d\mu \\ - \int_M \langle A_{ij}, \Delta^\perp A^{ij} \rangle |\nabla A|^2 \gamma^s d\mu$$

$$\begin{aligned}
 & -s \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^q \gamma \gamma^{s-1} d\mu \\
 = & \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu \\
 & + \int_M |A|^2 \langle \nabla^q A_{kl}, \Delta^\perp \nabla_q A^{kl} \rangle \gamma^s d\mu \\
 & - \int_M \langle A_{ij}, \Delta^\perp A^{ij} \rangle |\nabla A|^2 \gamma^s d\mu \\
 & + s \int_M |A|^2 \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^p \gamma \gamma^{s-1} d\mu \\
 (73) \quad & -s \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^q \gamma \gamma^{s-1} d\mu.
 \end{aligned}$$

We estimate

$$\begin{aligned}
 & \int_M |A|^2 \langle \nabla^q A_{kl}, \Delta^\perp \nabla_q A^{kl} \rangle \gamma^s d\mu \\
 & \leq \theta \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + \frac{1}{4} \int_M |\nabla A|^4 \gamma^s d\mu \\
 & \quad + c_\theta \int_M |A|^8 \gamma^s d\mu;
 \end{aligned}$$

and, recalling $s \geq 4$,

$$\begin{aligned}
 & s \int_M |A|^2 \langle \nabla^q A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^p \gamma \gamma^{s-1} d\mu \\
 & - s \int_M \langle A_{ij}, \nabla^p A^{ij} \rangle \langle A_{kl}, \nabla_{pq} A^{kl} \rangle \nabla^q \gamma \gamma^{s-1} d\mu \\
 & \leq \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu + \frac{1}{4} \int_M |\nabla A|^4 \gamma^s d\mu \\
 & \quad + cs^2(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
 \end{aligned}$$

Combining with the above we find

$$\begin{aligned}
 \int_M |\nabla A|^4 \gamma^s d\mu & \leq \theta \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu \\
 (74) \quad & + cs^2(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^4.
 \end{aligned}$$

Now let us estimate the second term on the right. By the Michael-Simon Sobolev inequality we find

$$\begin{aligned}
\int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu &\leq c \left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\quad + c \left(\int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{1}{2}} |\nabla_{(3)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\quad + c \left(\frac{3s}{4} (c_\gamma) \int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}-1} d\mu \right)^{\frac{4}{3}} \\
&\quad + c \left(\int_M |A|^{\frac{5}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\leq c \left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\quad + c \left(\int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{1}{2}} |\nabla_{(3)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\quad + c \left(s(c_\gamma) \int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}-1} d\mu \right)^{\frac{4}{3}} \\
(75) \quad &\quad + c \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{1}{3}} \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu.
\end{aligned}$$

We work on the first term by estimating:

$$\begin{aligned}
&c \left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
&\leq \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{1}{6}} \left(\int_M |\nabla A|^{\frac{8}{7}} |\nabla_{(2)} A|^{\frac{12}{7}} \gamma^{\frac{6s}{7}} d\mu \right)^{\frac{7}{6}} \\
&\leq \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{1}{6}} \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{1}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{6}} \\
&\leq \frac{1}{3} \|A\|_{4, [\gamma>0]}^{\frac{2}{3}} \int_M |\nabla A|^4 \gamma^s d\mu \\
(76) \quad &\quad + \frac{2}{3} \|A\|_{4, [\gamma>0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}}.
\end{aligned}$$

Estimating

$$\begin{aligned}
 & \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \\
 & \leq c \int_M |\nabla A| |\nabla_{(2)} A|^{\frac{2}{5}} |\nabla_{(3)} A| \gamma^{\frac{4s}{5}} d\mu \\
 & \quad + cs(c_\gamma) \int_M |\nabla A| |\nabla_{(2)} A|^{\frac{7}{5}} \gamma^{\frac{4s}{5}-1} d\mu \\
 & \leq c \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{1}{6}} \left[\left(\int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{6}} \right. \\
 & \quad \left. + s(c_\gamma) \left(\int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(2)} A|^{\frac{6}{5}} \gamma^{\frac{4s-6}{5}} d\mu \right)^{\frac{5}{6}} \right]
 \end{aligned}$$

and absorbing yields

$$\begin{aligned}
 \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu & \leq c \int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu \\
 & \quad + c(s(c_\gamma))^{\frac{6}{5}} \int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(2)} A|^{\frac{6}{5}} \gamma^{\frac{4s-6}{5}} d\mu \\
 & \leq c \int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu \\
 & \quad + \frac{1}{2} \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \\
 & \quad + c(s(c_\gamma))^{\frac{12}{5}} \int_M |\nabla A|^{\frac{12}{5}} \gamma^{\frac{4s-12}{5}} d\mu.
 \end{aligned}$$

Absorbing another time gives

$$\begin{aligned}
 \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu & \leq c \int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu \\
 (77) \quad & \quad + c(s(c_\gamma))^{\frac{12}{5}} \int_M |\nabla A|^{\frac{12}{5}} \gamma^{\frac{4s-12}{5}} d\mu.
 \end{aligned}$$

To close this estimate we must use the same technique again on the last term with c_γ . The first step is

$$\int_M |\nabla A|^{\frac{12}{5}} \gamma^{\frac{4s-12}{5}} d\mu$$

$$\begin{aligned}
&\leq c \int_M |A| |\nabla A|^{\frac{2}{5}} |\nabla_{(2)} A| \gamma^{\frac{4s-12}{5}} d\mu + cs(c_\gamma) \int_M |A| |\nabla A|^{\frac{7}{5}} \gamma^{\frac{4s-17}{5}} d\mu \\
&\leq \left(\int_M |\nabla A|^{\frac{12}{5}} \gamma^{\frac{4s-12}{5}} d\mu \right)^{\frac{1}{6}} \left[\left(\int_M |A|^{\frac{6}{5}} |\nabla_{(2)} A|^{\frac{6}{5}} \gamma^{\frac{4s-12}{5}} d\mu \right)^{\frac{6}{5}} \right. \\
&\quad \left. + cs(c_\gamma) \left(\int_M |A|^{\frac{6}{5}} |\nabla A|^{\frac{6}{5}} \gamma^{\frac{4s-18}{5}} d\mu \right)^{\frac{6}{5}} \right].
\end{aligned}$$

Absorbing gives

$$\begin{aligned}
&\int_M |\nabla A|^{\frac{12}{5}} \gamma^{\frac{4s-12}{5}} d\mu \\
(78) \quad &\leq \int_M |A|^{\frac{6}{5}} |\nabla_{(2)} A|^{\frac{6}{5}} \gamma^{\frac{4s-12}{5}} d\mu + c(s(c_\gamma))^{\frac{6}{5}} \int_M |A|^{\frac{6}{5}} |\nabla A|^{\frac{6}{5}} \gamma^{\frac{4s-18}{5}} d\mu.
\end{aligned}$$

Now this whole term that we are estimating is raised to the power $\frac{5}{4}$ and has a coefficient involving c_γ , which scales. Incorporating this, we find

$$\begin{aligned}
&\left((c_\gamma)^{\frac{18}{5}} \int_M |A|^{\frac{6}{5}} |\nabla A|^{\frac{6}{5}} \gamma^{\frac{4s-18}{5}} d\mu \right)^{\frac{5}{4}} \\
&\leq (c_\gamma)^{\frac{9}{2}} \left(\int_{[\gamma>0]} |A|^{\frac{12}{7}} d\mu \right)^{\frac{7}{8}} \left(\int_M |\nabla A|^4 \gamma^{\frac{8s-36}{3}} d\mu \right)^{\frac{3}{8}}.
\end{aligned}$$

Since $s \geq \frac{36}{5}$ we have $\frac{8s-36}{3} \geq s$, this term is estimated by

$$\begin{aligned}
&\left((c_\gamma)^{\frac{18}{5}} \int_M |A|^{\frac{6}{5}} |\nabla A|^{\frac{6}{5}} \gamma^{\frac{4s-18}{5}} d\mu \right)^{\frac{5}{4}} \\
&\leq (c_\gamma)^{\frac{9}{2}} \mu_\gamma(f)^{\frac{1}{2}} \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{3}{8}} \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{3}{8}} \\
(79) \quad &\leq \int_M |\nabla A|^4 \gamma^s d\mu + c(c_\gamma)^{\frac{36}{5}} \mu_\gamma(f)^{\frac{4}{5}} \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{3}{5}}.
\end{aligned}$$

In the above we used $\mu_\gamma := \mu|_{[\gamma>0]}$. Now let us move on to the first term in (78). We estimate it by

$$c(s(c_\gamma))^{\frac{12}{5}} \int_M |A|^{\frac{6}{5}} |\nabla_{(2)} A|^{\frac{6}{5}} \gamma^{\frac{4s-12}{5}} d\mu$$

$$\begin{aligned}
 &\leq \frac{1}{2} \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu + c(s(c_\gamma))^{\frac{24}{5}} \int_M |A|^{\frac{12}{5}} \gamma^{\frac{4s-24}{5}} d\mu \\
 &\leq \frac{1}{2} \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu + c(s(c_\gamma))^{\frac{24}{5}} \mu_\gamma(f)^{\frac{2}{5}} \|A\|_{4, [\gamma>0]}^{\frac{12}{5}}.
 \end{aligned}$$

Combining the above with (79), (78) and absorbing in (77) gives

$$\begin{aligned}
 \int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu &\leq c \int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu + \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{4}{5}} \\
 (80) \quad &+ c(c_\gamma)^{\frac{16}{5}} \left([(c_\gamma)^4 \mu_\gamma(f)]^{\frac{16}{25}} \|A\|_{4, [\gamma>0]}^{\frac{48}{25}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{2}{5}} \|A\|_{4, [\gamma>0]}^{\frac{12}{5}} \right).
 \end{aligned}$$

Note that $[(c_\gamma)^4 \mu_\gamma(f)]$ is scale invariant, and that in (80) the constant c depends on s , n and N . Incorporating the eventual $\frac{5}{4}$ power, we estimate the first term on the right in (80) by

$$\begin{aligned}
 &c \left(\int_M |\nabla A|^{\frac{6}{5}} |\nabla_{(3)} A|^{\frac{6}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
 &\leq c \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{1}{2}} \left(\int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \right)^{\frac{3}{4}} \\
 (81) \quad &\leq c \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 + \frac{1}{2} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu.
 \end{aligned}$$

Using a similar strategy as before, we estimate

$$\begin{aligned}
 &c \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 &\leq c \left(\int_M |A| |\nabla A| |\nabla_{(2)} A| \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 &\quad + cs(c_\gamma)^2 \left(\int_M |A| |\nabla A|^2 \gamma^{\frac{s-2}{2}} d\mu \right)^2 \\
 &\leq c \left(\int_M |A|^{\frac{12}{7}} |\nabla A|^{\frac{12}{7}} \gamma^{\frac{2s}{7}} d\mu \right)^{\frac{7}{6}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{6}} \\
 &\quad + cs(c_\gamma)^2 \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{4}{3}} \left(\int_M |A|^3 \gamma^{\frac{s-6}{2}} d\mu \right)^{\frac{2}{3}} \\
 &\leq \frac{1}{2} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \frac{1}{2} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2
 \end{aligned}$$

$$\begin{aligned}
& + c \left(\int_M |A|^{\frac{12}{7}} |\nabla A|^{\frac{12}{7}} \gamma^{\frac{2s}{7}} d\mu \right)^{\frac{7}{2}} + cs^3(c_\gamma)^6 \left(\int_M |A|^3 \gamma^{\frac{s-6}{2}} d\mu \right)^2 \\
& \leq \frac{1}{2} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \frac{1}{2} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& + c \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{3}{2}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& + cs^3(c_\gamma)^6 \left(\int_M |A|^3 \gamma^{\frac{s-6}{2}} d\mu \right)^2
\end{aligned}$$

and absorb to find

$$\begin{aligned}
\left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 & \leq \frac{1}{2} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
& + c \|A\|_{4, [\gamma>0]}^6 \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& + cs^3(c_\gamma)^6 \mu_\gamma(f)^{\frac{1}{2}} \|A\|_{4, [\gamma>0]}^6.
\end{aligned}$$

We combine this estimate with (81), (80) and absorb to obtain

$$\begin{aligned}
& \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& \leq \frac{1}{2} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + c \int_M |\nabla A|^4 \gamma^s d\mu \\
& + c \|A\|_{4, [\gamma>0]}^6 \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& + c(c_\gamma)^4 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \\
(82) \quad & \times \left(\|A\|_{4, [\gamma>0]}^6 + \|A\|_{4, [\gamma>0]}^3 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{10}} \|A\|_{4, [\gamma>0]}^{\frac{12}{5}} \right).
\end{aligned}$$

The estimate (82) is now combined with (76): In the above c depends only on n , N , and s .

We combine (81) above with the estimate and use Young's inequality to obtain

$$\left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}}$$

$$\begin{aligned}
 & + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \leq c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \int_M (|\nabla_{(3)} A|^2 + |\nabla A|^4) \gamma^s d\mu \\
 & \quad + c \|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \quad + c (c_\gamma)^4 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \\
 & \quad \times \left(\|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} + \|A\|_{4, [\gamma > 0]}^{\frac{11}{3}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{10}} \|A\|_{4, [\gamma > 0]}^{\frac{46}{15}} \right).
 \end{aligned}$$

This estimates the first term in (75). Now let us work on the second:

$$\begin{aligned}
 & \left(\int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{1}{2}} |\nabla_{(3)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \leq c \left(\int_M |A|^3 |\nabla_{(2)} A| \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \\
 & \leq c \left(\int_{[\gamma > 0]} |A|^4 d\mu \right)^{\frac{1}{3}} \left(\int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu \right)^{\frac{1}{3}} \left(\int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \\
 & \leq c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2) \gamma^s d\mu.
 \end{aligned}$$

Combining the above two estimates we find

$$\begin{aligned}
 & \left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + \left(\int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{1}{2}} |\nabla_{(3)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \quad + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \leq c (\|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} + \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \\
 & \quad \times \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4) \gamma^s d\mu \\
 & \quad + c \|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \quad + c (c_\gamma)^4 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \\
 (83) \quad & \times \left(\|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} + \|A\|_{4, [\gamma > 0]}^{\frac{11}{3}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{10}} \|A\|_{4, [\gamma > 0]}^{\frac{46}{15}} \right).
 \end{aligned}$$

This estimates the second term in (75). For the third term, we estimate:

$$\begin{aligned}
& \left(s(c_\gamma) \int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}-1} d\mu \right)^{\frac{4}{3}} \\
& \leq c(c_\gamma)^{\frac{4}{3}} \left(\int_M |A|^6 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{1}{3}} \int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \\
& \leq c(c_\gamma)^{\frac{4}{3}} \left(\int_{[\gamma>0]} |A|^4 d\mu \right)^{\frac{1}{6}} \left(\int_M |A|^8 \gamma^s d\mu \right)^{\frac{1}{6}} \\
& \quad \times \int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \\
& \leq c \|A\|_{4, [\gamma>0]}^{\frac{2}{3}} \int_M |A|^8 \gamma^s d\mu \\
(84) \quad & + c(c_\gamma)^{\frac{8}{5}} \|A\|_{4, [\gamma>0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}}.
\end{aligned}$$

In order to estimate the last term, we first calculate

$$\begin{aligned}
(c_\gamma)^2 \int_M |\nabla A|^2 \gamma^{\frac{5s-20}{6}} d\mu & \leq c(c_\gamma)^2 \int_M |A| |\nabla_{(2)} A| \gamma^{\frac{5s-20}{6}} d\mu \\
& \quad + c(c_\gamma)^3 \int_M |A| |\nabla A| \gamma^{\frac{5s-26}{6}} d\mu \\
& \leq \frac{1}{2} (c_\gamma)^2 \int_M |\nabla A|^2 \gamma^{\frac{5s-20}{6}} d\mu \\
& \quad + c(c_\gamma)^2 \int_M |A| |\nabla_{(2)} A| \gamma^{\frac{5s-20}{6}} d\mu \\
& \quad + c(c_\gamma)^4 \int_M |A|^2 \gamma^{\frac{5s-32}{6}} d\mu.
\end{aligned}$$

Absorbing, estimating, and using $s \geq \frac{32}{5}$, we find

$$\begin{aligned}
c(c_\gamma)^2 \int_M |\nabla A|^2 \gamma^{\frac{5s-20}{6}} d\mu & \leq c(c_\gamma)^2 \int_M |A| |\nabla_{(2)} A| \gamma^{\frac{5s-20}{6}} d\mu \\
& \quad + c(c_\gamma)^4 \int_M |A|^2 \gamma^{\frac{5s-32}{6}} d\mu \\
& \leq \frac{1}{2} \int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu
\end{aligned}$$

$$(85) \quad + c(c_\gamma)^4 \mu_\gamma(f)^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^2.$$

Returning now to the last term of (84), we estimate

$$\begin{aligned} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} &\leq c \left(\int_M |\nabla A| |\nabla_{(3)} A| \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} \\ &\quad + c(c_\gamma)^{\frac{6}{5}} \left(\int_M |\nabla A| |\nabla_{(2)} A| \gamma^{\frac{5s-14}{6}} d\mu \right)^{\frac{6}{5}} \\ &\leq \frac{1}{2} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} \\ &\quad + c \left(\int_M |\nabla A| |\nabla_{(3)} A| \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} \\ &\quad + c(c_\gamma)^{\frac{12}{5}} \left(\int_M |\nabla A|^2 \gamma^{\frac{5s-14}{6}} d\mu \right)^{\frac{6}{5}}. \end{aligned}$$

Absorbing and using (85), we find

$$(86) \quad \begin{aligned} \left(\int_M |\nabla_{(2)} A|^2 \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} &\leq c \left(\int_M |\nabla A| |\nabla_{(3)} A| \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} \\ &\quad + c(c_\gamma)^{\frac{24}{5}} \mu_\gamma(f)^{\frac{3}{5}} \|A\|_{4, [\gamma > 0]}^{\frac{12}{5}}. \end{aligned}$$

Note that (85) implies, using also $s \geq \frac{20}{3}$,

$$\begin{aligned} &c(c_\gamma)^{\frac{12}{5}} \left(\int_M |\nabla A|^2 \gamma^{\frac{5s-8}{3}} d\mu \right)^{\frac{3}{2}} \\ &\leq c(c_\gamma)^{\frac{12}{5}} \left(\int_M |A| |\nabla_{(2)} A| \gamma^{\frac{5s-20}{6}} d\mu \right)^{\frac{3}{2}} \\ &\quad + c(c_\gamma)^{\frac{27}{5}} \left(\int_M |A|^2 \gamma^{\frac{5s-32}{6}} d\mu \right)^{\frac{3}{2}} \\ &\leq c(c_\gamma)^{\frac{12}{5}} \left(\int_M |A|^{\frac{12}{7}} \gamma^{\frac{6s-40}{7}} d\mu \right)^{\frac{7}{8}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{8}} \\ &\quad + c(c_\gamma)^{\frac{27}{5}} \mu_\gamma(f)^{\frac{3}{4}} \|A\|_{4, [\gamma > 0]}^3 \\ &\leq (c_\gamma)^{-\frac{8}{5}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + (c_\gamma)^{\frac{32}{5}} \left(\int_M |A|^{\frac{12}{7}} \gamma^{\frac{6s-40}{7}} d\mu \right)^{\frac{7}{4}} \end{aligned}$$

$$\begin{aligned}
& + c(c_\gamma)^{\frac{27}{5}} \mu_\gamma(f)^{\frac{3}{4}} \|A\|_{4, [\gamma > 0]}^3 \\
\leq & (c_\gamma)^{-\frac{8}{5}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + (c_\gamma)^{\frac{32}{5}} \mu_\gamma(f) \|A\|_{4, [\gamma > 0]}^3 \\
& + c(c_\gamma)^{\frac{27}{5}} \mu_\gamma(f)^{\frac{3}{4}} \|A\|_{4, [\gamma > 0]}^3.
\end{aligned}$$

This yields the estimate

$$\begin{aligned}
& c \left(\int_M |\nabla A| |\nabla_{(3)} A| \gamma^{\frac{5s-8}{6}} d\mu \right)^{\frac{6}{5}} \\
& \leq c \left(\int_M |\nabla A|^2 \gamma^{\frac{5s-8}{3}} d\mu \right)^{\frac{3}{5}} \left(\int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \right)^{\frac{3}{5}} \\
& \leq c(c_\gamma)^{\frac{12}{5}} \left(\int_M |\nabla A|^2 \gamma^{\frac{5s-8}{3}} d\mu \right)^{\frac{3}{2}} + (c_\gamma)^{-\frac{8}{5}} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu \\
& \leq (c_\gamma)^{-\frac{8}{5}} \int_M |\nabla_{(3)} A|^2 \gamma^s d\mu + (c_\gamma)^{-\frac{8}{5}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
& \quad + c(c_\gamma)^{\frac{12}{5}} [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{4}} (1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{4}}) \|A\|_{4, [\gamma > 0]}^3.
\end{aligned}$$

which we combine with (86) and (84) to find

$$\begin{aligned}
& \left(s(c_\gamma) \int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}-1} d\mu \right)^{\frac{4}{3}} \\
& \leq c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \int_M (|\nabla_{(3)} A|^2 + |A|^8) \gamma^s d\mu \\
& \quad + c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
& \quad + c(c_\gamma)^4 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{5}} \\
(87) \quad & \times \left([(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{20}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{2}{5}} + \|A\|_{4, [\gamma > 0]}^{\frac{1}{15}} \right) \|A\|_{4, [\gamma > 0]}^3.
\end{aligned}$$

This estimates the third term in (75). Combining now (75), (87) and (83) we find

$$\int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}}$$

$$\begin{aligned}
 & + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 \leq & c \left(\int_M |A|^{\frac{1}{2}} |\nabla A| |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
 & + c \left(\int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{1}{2}} |\nabla_{(3)} A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & + c \left(s(c_\gamma) \int_M |A|^{\frac{3}{2}} |\nabla_{(2)} A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}-1} d\mu \right)^{\frac{4}{3}} \\
 & + c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \int_M |A|^2 |\nabla_{(2)} A|^2 \gamma^s d\mu \\
 \leq & c (\|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} + \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \\
 & \times \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu \\
 & + c \|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & + c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
 & + c(c_\gamma)^4 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \left(\|A\|_{4, [\gamma > 0]}^{\frac{11}{3}} + \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \right) \\
 & + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{3}{10}} \|A\|_{4, [\gamma > 0]}^{\frac{1}{15}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{4}} \\
 (88) \quad & + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{4}{5}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{10}} \|A\|_{4, [\gamma > 0]}^{\frac{1}{15}} \Big) \|A\|_{4, [\gamma > 0]}^3.
 \end{aligned}$$

We combine (88) above with our earlier estimate (74) to find

$$\begin{aligned}
 & \int_M (|A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4) \gamma^s d\mu + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
 & + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 \leq & (\theta + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \\
 & \times \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu
 \end{aligned}$$

$$\begin{aligned}
& + c \|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
& + c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
(89) \quad & + c(c_\gamma)^4 \left(1 + \|A\|_{4, [\gamma > 0]}^4 + [(c_\gamma)^4 \mu_\gamma(f)]^6 \right) \|A\|_{4, [\gamma > 0]}^3.
\end{aligned}$$

Note that we have interpolated terms inside the parentheses of the coefficient of the first and last terms.

It remains only to estimate the term $\int_M |A|^8 \gamma^s d\mu$, which we do so now with the Michael-Simon Sobolev inequality:

$$\begin{aligned}
\int_M |A|^8 \gamma^s d\mu & \leq c \left(\int_M |A|^5 |\nabla A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + c \left(\int_M |A|^7 \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
& + c \left((c_\gamma) \int_M |A|^6 \gamma^{\frac{3s-4}{4}} d\mu \right)^{\frac{4}{3}} \\
& \leq c \left(\int_M |A|^6 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^4 |\nabla A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \\
& + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^8 \gamma^s d\mu \\
& + c(c_\gamma)^{\frac{4}{3}} \|A\|_{4, [\gamma > 0]}^{\frac{8}{3}} \left(\int_M |A|^8 \gamma^s d\mu \right)^{\frac{2}{3}} \\
& \leq c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \left(\int_M |A|^8 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla A|^4 \gamma^s d\mu \right)^{\frac{1}{3}} \\
& + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^8 \gamma^s d\mu \\
& + c(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^{\frac{16}{3}} \\
(90) \quad & \leq c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M (|\nabla A|^4 + |A|^8) \gamma^s d\mu + c(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^{\frac{16}{3}}.
\end{aligned}$$

Combining this estimate with (89) and also the interpolation $2|A|^4 |\nabla A|^2 \leq |A|^8 + |\nabla A|^4$ we conclude

$$\int_M (|A|^2 |\nabla_{(2)} A|^2 + |A|^4 |\nabla A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu$$

$$\begin{aligned}
 & + \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} + \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \leq (\theta + c\|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \int_M (|\nabla_{(3)} A|^2 + |A|^2 |\nabla_{(2)} A|^2 + |\nabla A|^4 + |A|^8) \gamma^s d\mu \\
 & \quad + c\|A\|_{4, [\gamma > 0]}^{\frac{20}{3}} \left(\int_M |\nabla A|^3 \gamma^{\frac{s}{2}} d\mu \right)^2 \\
 & \quad + c\|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A|^{\frac{12}{5}} \gamma^{\frac{4s}{5}} d\mu \right)^{\frac{5}{4}} \\
 & \quad + c(c_\gamma)^4 \left(1 + \|A\|_{4, [\gamma > 0]}^4 + [(c_\gamma)^4 \mu_\gamma(f)]^6 \right) \|A\|_{4, [\gamma > 0]}^3,
 \end{aligned}$$

as required.

Next we consider (iv). We begin by estimating with the Michael-Simon Sobolev inequality

$$\begin{aligned}
 \int_M |A|^{10} \gamma^s d\mu & \leq c \left(\int_M |A|^{\frac{13}{2}} |\nabla A| \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} + c \left(\int_M |A|^{\frac{17}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |A|^{\frac{15}{2}} \gamma^{\frac{3s-4}{4}} d\mu \right)^{\frac{4}{3}} \\
 & \leq c \left(\int_M |A|^6 |\nabla A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^7 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\
 & \quad + c\|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |A|^{10} \gamma^s d\mu \\
 & \quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |A|^5 \gamma^{\frac{s-4}{2}} d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^{10} \gamma^s d\mu \right)^{\frac{2}{3}} \\
 & \leq \theta \int_M |A|^6 |\nabla A|^2 \gamma^s d\mu + c(\theta + \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \int_M |A|^{10} \gamma^s d\mu \\
 & \quad + c \left(\int_M |A|^7 \gamma^{\frac{s}{2}} d\mu \right)^2 + c(c_\gamma)^4 \left(\int_M |A|^5 \gamma^{\frac{s-4}{2}} d\mu \right)^2 \\
 & \leq \theta \int_M |A|^6 |\nabla A|^2 \gamma^s d\mu \\
 & \quad + c(\theta + \|A\|_{4, [\gamma > 0]}^4 + \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \int_M |A|^{10} \gamma^s d\mu \\
 & \quad + c(c_\gamma)^4 \|A\|_{4, [\gamma > 0]}^4 \int_M |A|^6 \gamma^{s-4} d\mu
 \end{aligned}$$

$$\begin{aligned} &\leq \theta \int_M |A|^6 |\nabla A|^2 \gamma^s d\mu + c(\theta + \|A\|_{4, [\gamma > 0]}^4) \int_M |A|^{10} \gamma^s d\mu \\ &\quad + c(c_\gamma)^8 \mu_\gamma(f_t)^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^2. \end{aligned}$$

Note that in the last step we used $s \geq 8$.

We shall move gradually higher in order. Next we estimate a first order term:

$$\begin{aligned} \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu &\leq c \left(\int_M |A|^{\frac{9}{2}} |\nabla_{(2)} A| |\nabla A|^{\frac{1}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\ &\quad + c \left(\int_M |A|^{\frac{7}{2}} |\nabla A|^{\frac{5}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\ &\quad + c \left(\int_M |A|^{\frac{11}{2}} |\nabla A|^{\frac{3}{2}} \gamma^{\frac{3s}{4}} d\mu \right)^{\frac{4}{3}} \\ &\quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |A|^{\frac{9}{2}} |\nabla A|^{\frac{3}{2}} \gamma^{\frac{3s-4}{4}} d\mu \right)^{\frac{4}{3}} \\ &\leq c \left(\int_M |\nabla_{(2)} A|^2 |A|^4 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla A| |A|^5 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c \left(\int_M |\nabla A|^5 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |A|^7 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |\nabla A|^2 |A|^4 \gamma^{s-2} d\mu \right)^{\frac{2}{3}} \\ &\quad \times \left(\int_M |\nabla A| |A|^5 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c \|A\|_{4, [\gamma > 0]}^4 \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu \\ &\leq \theta \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla A|^5) \gamma^s d\mu \\ &\quad + c \left(\int_M |\nabla A| |A|^5 \gamma^{\frac{s}{2}} d\mu \right)^2 \\ &\quad + c \|A\|_{4, [\gamma > 0]}^4 \int_M |A|^{10} \gamma^s d\mu \\ &\quad + c(c_\gamma)^2 \int_M |\nabla A|^2 |A|^4 \gamma^{s-2} d\mu \end{aligned}$$

$$\begin{aligned}
 & + c\|A\|_{4, [\gamma > 0]}^4 \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu \\
 \leq & \theta \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla A|^5) \gamma^s d\mu \\
 & + c\|A\|_{4, [\gamma > 0]}^4 \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu \\
 & + c\|A\|_{4, [\gamma > 0]}^4 \int_M |A|^{10} \gamma^s d\mu \\
 & + c(c_\gamma)^2 \int_M |\nabla A|^2 |A|^4 \gamma^{s-2} d\mu.
 \end{aligned}$$

Combining this with the estimate (recall $3s \geq 20$)

$$\begin{aligned}
 & c(c_\gamma)^2 \int_M |\nabla A|^2 |A|^4 \gamma^{s-2} d\mu \\
 & \leq \theta \int_M |\nabla A|^5 \gamma^s d\mu + c(c_\gamma)^{\frac{10}{3}} \int_M |A|^{\frac{20}{3}} \gamma^{\frac{3s-10}{3}} d\mu \\
 & \leq \theta \int_M |\nabla A|^5 \gamma^s d\mu + c(c_\gamma)^{\frac{10}{3}} \left(\int_M |A|^{10} \gamma^s d\mu \right)^{\frac{1}{2}} \\
 & \quad \times \left(\int_M |A|^{\frac{10}{3}} \gamma^{\frac{3s-20}{6}} d\mu \right)^{\frac{1}{2}} \\
 & \leq \theta \int_M |\nabla A|^5 \gamma^s d\mu + c(c_\gamma)^{\frac{10}{3}} \left(\int_M |A|^{10} \gamma^s d\mu \right)^{\frac{1}{2}} \left(\mu_\gamma(f)^{\frac{1}{6}} \|A\|_{4, [\gamma > 0]}^{\frac{10}{3}} \right)^{\frac{1}{2}} \\
 & \leq \theta \int_M (|\nabla A|^5 + |A|^{10}) \gamma^s d\mu + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{6}} \|A\|_{4, [\gamma > 0]}^{\frac{10}{3}}
 \end{aligned}$$

yields

$$\begin{aligned}
 \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu & \leq \theta \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla A|^5 + |A|^{10}) \gamma^s d\mu \\
 & + c\|A\|_{4, [\gamma > 0]}^4 \int_M |\nabla A|^2 |A|^6 \gamma^s d\mu \\
 & + c\|A\|_{4, [\gamma > 0]}^4 \int_M |A|^{10} \gamma^s d\mu \\
 & + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{6}} \|A\|_{4, [\gamma > 0]}^{\frac{10}{3}}.
 \end{aligned}$$

These estimates combine to yield

$$\begin{aligned}
 \int_M (|\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu &\leq \theta \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla A|^5 + |A|^{10}) \gamma^s d\mu \\
 (91) \qquad \qquad \qquad &+ c \|A\|_{4, [\gamma > 0]}^4 \int_M (|\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\
 &+ c (c_\gamma)^6 \left([(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{6}} \right) \\
 &\times \left(\|A\|_{4, [\gamma > 0]}^{\frac{10}{3}} + \|A\|_{4, [\gamma > 0]}^2 \right).
 \end{aligned}$$

Now we estimate

$$\begin{aligned}
 \int_M |\nabla A|^5 \gamma^s d\mu &\leq c \int_M |\nabla_{(2)} A| |\nabla A|^3 |A| \gamma^s d\mu + c(c_\gamma) \int_M |\nabla A|^4 |A| \gamma^{s-1} d\mu \\
 &\leq c \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu + \frac{\delta}{10} \int_M |\nabla A|^4 |A|^2 \gamma^s d\mu \\
 &\quad + \frac{1}{2} \int_M |\nabla A|^5 \gamma^s d\mu + c(c_\gamma) \int_M |\nabla A|^4 |A| \gamma^{s-1} d\mu \\
 &\leq c \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu + \delta \int_M |\nabla A|^5 \gamma^s d\mu \\
 &\quad + \delta \int_M |A|^{10} \gamma^s d\mu \\
 &\quad + \frac{3}{4} \int_M |\nabla A|^5 \gamma^s d\mu + c(c_\gamma)^5 \int_M |A|^5 \gamma^{s-5} d\mu
 \end{aligned}$$

so that absorbing for δ sufficiently small yields

$$\begin{aligned}
 \int_M |\nabla A|^5 \gamma^s d\mu &\leq c \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu + \theta \int_M |A|^{10} \gamma^s d\mu \\
 &\quad + c(c_\gamma)^5 \int_M |A|^5 \gamma^{s-5} d\mu.
 \end{aligned}$$

Now (recall $4s \geq 25$)

$$\begin{aligned}
 c(c_\gamma)^5 \int_M |A|^5 \gamma^{s-5} d\mu &\leq \theta \int_M |A|^{10} \gamma^s d\mu + c(c_\gamma)^{\frac{25}{4}} \int_M |A|^{\frac{15}{4}} \gamma^{\frac{4s-25}{4}} d\mu \\
 &\leq \theta \int_M |A|^{10} \gamma^s d\mu + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{16}} \|A\|_{4, [\gamma > 0]}^{\frac{15}{4}}.
 \end{aligned}$$

Combining this with the previous estimate we find

$$\begin{aligned} \int_M |\nabla A|^5 \gamma^s d\mu &\leq c \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu + \theta \int_M |A|^{10} \gamma^s d\mu \\ &\quad + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{16}} \|A\|_{4, [\gamma>0]}^{\frac{15}{4}}. \end{aligned}$$

Using this we estimate the RHS of (91) and absorb to find

$$\begin{aligned} &\int_M (|\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\ &\leq \theta \int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2) \gamma^s d\mu \\ &\quad + c \|A\|_{4, [\gamma>0]}^4 \int_M (|\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\ (92) \quad &\quad + c(c_\gamma)^6 \|A\|_{4, [\gamma>0]}^2 \left(1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}}\right) \left(1 + \|A\|_{4, [\gamma>0]}^{\frac{3}{4}}\right). \end{aligned}$$

Note that we interpolated some terms in the last product on the right hand side. Now we move on to terms involving $\nabla_{(2)} A$. We begin with

$$\begin{aligned} &\int_M |\nabla_{(2)} A|^2 |A|^4 \gamma^s d\mu \\ &\leq c \left(\int_M |\nabla_{(3)} A| |\nabla_{(2)} A|^{\frac{1}{2}} |A|^3 \gamma^{\frac{3s}{4}} d\mu + \int_M |\nabla_{(2)} A|^{\frac{3}{2}} |\nabla A| |A|^2 \gamma^{\frac{3s}{4}} d\mu \right. \\ &\quad \left. + \int_M |\nabla_{(2)} A|^{\frac{3}{2}} |A|^4 \gamma^{\frac{3s}{4}} d\mu + (c_\gamma) \int_M |\nabla_{(2)} A|^{\frac{3}{2}} |A|^3 \gamma^{\frac{3s-4}{4}} d\mu \right)^{\frac{4}{3}} \\ &\leq \left(\int_M |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c \left(\int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c \left(\int_M |\nabla_{(2)} A|^2 |A|^4 \gamma^s d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\quad + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^{s-4} d\mu \right)^{\frac{2}{3}} \left(\int_M |\nabla_{(2)} A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^{\frac{2}{3}} \\ &\leq c \int_M |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu \end{aligned}$$

$$\begin{aligned}
& + \theta \int_M (|\nabla_{(2)}A|^2 |\nabla A|^2 + |\nabla_{(2)}A|^2 |A|^4) \gamma^s d\mu \\
& + c \left(\int_M |\nabla_{(2)}A| |A|^4 \gamma^{\frac{s}{2}} d\mu \right)^2 + c(c_\gamma)^2 \int_M |\nabla_{(2)}A|^2 |A|^2 \gamma^{s-4} d\mu \\
\leq & c \int_M |\nabla_{(3)}A|^2 |A|^2 \gamma^s d\mu + (\theta + c \|A\|_{4, [\gamma > 0]}^4) \\
& \times \int_M (|\nabla_{(2)}A|^2 |\nabla A|^2 + |\nabla_{(2)}A|^2 |A|^4) \gamma^s d\mu \\
(93) \quad & + c(c_\gamma)^4 \int_M |\nabla_{(2)}A|^2 \gamma^{s-8} d\mu.
\end{aligned}$$

Since

$$\begin{aligned}
(c_\gamma)^2 \int_M |\nabla_{(k)}A|^2 \gamma^{s-4} d\mu & \leq \frac{1}{2} (c_\gamma)^2 \int_M |\nabla_{(k)}A|^2 \gamma^{s-4} d\mu \\
& + \theta \int_M |\nabla_{(k+1)}A|^2 \gamma^s d\mu \\
& + c(c_\gamma)^4 \int_M |\nabla_{(k-1)}A|^2 \gamma^{s-8} d\mu
\end{aligned}$$

implies

$$\begin{aligned}
(c_\gamma)^2 \int_M |\nabla_{(k)}A|^2 \gamma^{s-4} d\mu & \leq \theta \int_M |\nabla_{(k+1)}A|^2 \gamma^s d\mu \\
(94) \quad & + c(c_\gamma)^4 \int_M |\nabla_{(k-1)}A|^2 \gamma^{s-8} d\mu,
\end{aligned}$$

we have

$$\begin{aligned}
c(c_\gamma)^4 \int_M |\nabla_{(2)}A|^2 \gamma^{s-8} d\mu & \leq \theta \int_M |\nabla_{(4)}A|^2 \gamma^s d\mu + c(c_\gamma)^8 \int_M |A|^2 \gamma^{s-16} d\mu \\
& \leq \theta \int_M |\nabla_{(4)}A|^2 \gamma^s d\mu \\
& + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^2.
\end{aligned}$$

Combining this with the estimate (93) yields

$$\int_M |\nabla_{(2)}A|^2 |A|^4 \gamma^s d\mu$$

$$\begin{aligned}
 &\leq (\theta + c\|A\|_{4, [\gamma>0]}^4) \\
 &\quad \times \int_M (|\nabla_{(4)}A|^2 + |\nabla_{(2)}A|^2|\nabla A|^2 + |\nabla_{(2)}A|^2|A|^4)\gamma^s d\mu \\
 (95) \quad &\quad + c \int_M |\nabla_{(3)}A|^2|A|^2\gamma^s d\mu + c(c_\gamma)^6[(c_\gamma)^4\mu_\gamma(f)]^{\frac{1}{2}}\|A\|_{4, [\gamma>0]}^2.
 \end{aligned}$$

In order to estimate the remaining term involving $\nabla_{(2)}A$ we first note the following equality:

$$\begin{aligned}
 &-\int_M (\nabla_{ij}A_{kl}\nabla^j A^{kl})(\nabla^i|\nabla A|^2)\gamma^s d\mu \\
 &= -2 \int_M (\nabla_{ij}A_{kl}\nabla^j A^{kl})(\nabla^i\nabla_p A_{qr}\nabla^p A^{qr})\gamma^s d\mu \\
 &= -\frac{1}{2} \int_M |\nabla|\nabla A|^2|^2\gamma^s d\mu.
 \end{aligned}$$

In particular, this term has a sign. We use this to estimate

$$\begin{aligned}
 \int_M |\nabla_{(2)}A|^2|\nabla A|^2\gamma^s d\mu &\leq c \int_M |\nabla_{(3)}A||\nabla A|^3\gamma^s d\mu \\
 &\quad + c(c_\gamma) \int_M |\nabla_{(2)}A||\nabla A|^3\gamma^{s-1} d\mu \\
 &\leq \frac{1}{2} \int_M |\nabla_{(2)}A|^2|\nabla A|^2\gamma^s d\mu \\
 &\quad + c \int_M |\nabla_{(3)}A||\nabla A|^3\gamma^s d\mu \\
 (96) \quad &\quad + c(c_\gamma)^2 \int_M |\nabla A|^4\gamma^{s-2} d\mu.
 \end{aligned}$$

In order to control the last two terms on the right, we need two auxilliary estimates. The first is obtained by estimating

$$\begin{aligned}
 &(c_\gamma)^2 \int_M |\nabla A|^4\gamma^{s-2} d\mu \\
 &\leq c(c_\gamma)^2 \int_M |\nabla_{(2)}A||\nabla A|^2|A|\gamma^{s-2} d\mu + c(c_\gamma)^3 \int_M |\nabla A|^3|A|\gamma^{s-3} d\mu \\
 &\leq \frac{1}{2}(c_\gamma)^2 \int_M |\nabla A|^4\gamma^{s-2} d\mu + c(c_\gamma)^2 \int_M |\nabla_{(2)}A|^2|A|^2\gamma^{s-2} d\mu \\
 &\quad + c(c_\gamma)^6\|A\|_{4, [\gamma>0]}^4.
 \end{aligned}$$

Absorbing gives

$$(c_\gamma)^2 \int_M |\nabla A|^4 \gamma^{s-2} d\mu \leq c(c_\gamma)^2 \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^{s-2} d\mu \\ + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^4.$$

Estimating the first term on the right as in (93) (the only difference here is that we have $s - 2$ instead of $s - 4$), using also (94), we find

$$(c_\gamma)^2 \int_M |\nabla A|^4 \gamma^{s-2} d\mu \\ \leq \theta \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(2)} A|^2 |A|^4) \gamma^s d\mu \\ (97) \quad + c(c_\gamma)^6 (1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}}) (1 + \|A\|_{4, [\gamma > 0]}^2) \|A\|_{4, [\gamma > 0]}^2.$$

The second term in (96) is estimated as follows:

$$c \int_M |\nabla_{(3)} A| |\nabla A|^3 \gamma^s d\mu \leq c \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{4}} \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{4}} \\ \leq \theta \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{2}} \\ + c \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}}.$$

The first term will be estimated below, it is also useful in controlling the highest order term involving $\nabla_{(3)} A$. For the second, we calculate

$$\left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\ \leq \left(\int_M |\nabla_{(2)} A| |\nabla A|^2 |A| \gamma^{\frac{2s}{3}} d\mu + (c_\gamma) \int_M |\nabla A|^3 |A| \gamma^{\frac{2s-3}{3}} d\mu \right)^{\frac{3}{2}} \\ \leq \theta \left(\int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \right)^{\frac{3}{4}} \left(\int_M |\nabla A|^2 |A|^2 \gamma^{\frac{s}{3}} d\mu \right)^{\frac{3}{4}} \\ + (c_\gamma)^{\frac{3}{2}} \left(\|A\|_{4, [\gamma > 0]} \left[\int_M |\nabla A|^4 \gamma^{\frac{8s-12}{9}} d\mu \right]^{\frac{3}{4}} \right)^{\frac{3}{2}}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + \theta \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \\
 &\quad + \left(\int_M |\nabla A|^2 |A|^2 \gamma^{\frac{s}{3}} d\mu \right)^3 + (c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^6 \\
 &\leq \frac{1}{2} \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + \theta \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \\
 &\quad + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + (c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^6
 \end{aligned}$$

Note that in the above we used $\frac{8s-12}{9} \geq \frac{2s}{3}$. Absorbing yields

$$\begin{aligned}
 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} &\leq \theta \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \\
 &\quad + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\
 &\quad + (c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^6.
 \end{aligned}$$

This gives the following estimate for the second term in (96):

$$\begin{aligned}
 &c \int_M |\nabla_{(3)} A| |\nabla A|^3 \gamma^s d\mu + \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\
 &\leq \theta \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{2}} + \theta \int_M |\nabla_{(2)} A|^2 |\nabla A|^2 \gamma^s d\mu \\
 &\quad + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + (c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^6.
 \end{aligned}$$

Combining the second order estimates (95), (96), (97) together, and absorbing, we have the following partial estimate:

$$\begin{aligned}
 &\int_M (|\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2) \gamma^s d\mu + \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\
 &\leq (\theta + c \|A\|_{4, [\gamma > 0]}^4) \\
 &\quad \times \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla_{(2)} A|^2 |A|^4) \gamma^s d\mu
 \end{aligned}$$

$$(98) \quad \begin{aligned} & + \theta \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{2}} + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\ & + c(c_\gamma)^6 (1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}}) (1 + \|A\|_{4, [\gamma > 0]}^2) \|A\|_{4, [\gamma > 0]}^2. \end{aligned}$$

Let us now turn to controlling the highest order term. We first show the following estimate, which is also needed for the terms involving $\nabla_{(2)} A$ above:

$$\begin{aligned} & \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \leq c \left(\int_M |\nabla_{(4)} A| |\nabla_{(3)} A|^2 \gamma^{\frac{3s}{2}} d\mu + \int_M |\nabla_{(3)} A|^3 |A| \gamma^{\frac{3s}{2}} d\mu \right. \\ & \quad \left. + (c_\gamma) \int_M |\nabla_{(3)} A|^3 \gamma^{\frac{3s-2}{2}} d\mu \right)^{\frac{4}{3}} \\ & \leq c \left(\left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^{\frac{1}{2}} \left[\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right]^{\frac{1}{2}} \right. \\ & \quad \left. + \|A\|_{4, [\gamma > 0]} \left[\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right]^{\frac{3}{4}} + (c_\gamma) \int_M |\nabla_{(3)} A|^3 \gamma^{\frac{3s-2}{2}} d\mu \right)^{\frac{4}{3}} \\ & \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^{\frac{2}{3}} \left[\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right]^{\frac{2}{3}} \\ & \quad + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu + c(c_\gamma)^{\frac{4}{3}} \left(\int_M |\nabla_{(3)} A|^3 \gamma^{\frac{3s-2}{2}} d\mu \right)^{\frac{4}{3}} \\ & \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^2 + (\theta + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \quad + c(c_\gamma)^{\frac{4}{3}} \left[\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right]^{\frac{2}{3}} \left[\int_M |\nabla_{(3)} A|^2 \gamma^{s-4} d\mu \right]^{\frac{2}{3}} \\ & \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^2 + (\theta + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}}) \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \quad + c(c_\gamma)^4 \left[\int_M |\nabla_{(3)} A|^2 \gamma^{s-4} d\mu \right]^2. \end{aligned}$$

Absorbing yields the estimate

$$\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^2$$

$$(99) \quad \begin{aligned} & + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & + c(c_\gamma)^4 \left[\int_M |\nabla_{(3)} A|^2 \gamma^{s-4} d\mu \right]^2. \end{aligned}$$

Estimate (94) implies

$$\begin{aligned} (c_\gamma)^2 \int_M |\nabla_{(3)} A|^2 \gamma^{s-4} d\mu & \leq c \int_M |\nabla_{(4)} A|^2 \gamma^s d\mu + c(c_\gamma)^8 \int_M |A|^2 \gamma^{s-16} d\mu \\ & \leq c \int_M |\nabla_{(4)} A|^2 \gamma^s d\mu + c(c_\gamma)^8 \mu_\gamma(f)^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^2. \end{aligned}$$

Combining this with (99) we find

$$(100) \quad \begin{aligned} & \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu \right]^2 + c \|A\|_{4, [\gamma > 0]}^{\frac{4}{3}} \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \quad + c(c_\gamma)^{12} [(c_\gamma)^4 \mu_\gamma(f)] \|A\|_{4, [\gamma > 0]}^4 \\ & \leq c \left[\int_M |\nabla_{(4)} A|^2 \gamma^s d\mu + c \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}} \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{2}} \right. \\ & \quad \left. + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^2 \right]^2. \end{aligned}$$

We apply the auxilliary estimate (100) to control the following

$$\begin{aligned} & \int_M |\nabla_{(3)} A|^2 |A|^2 \gamma^s d\mu + \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \leq 2 \|A\|_{4, [\gamma > 0]}^2 \left(\int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \right)^{\frac{1}{2}} \\ & \leq c \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(4)} A|^2 \gamma^s d\mu + c \|A\|_{4, [\gamma > 0]}^{\frac{8}{3}} \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\ & \quad + c(c_\gamma)^6 [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}} \|A\|_{4, [\gamma > 0]}^4. \end{aligned}$$

Combining the above with (100), and the lower order estimates (92), (98), and interpolating some terms, we finally conclude

$$\int_M (|\nabla_{(3)} A|^2 |A|^2 + |\nabla_{(2)} A|^2 |A|^4 + |\nabla_{(2)} A|^2 |\nabla A|^2$$

$$\begin{aligned}
& + |\nabla A|^2 |A|^6 + |A|^{10} \gamma^s d\mu \\
& + \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} + \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\
& \leq (\theta + c \|A\|_{4, [\gamma > 0]}^4) \int_M (|\nabla_{(4)} A|^2 + |\nabla_{(2)} A|^2 |\nabla A|^2 + |\nabla_{(2)} A|^2 |A|^4 \\
& \quad + |\nabla A|^2 |A|^6 + |A|^{10}) \gamma^s d\mu \\
& + \|A\|_{4, [\gamma > 0]}^6 \left(\int_M |\nabla A|^4 \gamma^{\frac{2s}{3}} d\mu \right)^{\frac{3}{2}} \\
& + c(\theta + \|A\|_{4, [\gamma > 0]}^{\frac{2}{3}}) \|A\|_{4, [\gamma > 0]}^2 \int_M |\nabla_{(3)} A|^4 \gamma^{2s} d\mu \\
& + c(c_\gamma)^6 \|A\|_{4, [\gamma > 0]}^2 (1 + [(c_\gamma)^4 \mu_\gamma(f)]^{\frac{1}{2}}) (1 + \|A\|_{4, [\gamma > 0]}^2).
\end{aligned}$$

Finally let us consider (v). The estimate (18) has already been proved, it is the intermediate estimate (90). For (19), we note first that the equality (72) implies the estimate

$$\begin{aligned}
\int_M |\nabla A|^4 \gamma^s d\mu & \leq c \int_M |\nabla_{(2)} A| |\nabla A|^2 |A| \gamma^s d\mu \\
& + c(c_\gamma) \int_M |\nabla_{(2)} A| |\nabla A| |A|^2 \gamma^{s-1} d\mu \\
& \leq \frac{1}{4} \int_M |\nabla A|^4 \gamma^s d\mu + c \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
& + c(c_\gamma)^2 \int_M |\nabla A|^2 |A|^2 \gamma^{s-2} d\mu \\
& \leq \frac{1}{2} \int_M |\nabla A|^4 \gamma^s d\mu + c \int_M |\nabla_{(2)} A|^2 |A|^2 \gamma^s d\mu \\
& + c(c_\gamma)^4 \int_M |A|^4 \gamma^{s-4} d\mu.
\end{aligned}$$

The final estimate (19) follows by absorption. \square

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