Optimal control of a large dam using time-inhomogeneous Markov chains with an application to flood control

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Abstract: We consider a decision making framework for a dam modelled by a controllable time-inhomogeneous Markov chain with a compound Poisson process as an input, which approximates the inflow pattern of real dams with seasonal rainfall. The controls are responsible for output flow explicitly and implicitly via a price on water consumption. We consider the problem with system balance and demand shortfall performance criteria under constraints. The state space of the dam is discretized while the time domain remains continuous. We also add a controllable water release to reduce the probability of the dam overflowing and give the procedure for determining if an optimal solution exists by showing the consistency of constraints.

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1. INTRODUCTION

This paper expands on previous work that develops a decision making framework for the optimal control of water resources, especially flood control, which a number of recent examples have shown is often not dynamic enough to react to rapidly rising water levels (see South East Queensland Water (2011) and Gray (2016)). Stochastic discrete-continuous systems are an example of stochastic hybrid systems, where the state changes abruptly and these changes are controlled by transition rates. This approach has many applications in natural resources, gas, electricity and water distribution systems (for example see Williams (2009) and references therein). We treat a large dam as a stochastic hybrid system, with the dam volume being discretized while the time domain remains continuous.

This type of problem is usually treated as a Markov decision processes (MDP). Here we introduce some major extensions to the model outlined in Miller and McInnes (2011a) and we clarify the procedure for checking consistency of constraints previously reported in McInnes and Miller (2013). We introduce a time-inhomogeneous inflow process to the dam model as well as a time and state dependent water release control. We also include an overflow state to explicitly take into account the probability of overflow. The optimality of this solution is in reference to the standard practice of allowing only small volume environmental flows and increasing these only in the case of an actual flooding event (for a typical policy see South Gippsland Water (2015)). In section 2 we detail the dynamics of the new model and give the form of the generator of the continuous-time controllable Markov chain (CCMC) which describes the dam process. In section 3 we describe how the price control dependent consumption, \( C(t, p) = C(t, p(t, X)) \), is derived. Section 4 deals with the solution of the problem via dynamic programming, section 5 shows the procedure to check consistency of constraints and in section 6 we give a numerical example of this.

2. DAM MODEL

We assume that we can approximate the level of a large finite dam by discretizing the volume of the dam into \( N \in \mathbb{N} \) levels and denote the level at any time \( t \in [0, T] \), \( T < \infty \), by an integer valued random variable \( L_t \in \{0, ..., N\} \) Delebecque and Quadrat (1981). Figure 1 gives a stylised depiction of the approximate flow process with the overflow level marked.

If we let each level be represented by the set of unit vectors \( S = \{e_0, e_1, ..., e_N\} \in \mathbb{R}^{N+1} \), then we can define a random vector \( X_t \in S \) on \([0, T]\) to represent this level at any time \( t \). Note that this means that \( I \{ L_t = i \} = I \{ X_t = e_i \} \) where...
I is an indicator function. All processes are taken to be defined on the standard probability space, \( \{ \Omega, \mathcal{F}, \mathbb{P} \} \).

2.1 Inflows and outflows

**Inflows:** We assume that the process of inflows to the dam can be approximated by a time-inhomogeneous compound Poisson process, \( I_t \). We further assume that this natural inflow is the result of rain events which arrive randomly according to the time-inhomogeneous counting process \( R_t \) with intensity \( \lambda(t) \). The resulting distribution of jumps in the dam level is given by \( Z_t \). If we designate the maximum jump size as \( Z_{\text{max}} \in \mathbb{N} \), then \( Z_t = \{ 1, 2, 3, ..., Z_{\text{max}}-1, Z_{\text{max}} \} \), with \( \mathbb{P} \{ Z_t = j \} = q_j(t) \) independent of \( R_t \) and the state of the dam. Then, if \( \tau_k \) is the time of the \( k \)th jump, \( I_t = \sum_{k=0}^{\tau_k} R_t \).

The semi-martingale representation of \( I_t \) is

\[
I_t = \int_0^t \lambda(s) \mathbb{E}[Z_s] \, ds + M_t^{(i)},
\]

where \( M_t^{(i)} \) is a square-integrable martingale. The expectation of this process is \( \mathbb{E}[I_t] = \int_0^t \lambda(s) \mathbb{E}[Z_s] \, ds \).

If we consider the jumps of this process, then the size of each jump is given by the random variable \( Z_t \). Relating this to the level of the dam, it is clear that if a jump occurs at time \( \tau \) when the dam is at level \( L_t \) and \( L_t + Z_\tau \geq N \), then the dam is overflowing. It follows that we should include jumps greater than \( N - L_t \) in an inflow state, \( N + 1 \). Applying this we can represent the inflows in the following way, where \( \tau \) is the jump instant of \( I_t \):

\[
I_t = \sum_{\tau \leq t} Z_\tau.
\]

Now, \( Z_\tau \) can take the values from 1, ..., \( Z_{\text{max}} \), so we define a sum of indicators of \( i \leq Z_\tau \) and then rewrite (2) as

\[
I_t = \sum_{\tau \leq t} \sum_{i=1}^{Z_{\text{max}}} I \{ i = Z_\tau \} i.
\]

**Outflows:** Outflows are controlled explicitly by controlled releases and implicitly via a price on water. Outflows from the dam are assumed to comprise of natural losses due to evaporation, the consumption of the various dam users, controlled water releases as well as outflows if the inflows exceed the dam capacity. We approximate the natural losses by a general counting process with state dependent intensity, \( \mu(t) = \mu(t, X) \). The consumption is another counting process which depends on a price control, \( p(t) = p(t, X) \), which depends on the current state of the dam, and its intensity will be denoted, \( C(t, p) = C(t, p(t, X)) \). Here \( p(t) \) is taken to be a \( \mathcal{F}_t \)-predictable control in the compact set \([p_{\text{min}}, p_{\text{max}}]\). A state and time dependent controlled counting process with controllable intensity \( \nu(t, X) = \nu(t) \), represents controlled water releases and is a \( \mathcal{F}_t \)-predictable control in the compact set \([\nu_{\text{min}}, \nu_{\text{max}}]\), where \( \nu_{\text{min}} \) and \( \nu_{\text{max}} \) are the minimum and maximum release rates respectively. The semi-martingale form of these processes is given by

\[
O_t = \int_0^t \left( \mu(s) + C(s, p) + \nu(s) \right) I \{ L_s > 0 \} \, ds + M_t^{(o)},
\]

where \( M_t^{(o)} \) is a square-integrable martingale. Here, as with the inflows, we can rewrite the outflow process as

\[
O_t = \sum_{\eta \leq t} I \{ L_{\eta} > 0 \} \Delta O_{\eta},
\]

where the \( \eta \) are the jump instants for the outflow process and \( \Delta O_{\eta} = 1 \).

**Remark 1.** We assume that \( I_t \) and \( O_t \) are processes whose jumps do not occur at the same instant. This implies that the mutual quadratic variation, \( \langle M^{(i)}, M^{(o)} \rangle_t \), is zero.

2.2 Semi-Martingale model of the process \( X_t \)

**Proposition 2.** The infinitesimal generator, \( A(t, p(t), \nu(t)) \), of the controllable Markov chain, \( X_t \), has the form (with dependencies omitted)

\[
A(t, p, \nu) = \begin{pmatrix}
\lambda & C + \mu + \nu & \ldots & 0 & 0 \\
\lambda q_1 & (\lambda + C + \mu + \nu) & \ldots & 0 & 0 \\
\lambda q_2 & \lambda q_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda q_{\text{max}} & \lambda q_{\text{max}} & \ldots & (\lambda + C + \mu + \nu) & 0 \\
\lambda \sum_{k=N}^{\infty} q_k & \lambda \sum_{k=N}^{\infty} q_k & \ldots & \lambda & 0
\end{pmatrix}
\]

**Remark 3.** In general we assume the control as predictable process depending on entire history of the process up to time \( t \).

**Remark 4.** Once in the overflow state the process stops, that is, the \((N + 1)^{th}\) state is absorbing. In this model, control of transition probabilities stops in this state and other direct control methods must be used.

**Proof.** We omit the proof for brevity but note that it is similar to a proof given for the form of the infinitesimal generator of a controllable queuing system (see Miller (2009) section 3, especially section 3.1 and the proof of Proposition 2). Here we have a complex inflow process, given by a time-inhomogeneous compound Poisson process and the proof is necessarily more complicated by having to consider jumps of more than unit size.

3. CONTROLLED WATER USE VIA PRICE CONTROL AND CONTROLLABLE RELEASE

To optimally regulate the demand on water from the dam, we introduce an optimal price on water for each state of the dam at any time. The detailed model of the price \( p(t) = p(t, X) \in [p_{\text{min}}, p_{\text{max}}] \) control, basing on the customers demands and their utility functions is already given in Miller and McInnes (2011a).

With total constraints taking into account various demands of customers Miller and McInnes (2011a) one can now define maximum and minimum optimal demand by setting \( p(t) = p_{\text{min}} \) and \( p(t) = p_{\text{max}} \) respectively, giving us \( C_{\text{max}}(t) \) and \( C_{\text{min}}(t) \).

\[
C_{\text{max}}(t) = \sum_{i=1}^{n} \max(x_i(t) - \frac{p_{\text{min}}}{2}, 0), \quad (5)
\]

and
Recalling that \( p(t) \) is a vector, we now have a vector of optimal demand functions such that \( C_{\min}(t) \leq C(t, p(t, X(t))) \leq C_{\max}(t) \).

4. DYNAMIC PROGRAMMING EQUATION AND ITS SOLUTION

Here we outline the method of solution for the optimal controls in a single large dam. This method has been developed in Miller (2009) for control of a server router, and in Miller and McInnes (2011a), and Miller and McInnes (2011b) for single and multiple dams in a system. In general, this can be extended to a system of dams by using the methodology developed in under the same assumptions as for a single dam Miller et al. (2015).

We start with the general performance criterion

\[
\min_{p(t), \nu(t)} J[p(t), \nu(t)],
\]

where

\[
J[p(t), \nu(t)] = \mathbb{E} \left[ \phi_0(X_T) + \int_0^T f_0(s, p(s), \nu(s), X_s) ds \right].
\]

If \( \langle \cdot, \cdot \rangle \) is the inner product and \( \phi_0 \in \mathbb{R}^n \), then \( \phi_0(X_T) = \langle \phi_0, X_T \rangle \), and \( f_0(s, p(s), \nu(s), X_s) = \langle f_0(s, p(s), \nu(s), X_s), X_s \rangle \). The term \( f_0(s, p(s), \nu(s), X_s) \) is the running cost function of the chain at time \( s \) in state \( X_s \), and so we define

\[
\phi_0(s, p, \nu) = \langle f_0(s, p(s), \nu(s), X_s), X_s \rangle
\]

as the vector of running cost functions of the Markov chain.

**Assumption 5.** For each \( i, \ i = 1, ..., N \) the components of the running cost function \( f_0(s, p, \nu) \) are continuous on \( [0, T] \times [p_{min}, p_{max}] \times [0, \nu_{max}] \) and bounded below.

From this assumption we can consider the value function

\[
V(t, x) = \min_{p(t), \nu(t)} J[p(t), \nu(t)] | X_t = x,
\]

which gives the infimal cost to go from state \( X_t = x \) at some time \( t < T \) to state \( X_T \), and be certain that this infimum exists.

Now, recalling that our state space is made up of the unit vectors in \( \mathbb{R}^{N+1} \), we can describe the value function for each state as \( V(t, x) = \langle \phi(t), x \rangle \), where \( \phi(t) = (\phi_1(t), \phi_2(t), ..., \phi_{N+1}(t))^T \in \mathbb{R}^{N+1} \) is a measurable function giving the cost to go for each state.

We now define \( \hat{\phi}(t) \) as having the same form as \( \phi(t) \) and form the dynamic programming equation with respect to \( \hat{\phi}(t) \) (Elliott et al. (1995)):

\[
0 = \langle \hat{\phi}'(t), x \rangle - \min_{p \in P, \nu \in [0, \nu_{max}]} \left[ \langle \hat{\phi}(t), A(t, p, \nu) x \rangle + \langle f_0(t, p, \nu, x), x \rangle \right] = \langle \hat{\phi}'(t), x \rangle - \min_{p \in P, \nu \in [0, \nu_{max}]} H(t, \hat{\phi}(t), p, \nu, x) = \langle \hat{\phi}'(t), x \rangle - \mathcal{H}(t, \hat{\phi}(t), x),
\]

with boundary condition \( \hat{\phi}(T) = \phi_0 \). As \( H(t, \hat{\phi}(t), p, \nu, x) \) is continuous in \( (t, p, \nu) \) and affine in \( \phi \), for any \( (t, x) \in [0, T] \times S \), \( \mathcal{H}(t, \hat{\phi}(t), x) \) is Lipschitz in \( \phi \) (Miller (2009)).

**Proposition 6.** Given Assumption 5, equation (9) has a unique solution on \( [0, T] \).

**Remark 7.** Now letting \( x = e_i, \ i = 1, ..., N \), we obtain a system of ordinary differential equations

\[
\frac{d\hat{\phi}(t)}{dt} = -\mathcal{H}(t, \hat{\phi}(t), e_i), \ i = 1, ..., N. \tag{10}
\]

The following theorem describes the connection between the value function, \( V(t, x) \), and the solution of the system (10) as well as some key features of the optimal controls (Elliott et al. (1995), Miller (2009)).

**Theorem 8.** Let \( \hat{\phi}(t) \) be the solution of system (10), then for each \( (t, x) \in [0, T] \times S \) there exists \( p_0(t, x) \in P \) and \( \nu_0(t, x) \in [0, \nu_{max}] \) such that \( H(t, \hat{\phi}(t), p_0(t, x), \nu_0(t, x)) \) achieves a minimum at \( (p_0(t, x), \nu_0(t, x)) \). Then

1. There exists an \( \mathcal{F}_t \)-predictable optimal control, \( (\hat{p}(t, X_0), \hat{\nu}(t, X_0)) \) such that \( V(t, x) = \hat{\phi}(t, x) \).
2. The optimal control can be chosen as Markovian, that is

\[
(\hat{p}(t, X_0), \hat{\nu}(t, X_0)) = (p_0(t, X_t-), \nu_0(t, X_t-)) = \arg \min_{p, \nu} H(t, \hat{\phi}, p, \nu, X_t-).
\]

The system of equations (10) is solved numerically which gives the optimal values of water prices and the water release intensity at any possible state and time.

5. CONSISTENCY OF CONSTRAINTS

We now consider how to test the consistency of constraints. We need to show that there exist controls \( (p, \nu) \) such that for a corresponding system of constraints \( J_m \), for \( m = 1, ..., M \), the \( J_m \leq 0, \forall m \). Note that \( \gamma_0 \) is the multiplier of the criterion describing the objective function and so is set to zero when considering the consistency of the system of constraints. The following proposition gives a criterion for the consistency of the system of constraints Miller et al. (2009).

**Proposition 9.** Let

\[
\Gamma_0 = \left\{ \gamma : \gamma_0 = 0, \gamma_1 \geq 0, ..., \gamma_M \geq 0; \sum_{m=1}^{M} \gamma_m = 1 \right\},
\]

the vector of values of the criteria be \( \bar{J}(U(\cdot)) = (J_0(U(\cdot)), ..., J_M(U(\cdot))) \) and \( \mathcal{H}(\gamma, J) = (\gamma, J) \). Then, the system of inequality constraints \( J_m(U(\cdot)) \leq 0 \) is consistent if and only if

\[
\max_{\Gamma_0} \min_{J \in \mathcal{J}} \mathcal{H}(\gamma, J) \leq 0. \tag{11}
\]

**Proof.** The proof is omitted for brevity but is given in Miller et al. (2009).

The next proposition gives a stronger condition for consistency.

**Proposition 10.** The system of inequality constraints, \( J_m(U(\cdot)) \leq 0 \), is strongly consistent and satisfies the Slater condition if and only if

\[
\max_{\Gamma_0} \min_{J \in \mathcal{J}} \mathcal{H}(\gamma, J) < 0. \tag{12}
\]
This is equivalent to the statement that there exists $U^0$ such that $J_m(U^0) < 0$, $\forall m = 1, \ldots, M$ (Miller et al. (2009)).

**Proof.** The proof is also given in Miller et al. (2009).

We do not deal with the procedure for finding the optimal control solutions for the dam problem, which is a difficult numerical problem (see Miller et al. (2009)), but simply note that the consistency of the constraints implies that an optimal solution exists. and Lee and Marcus (1967).

### 6. EXAMPLE SHOWING CONSISTENCY OF CONSTRAINTS

#### 6.1 Performance criteria

The first criterion is the mean square difference of some fixed proportion of the water demanded by customers and water optimally supplied, representing a measure of how well the customers demanded water needs are being met.

This is subject to the square of this difference being less than some amount, say $\alpha > 0$, with $\alpha$ being greater than the minimum possible value of this criterion. Let the customer demand $(1 - r)\sum_{i=1}^{n} x_i(t) = C(t)$, then

$$J_1[p(\cdot)] = \mathbb{E}p \left\{ \int_0^T (C(s, p(s)) - C(s))^2 ds \right\},$$

with $J_1[p(\cdot)] \leq \alpha$.

The second criterion is the mean square difference between inflows and outflows, and is an expression of the requirement for balance between inflows and outflows in the system. This should be less than some given quantity, say $\beta > 0$, with $\beta$ greater than the minimum possible value of the criterion:

$$J_2[p(\cdot), \nu(\cdot)] = \mathbb{E}^{p, \nu} \left\{ \int_0^T (\lambda s - \mu(s, X(s))) - C(s, p(s)(X(s))) - \nu(s, X(s)))^2 ds \right\},$$

with $J_2[p(\cdot), \nu(\cdot)] \leq \beta$.

#### 6.2 System of ODE’s

To find the form of the controls, we set $\gamma_0 = 0$, as in Proposition 9 and consider the slightly modified set of differential equations (15). Here we have only two criteria, so the multiplier for the first will be $\gamma$ and the second $(1 - \gamma)$, such that $\gamma + (1 - \gamma) = 1$ as required by Proposition 9. The resulting equation is

$$\min_{p(t, e_i), \nu(t, e_i)} H(t, \phi(t), p(t, e_i), \nu(t, e_i), e_i, \gamma) =$$

$$\phi_{t-1}(t)(C(t, p(t, e_i) + \mu(t, e_i) + \nu(t, e_i)) - \phi_i(t)(\lambda(t)$$

$$+ C(t, p(t, e_i) + \mu(t, e_i) + \nu(t, e_i)) + \sum_{j=1}^{Z} \phi_{i+j}(t) g_j(t)$$

$$+ [\gamma(\tilde{C}(t) - C(t, p(t, e_i))]^2 - \alpha] + (1 - \gamma)((\lambda(t) - \mu(t, e_i)$$

$$- C(t, p(t, e_i) - \nu(t, e_i))^2 - \beta]$$

for $i = 1, \ldots, N - 1$. In practice we minimise over $C(t, p(t, e_i))$ because we can find the explicit form of this from the solutions of the ODE system and extract $p(t, e_i)$ from this. To minimise we take the partial derivatives of equation (15) with respect to $C(t, p(t, e_i))$ and $\nu(t, e_i)$ to get the following system equations:

$$0 = \frac{\partial H(t, \phi(t), p(t, \nu, e_i, \gamma)}{\partial C(t, p(t, e_i))}$$

$$= \phi_{t-1}(t) - \phi_i(t) - 2\gamma(\tilde{C}(t) - C(t, p(t, e_i)))$$

$$- 2(1 - \gamma)(\lambda(t) - \mu(t, e_i) - C(t, p(t, e_i)) - \nu(t, e_i))$$

$$0 = \frac{\partial H(t, \phi(t), p(t, \nu, e_i, \gamma)}{\partial \nu(t, e_i)}$$

$$= \phi_{t-1}(t) - \phi_i(t) - 2(1 - \gamma)(\lambda(t) - \mu(t, e_i)$$

$$- C(t, p(t, e_i)) - \nu(t, e_i))$$

These equations are from the convex optimisation procedure and apply only if the controls at the stationary points are inside the control set. If they are, then the partial derivatives are identical except for the term $-2\gamma(\tilde{C}(t) - C(t, p(t, e_i)))$, which implies that this term must be zero, or $\tilde{C}(t) = C(t, p(t, e_i))$. Substituting this into either of the above gives

$$\nu(t, e_i) = \frac{\phi_i(t) - \phi_{t-1}(t)}{2(1 - \gamma)} + \lambda(t) - \mu(t, e_i) - \tilde{C}(t).$$

Let $C^*(t, e_i)$ and $\nu^*(t, e_i)$ be the form of these minimising controls, then the solutions are

$$(C^*(t, e_i), \nu^*(t, e_i)) = (\tilde{C}(t), \phi_i(t) - \phi_{t-1}(t)$$

$$\frac{2(1 - \gamma)}{\lambda(t) - \mu(t, e_i) - \tilde{C}(t)}$$

However, $C(t, p(t, e_i)) \in [C_{min}(t), C_{max}(t)]$ and $\nu(t, e_i) \in [\nu_{min}, \nu_{max}]$, so $C^*(t, e_i)$ and $\nu^*(t, e_i)$ are either within these bounds or they are on the boundary. In the case of $C^*(t, e_i)$, since $\tilde{C}(t)$ is known beforehand, this means that for each $t \in [0, T]$ we set $C(t, p(t, e_i))$ as

$$C(t, p(t, e_i)) = \begin{cases} C_{min}(t), & \tilde{C}(t) \leq C_{min}(t) \\ C(t), & C_{min}(t) < \tilde{C}(t) < C_{max}(t) \\ C_{max}(t), & \tilde{C}(t) \geq C_{max}(t) \end{cases}$$

The control $\nu^*(t, e_i)$ depends on the solution of the system of ODE’s. In this case, when we solve the ODE system numerically, for each $t \in [0, T]$ we set the definition of $\nu(t, e_i)$ to be

$$\nu(t, e_i) = \begin{cases} \nu_{min}, & \nu^*(t, e_i) \leq \nu_{min} \\ \nu^*(t, e_i), & \nu_{min} < \nu^*(t, e_i) < \nu_{max} \\ \nu_{max}, & \nu^*(t, e_i) \geq \nu_{max} \end{cases}$$

The control $\nu^*(t, e_i)$ depends on the solution of the system of ODE’s and on the orientation of the level curves of (15). When solving this optimisation problem numerically, we solve for

$$(C^*(t, e_i), \nu^*(t, e_i)) = (\tilde{C}(t), \phi_i(t) - \phi_{t-1}(t)$$

$$\frac{2}{\lambda(t) - \mu(t, e_i) - \tilde{C}(t)}$$

for each time $t \in [0, T]$ and then check to see whether or not it is in the control set. If it is not, then we set the controls at that time to be the coordinates of the point on the boundary of the control set which is touched by the level curves of (15). In this particular case, the axes of the level curves are not parallel to the edges of the admissible control set, so we have rotated the axes by the coordinate transformation $R(t) = C(t)$ and $S(t) = C(t) + \nu(t)$. 

### Table 1. Model parameters, inflows and demands

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Inflows &amp; Demands</th>
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<tr>
<td>$\alpha$</td>
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</tr>
<tr>
<td>$\beta$</td>
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</tr>
<tr>
<td>$\gamma_0$</td>
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<tr>
<td>$\nu_{min}$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\nu_{max}$</td>
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</tbody>
</table>

Figure 4 shows the values of the constrained performance criteria of ODE’s and on the orientation of the level curves of (15). We see that the probability of overflow increases from $0.1$ to $0.99$ at time $t = 0$, starting from the point $C_{min}(t) = 20$. The control set $C_{max}(t) = 21$ with the shaded rectangular admissible control set. A quadratic optimisation procedure would be abandoned all attempts to balance the flows of the dam with the shaded rectangular region.
The admissible control region is then $C_{\min}(t) \leq C(t) \leq C_{\max}(t)$ and $C_{\min}(t) + \nu_{\min} \leq C(t) + \nu(t) \leq C_{\max}(t) + \nu_{\max}$. Figure 2 shows the level curves of (15), without coordinate transformation, when the dam is in level 10 at time $t = 0.5$ with the shaded rectangular admissible control region. The minimum is clearly not contained in the admissible control set and in this case it is obvious that axes of the level curves are not parallel to the edges of the control set. A quadratic optimisation procedure would be needed to check the point of intersection with the level curves. Figure 3 shows the level curves of (15) when the dam is in level 10 at time $t = 0.5$ with the transformed coordinates and the transformed control set. Now there is no need to introduce an optimisation procedure because the axes are aligned and we can simply assign $C(t)$ first, according to its relation to $C_{\min}(t)$ and $C_{\max}(t)$, and then use this to find the value of $\nu(t)$, depending on the relation of $C(t) + \nu(t)$ to the appropriate control boundaries.

![Fig. 2. Level curves of (15) with $C(t)$ on the horizontal axis and $\nu(t)$ on the vertical axis for dam level 10 at $t = 0.5$. The admissible control region is the shaded rectangular region.](image1)

![Fig. 3. Level curves of (15) with $C(t)$ on the horizontal axis and $C(t) + \nu(t)$ on the vertical axis for dam level 10 at $t = 0.5$. The admissible control region is the shaded rectangular region.](image2)

### 6.3 Numerical example

For this example we have taken the increment of $\gamma$ as $1/100$ so that we end up with 101 solutions for (15). All modelling was done with Mathematica 8 and used default ODE solvers to obtain the solution to the ODE system. The model has the following equations and parameters as inputs:

Figure 4 shows the values of the constrained performance criteria against $\gamma$ for various values of $\alpha$ and $\beta$. The solid, thin line is for $\alpha = 4$ and $\beta = 100$ at time $t = 0. It clearly shows that the values of the criteria are greater than zero for all $\gamma$ and so the constraints are inconsistent in this case. If we change the value of $\beta$ to $\beta = 225$, while keeping $\alpha = 4$ constant, we get the solid, bold line. This shows that the values of the criteria are less than zero for all $\gamma$ and so the constraints are consistent in this case. Because of the relatively small value of $\alpha$, changing it makes little difference to the overall picture. The dashed line shows that the values of the criteria when $\alpha = 25$ and $\beta = 100$ are still mostly above zero, which means that we should constrict our search for controls to the set of values of $\gamma$ where the values of the weighted mixed criteria are less than zero.

Using the control solution for all states, we can set an initial state and calculate the probability of being in a given state at any time $t$. Considering the scenario with $\alpha = 25$ and $\beta = 100$, we find that the value of $\gamma$ which gives us a value just below zero is $\gamma = 0.66$, as can be seen in figure 4. This is not the optimal solution because we have constrained $\gamma$ to the interval $[0, 1]$ but it will indicate the type of result we can expect. After solving the system using this value of $\gamma$ for all states, we calculated the probabilities of going into the overflow state, $L = 21$, starting from the initial state, $L = 20$, just below overflow, with controlled releases and only minimum release. Figure 5 shows the probability of overflow with controlled releases, the bold line, and with only minimum releases, the dashed line. We see that the probability of overflow increases from zero to about 50% with minimum releases and to a little under 30% with controlled releases. This indicates that on average we can decrease the overflow probability by about 20%, which is significant. On the other hand, if we allow continuous maximum releases we get figure 6. We can see that the control solution is the same but the maximum release scenario results in a little under 10% better chance of avoiding overflow on average. This comes at the cost of abandoning all attempts to balance the flows of the dam for future needs, so the control solution is preferable from that perspective considering that this solution is less than optimal.

### 7. CONCLUSION

In this paper we considered a decision making framework for a dam modelled by a controllable time-inhomogeneous Markov chain with a compound Poisson process as an input and considered the problem with various performance criteria under constraints, demonstrating how we can show that optimal solutions exist for the constrained optimisation problem. This framework indicates what should be done on average to attain a certain control objective and is a significant addition to the field, since we do not insist on stationarity or stability of the underlying process. We added a controllable water release to reduce the probabil-

<table>
<thead>
<tr>
<th>Parameters</th>
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<th>Inflows &amp; Demands</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 20$</td>
<td>$n = 10$</td>
<td>$\lambda(t) = 3(\cos(4\pi t)) + 1$</td>
</tr>
<tr>
<td>$M = 10$</td>
<td>$r = 0.1$</td>
<td>$\mu_{X+1}(t) = \sin(2\pi t) + 1$</td>
</tr>
<tr>
<td>$p_{\max} = 3.00$</td>
<td>$K = 50$</td>
<td>$x_{1}(t) = \cos(2\pi t) + 5$</td>
</tr>
<tr>
<td>$p_{\min} = 1.5$</td>
<td>$\nu_{\max} = 15$</td>
<td>$x_{2}(t) = 0.3\cos(2\pi t) + 4$</td>
</tr>
<tr>
<td>$\nu_{\min} = 1$</td>
<td>$x_{3}(t) = 0.5\cos(2\pi t) + 5$</td>
<td></td>
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</tbody>
</table>
ity of the dam overflowing to previous models, making this model a useful extension of our previous work. The reduced probability of flooding is significant, but we note that it would be better than that shown in figure 5 if we start in a more realistic state, since the starting state for these calculations was that just prior to flooding. It is certainly a better solution than allowing low volume releases for environmental flow and only increasing them in the case of actual flooding. In future research we plan to compare this model with other flood mitigation models, such as that proposed by Nasir et al. (2016).

8. ACKNOWLEDGEMENTS

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Fig. 4. Criteria value as a function of $\gamma$ with: (1) Bold - $\alpha = 4$ and $\beta = 100$, (2) Thin - $\alpha = 4$ and $\beta = 225$, and (3) Dashed - $\alpha = 25$ and $\beta = 100$.

Fig. 5. Overflow probabilities with controlled releases (bold) and with minimum releases (dashed) starting in state $L = 20$.

Fig. 6. Overflow probabilities with controlled releases (bold) and with maximum releases (dashed) starting in state $L = 20$.

REFERENCES


