IET Control Theory & Applications

Research Article

Saturated fault tolerant control based on partially decoupled unknown-input observer: a new integrated design strategy

ISSN 1751-8644
Received on 28th November 2018
Revised 30th April 2019
Accepted on 7th June 2019
E-First on 30th July 2019
doi: 10.1049/iet-cta.2018.6349
www.ietdl.org

Mojtaba Hashemi[1], Ali Kamali Egoli[1] ✉, Mahyar Naraghi[1], Chee Pin Tan[2]

[1] *Mechanical Engineering, AmirKabir University of Technology, Hafez Ave., Tehran, Iran*

[2] *School of Engineering and Advanced Engineering Platform, University Malaysia, Monash, 47500 Bandar Sunway, Selangor, Malaysia*

✉ *E-mail: alikamalie@aut.ac.ir*

Abstract: This study presents a fault tolerant control (FTC) scheme based on fault estimation (FE) for a system subject to input saturation, uncertainty, $\mathbf{L}_2$-bounded disturbance and additive faults. In this study, the saturation is represented in polytopic form, and the disturbance is partitioned into matched and unmatched parts. For the FE scheme, a partially-decoupled unknown input observer is introduced to completely compensate for the matched part and only the unmatched part affects the FE performance. The FE and FTC schemes are integrated to ensure that the resulting closed-loop system is stable with $\mathbf{L}_2$-gain performance. Furthermore, all sufficient conditions for robust performance are derived and cast as linear matrix inequalities (LMIs). Usage of the Young relation removes equality constraints, and thus all LMIs are solvable in a single step. Numerical simulations are provided to verify the effectiveness of the proposed scheme.

1 Introduction

The increasing need for safety, quality and reliability in complex systems can be met by compensating for unexpected faults. Faults in any subsystem could lead to deviation from normal behaviour, loss of performance, and at worst may cause instability. Rapid and accurate fault diagnosis and fault tolerant control (FTC) systems can increase system safety and lower the running cost, by tolerating component malfunctions while maintaining desirable performance and stability. FTC design methods can generally be classified into two main approaches: passive and active methods. The passive strategy is a type of robust control in which faults are considered as an additional source of uncertainty affecting the closed-loop system [1] and therefore, the design is conservative because the controller is designed for a fault that does not exist all the time. On the other hand, an active FTC (AFTC) scheme can address the problem of unpredictable faults, utilising the on-line information provided by fault detection and diagnosis (FDD) or fault estimation (FE) techniques. For this reason, AFTC has been the focus of attention in industrial and academic communities. However, there are still many challenging practical issues that need to be addressed, such as integrating FDD/FE with FTC and coping with control input constraints [2] where amplitude and rate limitations of actuators can lead to reduction of controller performance. A fault increases actuator effort to compensate for undesired effects, making actuator saturation more likely. However, less attention has been paid to this vital subject in the FE/FTC literature, especially when the system subject to disturbance and uncertainty. Saturation in FTC is usually addressed by: reference management techniques [3–8] and considering stability explicitly [9–15]. In the first case the FTC modifies the reference trajectory to reduce the controller effort when faults occur, whereas in the second, the saturation is considered as a hard constraint that the FTC adheres to.

In the afore-mentioned techniques, either all states must be measured, or only the measured outputs are available for FTC. This limitation can be overcome by using observer-based FTC (or FTC based on FE), where the observer/FE can provide information of the states using only the measured outputs. However, FTC based on observer/FE for practical systems (those experiencing saturation, faults and/or disturbances) have hardly been researched - this is due to the so-called bi-directional effects between the FE and FTC (due to mutual perturbations between them). These bi-directional effects are mainly caused by uncertainty and disturbance, which will degrade the performance of the FE scheme, which will in turn degrade the FTC performance. The concept of bi-directional interactions between the FE observer and FTC system is illustrated in [16] and is developed for Lipschitz non-linear systems in [17]; also, in both [16, 17] the disturbance is attenuated by $\mathbf{H}_\infty$ FE/FTC integrated design (where no components are decoupled) and therefore the the bi-directional effects are not reduced as much. The FE scheme plays a very important role in the bi-directional effect and hence the FE must be robust against these perturbations. Significant studies on the subject of robust FE design methods to compensate for or attenuate the effect of disturbance, have been established, e.g. adaptive observer [18, 19], the sliding mode observer [20], the extended state observer [21] and the unknown input observer [22]. Recently, a novel method was proposed to decouple partial disturbances and to decouple the effect of disturbances in FE, but some strong requirements (that relate to the rank of matrices) need to be satisfied. Gao *et al.* [23] relaxed the requirement by decoupling the FE scheme from certain components of disturbance, and attenuating the remaining components; however they did not consider the parametric uncertainty in the system. Hence, FE/FTC for systems with actuator saturation has not been well-addressed and is limited to few works [24–26]. In [24], FE and FTC are designed to avoid actuator saturation, and disturbances were not considered, whereas Fan *et al.* [25] considered disturbance, uncertainty, multiplicative faults and actuator saturation, and proposed an FE-based FTC for linear systems and solved bilinear matrix inequalities (BMIs) to design the scheme. Further, Lan *et al.* [26] proposed a sliding mode FE for FTC in a non-linear system (in particular a 3-DOF helicopter), where the fault is only considered in the controller signal as a additive term, and the stabilisation is solved by linear matrix inequalities (LMIs) in a single step. Therefore, it can be seen that existing results on FE-based FTC for systems subject to input saturation did not consider uncertainty, additive faults and unmatched disturbances, and this is one of the motivations for this paper.

In this paper, we propose a systematic and easily implementable FE/FTC design for systems with faults, actuator saturation, uncertainty and unmatched disturbances. The FE scheme is an partially-decoupled unknown input observer (PDUIO), and FTC is achieved using a state-feedback controller. In the FE scheme, we decompose the disturbances to matched and unmatched components, where the matched part is fully decoupled from the residual signal, and the unmatched part will contribute to the bi-directional effects. We also derive sufficient conditions for the overall FTC and FE systems based on Lyapunov criteria, not only to achieve stability, but to also guarantee performance in an $\mathbf{L}_2$ sense. We found those conditions to be in the form of BMIs. We then cast them to be novel LMIs conditions that are more tractable and do not require the equality constraint for the design procedure and are solvable in a single step. Quadratic polytopic differential inclusion (QPDI) is invoked to model the saturation non-linearity as (polytopic or linear) differential inclusions thus making it possible to use linear design methods. This modelling method allows saturation to occur, unlike the previously aforementioned references.

This paper is organised as follows: In Section 2, the problem is defined, including system modelling with additive fault, parameter uncertainty and input saturation in the presence of disturbance. In Section 3, observer and controller parameters are designed with the joint design strategy in a single step. A numerical example is presented in Section 4 and then the paper is concluded in Section 5.

1.1 Notation

Following notations are used in this paper: the super-script symbol + represents the pseudo-inverse of matrix, $\| \ * \ \|_p$ represents the p-norm in the Euclidean space; $\ell_2[0,\infty)$ denotes the 2-norm space; * is used for the blocks induced by symmetry, He(X) is defined as $\mathrm{He}(\mathbf{X}) = \mathbf{X} + \mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}_i$ is i^{th} row of matrix $\mathbf{X}$. Also, $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, and 0 denotes the zero matrix with compatible dimension. For simplicity, time t is dropped in the derivation of the following equations.

2 Definitions and problem formulation

2.1 Definitions and assumptions

The uncertain linear system studied in this paper is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} + \Delta\mathbf{A}(t))\mathbf{x}(t) + \mathbf{B}\mathrm{sat}(\mathbf{u}(t)) + \mathbf{E}\omega(t) + \mathbf{F}\mathbf{f}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\,. \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathfrak{R}^n$ is the state vector, $\mathbf{y} \in \mathfrak{R}^{\mathrm{m}}$ is the output vector, $\mathbf{u} \in \mathfrak{R}^r$ is the input vector, $\omega \in \mathfrak{R}^p$ is the unmatched disturbance and $\mathbf{f} \in \mathfrak{R}^l$ denotes a time varying additive fault. The term $\omega \in \mathfrak{R}^p$ is a disturbance which is not a fault, but rather represent any mismatch between the model (1) and actual system, such as un-modelled dynamics or other external unknown signals that are not faults. System matrices **A, B, C E** and **F** are deterministic with proper dimensions. The uncertainty in the system matrix is described by $\Delta\mathbf{A}(t) = \mathbf{G}_{\mathrm{A}}\mathbf{F}_{\mathrm{A}}(t)\mathbf{E}_A$ where $\mathbf{F}_{\mathrm{A}}(t) = \mathbb{R}^{r \times s}$ represents bounded parameter uncertainties satisfying $\mathbf{F}_A^{\mathrm{T}}\mathbf{F}_A \leq \mathbf{I}$; constant real matrices $\mathbf{G}_A$, $\mathbf{E}_A$ are of appropriate dimensions and specify the structure of these uncertainties. In addition, sat: $\mathbb{R}^r \to \mathbb{R}^r$ is a standard saturation function defined as: $\mathrm{sat}(\mathbf{u}(t)) = [\mathrm{sat}(\mathbf{u}_1(t)), \mathrm{sat}(\mathbf{u}_2(t)), \ldots, \mathrm{sat}(\mathbf{u}_r(t))]^{\mathrm{T}}$ where, $\mathrm{sat}(u_i(t)) = \mathrm{sign}(u_i(t)) \cdot \min\{u_{\max,i}, |u_i(t)|\}, i \in [1, r]$. In this paper, 'sat(·)' is utilised to represent both the scalar valued and the vector valued saturation function. In addition, the following assumptions are introduced.

Assumption 1: The pair $(\mathbf{C}, \mathbf{A})$ is observable and the pair $(\mathbf{A}, \mathbf{B})$ is controllable.

Assumption 2: The fault is bounded with known limits, i.e. $\| \mathbf{f} \|_2 \leq f_0$. Also, without loss of generality, it is assumed that the fault has a finite first-order time derivative, which belongs to $\ell_2[0,\infty)$.

Assumption 3: The disturbance ω belongs to $\ell_2[0,\infty)$.

Assumption 4:

$$\mathrm{rank}\left(\begin{bmatrix} \mathbf{A} - s\mathbf{I} & \mathbf{F} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) = n + l, \quad \forall s, \tag{2}$$

where s is a complex number with a non-negative real part.

Assumption 5: rank(**CF**)=rank(**F**).

Remark 1: Assumption 1 is the standard requirement for controllable systems. Assumptions 2 and 3 imply that the considered fault and disturbance belong to the $\mathbf{L}_2$-norm bounded set and also the fault is norm-bounded with a known upper bound. Assumptions 4 and 5 are necessary conditions for state and FE [27]. Assumption 5 implies that $m \geq l$ (the number of measurements is equal to or greater than the number of faults) and Assumption 4 is equivalent to the statement that the invariant zeros of (**A**, **F**, **C**) lie in the open left-half complex plane [27].

2.2 Problem statement

A set is said to be invariant if all the trajectories starting from it will remain inside it. The notion of the invariant set plays an important role in studying the stability and other performance criteria of a system. An ellipsoidal set is commonly used as the invariant set and can be defined as follows:

$$\Omega(\mathbf{P}, \rho) = \left\{ \mathbf{x} | \ \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} \leq \rho, \quad \mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0 \right\}. \tag{3}$$

Definition 1: Throughout this paper, we are concerned with the following problem: given a set of admissible initial states Ω_0 and a set defined for a specified class of signal ω, determine the control law (21) and observer gains such that

(i) The trajectories of the closed-loop system are bounded, i.e. the trajectory of system enters into, and remains in for all time, Ω_1.
(ii) If the disturbance is vanishing, i.e. $\omega \to 0$, then $\lim_{t \to \infty} \mathbf{x}(t) = 0$.
(iii) The L_2-gain from ω to $\mathbf{y}(t)$ is minimised, i.e. γ is minimised where

$$\int_0^\infty \mathbf{y}^{\mathrm{T}}(t)\mathbf{y}(t)\,\mathrm{d}t < \gamma^2 \int_0^\infty \omega^{\mathrm{T}}(t)\omega(t)\,\mathrm{d}t \tag{4}$$

The following lemmas are utilised to demonstrate the main results of this paper.

Lemma 1: Let $\mathbf{u}, \mathbf{v} \in \mathfrak{R}^r$ and suppose that $|v_i| \leq u_{i,\max}, \ \ i \in [1, 2^r]$, then

$$\mathrm{sat}(\mathbf{u}) \in \mathrm{Co}\left\{ \Delta_j\mathbf{u} + \Delta_j^{-}\mathbf{v}, \ \ j \in [1, 2^r] \right\} \tag{5}$$

where Co{ · } denotes the convex hull, and $\Delta_j \in \mathfrak{R}^{m \times m}$ is a diagonal matrix whose elements are 0 or 1 and $\Delta_j^{-} = \mathbf{I}_m - \Delta_j$ [28].

Remark 2: In order to intuitively illustrate Lemma 1, let us consider first the mono-input case, i.e. $r = 1$. In this case, let **v** and **u** be scalars. Suppose now that $-u_{\min} \leq \mathbf{v} \leq u_{\max}$. Hence, it follows that the scalar function sat(**u**) can be computed as a convex combination of **u** and **v**, i.e: $\mathrm{sat}(\mathbf{u}) = \lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ with $0 \leq \lambda \leq 1$, or, equivalently $\mathrm{sat}(\mathbf{u}) \in Co\{\mathbf{u}, \mathbf{v}\}$. Following from Lemma 1, consequently saturation function can be modelled such as $\mathrm{sat}(\mathbf{u}) = \sum_{j=1}^{2^r} \lambda_j\left(\Delta_j\mathbf{u} + \Delta_j^{-}\mathbf{v}\right)$.

Lemma 2 ((Young's inequality)): For given matrices **X** and **Y** of appropriate dimensions and for any symmetric invertible matrix $\mathbf{S} > 0$, the following inequality is established:

$$\mathbf{X}^{\mathrm{T}}\mathbf{Y} + \mathbf{Y}^{\mathrm{T}}\mathbf{X} \leq \mathbf{X}^{\mathrm{T}}\mathbf{S}\mathbf{X} + \mathbf{Y}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{Y} \tag{6}$$

Lemma 3: The matrix $\mathbf{E}$ and the disturbance $\omega(t)$ (without loss of generality) can be decomposed into

$$\mathbf{E} = [\mathbf{E}_1 \;\; \vdots \;\; \mathbf{E}_2], \quad \omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}. \tag{7}$$

where $\omega_1(t) \in \Re^{p_1}$ is selected in order that $\operatorname{rank}(\mathbf{CE}_1) = \operatorname{rank}(\mathbf{E}_1) = p_1$. Therefore, the system (1) becomes

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} + \Delta\mathbf{A}(t))\mathbf{x}(t) + \mathbf{B}\operatorname{sat}(\mathbf{u}(t)) + \mathbf{Ff}(t) \\ &\quad + \mathbf{E}_1\omega_1(t) + \mathbf{E}_2\omega_2(t) \\ \mathbf{y}(t) &= \mathbf{Cx}(t) \end{aligned} \tag{8}$$

Note that in Lemma 3, $\omega_1(t)$ is the *matched* part and $\omega_2(t)$ is the *unmatched* part of the disturbance where it will be shown later that ω_1 is completely decoupled from the state and FEs. Classical control theory only considers one 'equivalent' disturbance resulting from different sources such as external disturbances, non-linearities, uncertainties and un-modelled dynamics. However, sensor technology and data processing development have shown that multiple disturbances exist in most practical systems, which can be formulated into different mathematical models [29]. The effect of these factors can be broken down into those sensed by sensors (matched part) and those that have no effect on the measurement (unmatched part).

3 FTC scheme

The FTC scheme proposed in this paper consists of the PDUIO which estimates the state and fault, and a state feedback controller with the adaptive part for the fault compensation. Fig. 1 illustrates the structure of the closed-loop system, where the control saturation is explicitly taken into consideration. Therefore, the objective is to find the feedback gain and observer parameters to address the problem statement. It is worth remarking that, some works assume that the disturbance $\omega(t)$, enter the system together with the control input $\mathbf{u}(t)$ [30, 31]. Furthermore, they are mostly concerned with stabilisation in a global or semi-global context. Thus, when the open-loop system is unstable, and/or if the system is subject to disturbances that do not enter the system additively (which is common in practice), the approaches above cannot be applied. The structure for FE and fault compensation in this paper is similar to the structure in [24, 25], but here the state estimation observer utilises the saturated control input instead.

The PDUIO has the following structure:

$$\begin{aligned} \dot{\chi}(t) &= \mathbf{N}\chi(t) + \mathbf{TB}\operatorname{sat}(\mathbf{u}(t)) + \mathbf{Ly}(t) + \mathbf{TF}\hat{\mathbf{f}}(t) \\ \hat{\mathbf{x}}(t) &= \chi(t) + \mathbf{Gy}(t) \end{aligned} \tag{9}$$

where $\hat{\mathbf{x}}(t) \in \Re^n$ is the estimate of state $\mathbf{x}(t)$ and $\hat{\mathbf{f}}(t) \in \Re^l$ is the estimate of fault $\mathbf{f}(t)$. $\chi(t)$ is the observer state vector, and the matrices $\mathbf{L}$, $\mathbf{N}, \mathbf{T}$ and $\mathbf{G}$ are observer parameters to be determined later. To generate $\hat{\mathbf{f}}(t)$, an adaptive FE law is used, which has been verified to be effective in time-varying FE [32], as follows:

$$\begin{aligned} \dot{\hat{\mathbf{f}}}(t) &= \Gamma(\mathbf{F}_1\dot{\mathbf{r}}(t) + \mathbf{F}_2\mathbf{r}(t)), \\ \mathbf{r}(t) &= \mathbf{Ce}_{\mathbf{x}}(t) = \mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \end{aligned} \tag{10}$$

In conventional adaptive schemes [24], the matrix $\mathbf{F}_1$ is set to identity; in this paper it is left to be variable and hence is additional design freedom. The adaptive law, $\Gamma \in \Re^{l\times l}$ is a constant symmetric positive definite learning rate matrix, and $\mathbf{F}_1$, $\mathbf{F}_2 \in \Re^{l\times n}$ are design matrices and $\mathbf{r}(t)$ represents the residual vector. Define the fault and state estimation errors as

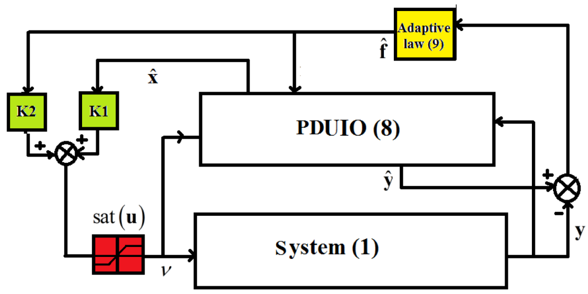

Fig. 1 *Closed-loop block diagram of FE and FTC system*

$$\begin{aligned} \mathbf{e}_{\mathbf{x}}(t) &= \mathbf{x}(t) - \hat{\mathbf{x}}(t) \\ \mathbf{e}_{\mathbf{f}}(t) &= \mathbf{f}(t) - \hat{\mathbf{f}}(t) \end{aligned} \tag{11}$$

Hence, the dynamics of the estimation error can be obtained by differentiating (11) and substituting from (1), (9) and (10) to get (see (12)) , where, $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$. If the following are satisfied:

$$(\mathbf{GC} - \mathbf{I})\mathbf{E}_1 = 0 \tag{13}$$

$$\mathbf{T} = \mathbf{I} - \mathbf{GC} \tag{14}$$

$$\mathbf{N} = \mathbf{A} - \mathbf{GCA} - \mathbf{L}_1\mathbf{C} \tag{15}$$

$$\mathbf{L}_2 = \mathbf{NG} \tag{16}$$

then, the estimation error (12) becomes

$$\dot{\mathbf{e}}_{\mathbf{x}}(t) = \mathbf{T}\Delta\mathbf{A}(t)\mathbf{x}(t) + \mathbf{Ne}_{\mathbf{x}}(t) + \mathbf{TFe}_{\mathbf{f}}(t) + \mathbf{TE}_2\omega_2(t) \tag{17}$$

and

$$\begin{aligned} \dot{\mathbf{e}}_{\mathbf{f}}(t) &= \dot{\mathbf{f}}(t) - \dot{\hat{\mathbf{f}}}(t) = \dot{\mathbf{f}}(t) - (\Gamma\mathbf{F}_1\mathbf{CN} + \Gamma\mathbf{F}_2\mathbf{C})\mathbf{e}_{\mathbf{x}}(t) \\ &\quad - \Gamma\mathbf{F}_1\mathbf{CT}\Delta\mathbf{A}(t)\mathbf{x}(t) - \Gamma\mathbf{F}_1\mathbf{CTFe}_{\mathbf{f}}(t) - \Gamma\mathbf{F}_1\mathbf{CTE}_2\omega_2(t) \end{aligned} \tag{18}$$

Notice that the errors (17) and (18) are not affected by the matched disturbance $\omega_1(t)$. Define a new variable $\mathbf{x}_e(t) = \begin{bmatrix} \mathbf{e}_{\mathbf{x}}^{\mathrm{T}}(t) & \mathbf{e}_{\mathbf{f}}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$ which includes both the state estimation and FE errors. Following from (17) and (18), it is obtained that

$$\dot{\mathbf{x}}_e(t) = \mathbf{A}_e\mathbf{x}_e(t) + \mathbf{B}_e\omega_e(t) + \mathbf{B}_\Delta\Delta\mathbf{A}(t)\mathbf{x}(t) \tag{19}$$

where

$$\begin{aligned} \mathbf{A}_e &= \begin{bmatrix} \mathbf{N} & \mathbf{TF} \\ -\Gamma\mathbf{F}_1\mathbf{CN} - \Gamma\mathbf{F}_2\mathbf{C} & -\Gamma\mathbf{F}_1\mathbf{CTF} \end{bmatrix}, \quad \mathbf{B}_e = \begin{bmatrix} \mathbf{TE}_2 & \mathbf{0} \\ -\Gamma\mathbf{F}_1\mathbf{CTE}_2 & \mathbf{I} \end{bmatrix}, \\ \mathbf{B}_\Delta &= \begin{bmatrix} \mathbf{T} \\ -\Gamma\mathbf{F}_1\mathbf{CT} \end{bmatrix}, \quad \omega_e = \begin{bmatrix} \omega_2 \\ \dot{\mathbf{f}} \end{bmatrix} \end{aligned} \tag{20}$$

The error dynamics (19) are dependent on system states due to uncertainty. If $\Delta\mathbf{A}(t) = 0$, $\omega_e = 0$ and $\mathbf{A}_e$ is Hurwitz, then there exists a stable and unbiased observer for states and FE.

Remark 3: These (13)–(16) are not assumptions but are design tasks. The solution to equation (13) is as follows: $\mathbf{G} = \mathbf{E}_1(\mathbf{CE}_1)^+$. The solution for $\mathbf{G}$ above exists because to design $\operatorname{rank}(\mathbf{CE}_1) = \operatorname{rank}(\mathbf{E}_1)$ (Lemma 3). After $\mathbf{G}$ is calculated from (13), then $\mathbf{T}$ can be calculated from (14). Following that, the LMIs in Corollary can be solved to get $\mathbf{L}_1$, and consequently $\mathbf{N}$ can be calculated by (15), and $\mathbf{L}_2$ can be calculated from (16).

$$\begin{aligned} \dot{\mathbf{e}}_{\mathbf{x}}(t) &= (\mathbf{I} - \mathbf{GC})\Delta\mathbf{A}(t)\mathbf{x}(t) + (\mathbf{A} - \mathbf{GCA} - \mathbf{L}_1\mathbf{C})\mathbf{e}_{\mathbf{x}}(t) \\ &\quad + [(\mathbf{A} - \mathbf{GCA} - \mathbf{L}_1\mathbf{C}) - \mathbf{N}]\mathbf{z}(t) + [\mathbf{L}_2 - (\mathbf{A} - \mathbf{GCA} - \mathbf{L}_1\mathbf{C})\mathbf{G}]\mathbf{y}(t) \\ &\quad + [-\mathbf{T} + (\mathbf{GC} - \mathbf{I})]\mathbf{B}\operatorname{sat}(\mathbf{u}(t)) + (-\mathbf{GC} + \mathbf{I})\mathbf{E}_1\omega_1(t) + \mathbf{TE}_2\omega_2(t) + \mathbf{TFe}_{\mathbf{f}}(t) \end{aligned} \tag{12}$$

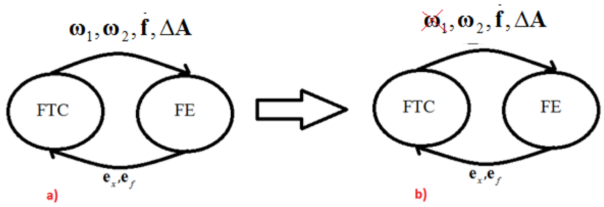

Fig. 2 *Interaction between (bi-directional effects) FE and FTC*
(a) Without PDUIO, ***(b)*** With PDUIO

The key challenge therefore is to design a feedback controller in order that the control input constraints are satisfied. Since the adaptive observers (9) and (10) can provide the estimated states and actuator fault parameters/magnitudes, a fixed-gain feedback control law is designed to compensate for the effect of the actuator faults as follows:

$$u(t) = \mathbf{K}\tilde{\mathbf{x}} = [\mathbf{K}_1 \quad \mathbf{K}_2]\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \hat{\mathbf{f}}(t)\end{bmatrix} \tag{21}$$

where $\mathbf{K} \in \Re^{r\times(l+n)}$ is the feedback gain matrix.

Remark 4: A complex robustness problem arises when considering observer-based FE/FTC designs due to the bi-directional effect of FE and FTC. The PDUIO estimates the fault and states based on output measurements of the system (that is perturbed by uncertainty and disturbances) and actuator output (Fig. 1). Also, the controller output is a linear combination of fault and states estimation as depicted in Fig. 1 that lead to the system affected by the precision of estimations. Hence, in this study, system uncertainty $\Delta\mathbf{A}(t)$, matched and unmatched disturbances ω_1, ω_2 with fault derivative $\dot{\mathbf{f}}$ are system perturbations that will negatively affect the FE. Fault and states estimation errors are FE perturbations that will have negative effect on FTC. Fig. 2 shows the interactions (bi-directional effect) between the FE and FTC schemes, which also shows the PDUIO partially alleviating the problem by compensating for the matched disturbance.

3.1 Polytopic models for input saturation

In this paper, a QPDI model is used for representing the saturated closed-loop system. The main advantage in this case is that stability and stabilisation conditions can be obtained in a more tractable way, since the hard saturation non-linearity disappears in the QPDI representation which represents the saturated system only locally. In this paper, we introduce the convex hull of the saturated controller $\mathbf{K}$ and a virtual non-saturating controller $\mathbf{H}$. The saturation in the system is addressed by a convex combination as in Lemma 1. Based on Lemma 1, if control matrices $\mathbf{K}$, $\mathbf{H}$ are given such that

$$\mathbf{H}\tilde{\mathbf{x}}(t) = \underbrace{[\mathbf{H}_1 \quad \mathbf{H}_2]}_{\mathbf{H}}\begin{bmatrix}\hat{\mathbf{x}}(t)\\ \hat{\mathbf{f}}(t)\end{bmatrix} \tag{22}$$

is unsaturated, then a linear set can be defined as follows:

$$\mathrm{L}(\mathbf{H}) = \left\{\left\|\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{f}}\end{bmatrix}\right\| \left|\mathbf{H}_{1i}\hat{\mathbf{x}} + \mathbf{H}_{2i}\hat{\mathbf{f}}\right| \le u_{i,\max}, \quad i \in [1, m]\right\} \tag{23}$$

If $\tilde{\mathbf{x}} \in \mathrm{L}(\mathbf{H})$ then

$$\mathrm{sat}(\mathbf{K}\tilde{\mathbf{x}}) \in \mathrm{Co}\{\boldsymbol{\Delta}_i\mathbf{K}\tilde{\mathbf{x}} + \boldsymbol{\Delta}_i^{-}\mathbf{H}\tilde{\mathbf{x}}, \ i \in [1, 2^m]\} \tag{24}$$

Therefore, the saturation function is covered by the convex hull of two controllers. There exist $\theta_j \ge 0$, such that $\sum_{j=1}^{2^r}\theta_j = 1$, and therefore

$$\mathrm{sat}(\mathbf{K}\tilde{\mathbf{x}}) = \sum_{j=1}^{2^r}\theta_j(\boldsymbol{\Delta}_j\mathbf{K} + \boldsymbol{\Delta}_j^{-}\mathbf{H})\tilde{\mathbf{x}} \tag{25}$$

It is easy to show that by substituting for the saturation model (25) into (8), the closed-loop dynamics (8) becomes

$$\begin{aligned}\dot{\mathbf{x}} = \sum_{j=1}^{2^m}\theta_j\big((\mathbf{A}_{\boldsymbol{\Delta}} + \bar{\mathbf{B}}_1)\mathbf{x}(t) - \bar{\mathbf{B}}_1\mathbf{e}_{\mathbf{x}}(t) - \bar{\mathbf{B}}_2\mathbf{e}_{\mathbf{f}}(t)\\ +(\bar{\mathbf{B}}_2 + \mathbf{F})\mathbf{f}(t)\big) + \mathbf{E}_1\omega_1(\mathrm{t}) + \mathbf{E}_2\omega_2(t)\end{aligned} \tag{26}$$

where $\mathbf{A}_{\boldsymbol{\Delta}} = \mathbf{A} + \Delta\mathbf{A}(t)$, $\bar{\mathbf{B}}_1 = \mathbf{B}\boldsymbol{\Delta}_j\mathbf{K}_1 + \mathbf{B}\boldsymbol{\Delta}_j^{-}\mathbf{H}_1$ and $\bar{\mathbf{B}}_2 = \mathbf{B}\boldsymbol{\Delta}_j\mathbf{K}_2 + \mathbf{B}\boldsymbol{\Delta}_j^{-}\mathbf{H}_2$.

It was assumed that $\mathbf{B}\mathbf{K}_2 + \mathbf{F} = 0$; this is not a strong assumption, since $\mathbf{K}_2$ can be readily selected to satisfy this assumption. For this purpose, $\mathbf{K}_2 = -\mathbf{B}^{+}\mathbf{F}$ can be set, where $\mathbf{B}^{+}$ is the general pseudo inverse of $\mathbf{B}$. If $\mathbf{F} = \mathbf{B}$, then $\mathbf{K}_2$ can be selected as $\mathbf{K}_2 = -\mathbf{I}_{r\times l}$. It is clear that if $\mathbf{H}_2 = \mathbf{K}_2$, then $\mathbf{B}\boldsymbol{\Delta}_j\mathbf{K}_2 + \mathbf{B}\boldsymbol{\Delta}_j^{-}\mathbf{H}_2 + \mathbf{F} = 0$ and thus the closed-loop dynamics become

$$\begin{aligned}\dot{\mathbf{x}}(t) = \sum_{j=1}^{2^m}\theta_j\big((\mathbf{A}_{\boldsymbol{\Delta}} + \bar{\mathbf{B}}_1)\mathbf{x}(t) - \bar{\mathbf{B}}_1\mathbf{e}_{\mathbf{x}}(t) - \bar{\mathbf{B}}_2\mathbf{e}_{\mathbf{f}}(t)\big)\\ +\mathbf{E}_1\omega_1(\mathrm{t}) + \mathbf{E}_2\omega_2(t)\end{aligned} \tag{27}$$

Define a new variable $\mathbf{z}(t) = [\mathbf{x}(t) \quad \mathbf{e}_{\mathbf{x}}(t) \quad \mathbf{e}_{\mathbf{f}}(t)]^{\mathrm{T}}$. From (17), (18) and (27), it is obtained that

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{A}_z\mathbf{z}(t) + \mathbf{B}_z\tilde{\omega}\\ \mathbf{y}(t) &= \mathbf{C}_z\mathbf{z}(t)\end{aligned} \tag{28}$$

where

$$\mathbf{A}_z = \left[\begin{array}{c|cc}\sum_{j=1}^{2^r}\theta_j(\mathbf{A}_{\Delta} + \bar{\mathbf{B}}_1) & -\sum_{j=1}^{2^r}\theta_j\bar{\mathbf{B}}_1 & -\sum_{j=1}^{2^r}\theta_j\bar{\mathbf{B}}_2\\ \hline \mathbf{T}\,\Delta\mathbf{A}(t) & \multicolumn{2}{c}{\multirow{2}{*}{$[\mathbf{A}_e]$}}\\ -\boldsymbol{\Gamma}\mathbf{F}_1\mathbf{C}\mathbf{T}\,\Delta\mathbf{A}(t) & & \end{array}\right] \tag{29}$$

and

$$\mathbf{B}_z = \left[\begin{array}{c|c}\mathbf{E}_1 & \mathbf{E}_2 \quad \mathbf{0}\\ \hline \mathbf{0} & \multirow{2}{*}{$[\mathbf{B}_e]$}\\ \mathbf{0} & \end{array}\right], \quad \tilde{\omega} = \begin{bmatrix}\omega_1\\ \omega_e\end{bmatrix}, \quad \mathbf{C}_z = [\mathbf{C} \mid \mathbf{0} \quad \mathbf{0}] \tag{30}$$

It is clear that if $\tilde{\omega} = 0$, the augmented system (28) becomes asymptotically stable if matrix $\mathbf{A}_z$ is Hurwitz, otherwise $(\tilde{\omega} \ne 0)$ the augmented system should be robustly stable. The matrix $\mathbf{A}_z$ is not an upper/lower triangular matrix, due to the uncertainty $\Delta\mathbf{A}(t)$; therefore, the separation principle is violated in this case. In this study, the invariant sets are used to describe the boundedness of the state vector and restricted $\mathbf{L}_2$-gain criteria.

3.2 FE/FTC design for reducing bi-directional effects

In this section, to design the FTC and FE parameters, the results of Lemmas 1–3 are employed and all criteria are cast as LMI conditions. The feedback controller and observer gains cannot be designed separately with methods such as pole placement due to violation of the separation principle. To use the results of aforementioned lemmas for the new augmented system in (28), the following augmented system (31) is considered instead of the original one. It will be proven that if stability and performance condition is satisfied by (31), the aforementioned conditions hold for (28) too

$$\dot{\mathbf{z}}(t) = \mathbf{A}_z^i(t)\mathbf{z}(t) + \mathbf{B}_z\tilde{\omega}$$
$$\mathbf{y}(t) = \mathbf{C}_z\mathbf{z}(t) \quad (31)$$

where

$$\mathbf{A}_z^i = \begin{bmatrix} \mathbf{A}_\Delta + \bar{\mathbf{B}}_1 & -\bar{\mathbf{B}}_1 & -\bar{\mathbf{B}}_2 \\ \mathbf{T}\Delta\mathbf{A}(t) & \mathbf{N} & \mathbf{TF} \\ -\mathbf{\Gamma F}_1\mathbf{CT}\Delta\mathbf{A}(t) & -\mathbf{\Gamma}(\mathbf{F}_1\mathbf{CN} - \mathbf{F}_2\mathbf{C}) & -\mathbf{\Gamma F}_1\mathbf{CTF} \end{bmatrix}, \quad (32)$$

The bound of disturbance energy for the augmented system (31) is obtained based on the following inequality:

$$\int_0^t \tilde{\omega}^{\mathrm{T}}(t)\tilde{\omega}(t)\,\mathrm{d}t \le \alpha + \beta \quad (33)$$

Theorem 1: If there exist a symmetric positive definite matrix $\mathbf{P}$, matrices $\mathbf{K}$ and $\mathbf{H}$ and scalars $\eta, \gamma > 0$, such that

$$\mathbf{A}_z^{i\,\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_z^i + \frac{1}{\eta}\mathbf{P}\mathbf{B}_z\mathbf{B}_z^{\mathrm{T}}\mathbf{P} + \frac{\eta}{\gamma^2}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z \le 0 \quad (34)$$

$$[\mathbf{H}_{1i} \ \ -\mathbf{H}_{1i} \ \ -\mathbf{H}_{2i}]\mathbf{P}^{-1}[\mathbf{H}_{1i} \ \ -\mathbf{H}_{1i} \ \ -\mathbf{H}_{2i}]^{\mathrm{T}} \le \frac{c_i^2}{\eta(\alpha+\beta)} \quad (35)$$

$$f_0 < \min\left\{\frac{u_{\max,i}}{\|\mathbf{K}_{2i}\|_1}\right\} \quad \forall i = 1,\dots,m \quad (36)$$

then, $\Omega_1(\mathbf{P}, \eta(\alpha+\beta))$ is an invariant set of system (28) where the initial condition is set to zero, i.e. $\mathbf{z}(0) = 0$, and $\Omega_1(\mathbf{P}, \eta(\alpha+\beta)) \subset \mathrm{L}(\mathbf{H})$. The restricted $\mathbf{L}_2$-gain from disturbance $\tilde{\omega}$ to output $\mathbf{y}$ is less than or equal to γ in this set (where $c_i = u_{\max,i} - f_0\|\mathbf{H}_{2i}\|_1 > 0$).

Proof: Refer to Appendix 1. □

The zero initial condition is necessary to satisfy the performance criteria and the constraints in Theorem 1. This requirement is a rigorous precondition (due to the presence of state and FE errors) so that the augmented system (28) will have an invariant set. The set Ω_1 is affected by the state estimation error, where if the estimation increases, then the volume of Ω_1 also increases, which is not desirable. The next remark illustrates a necessary condition for the augmented system with non-zero initial condition.

Remark 5: In the case of non-zero initial condition, if all the conditions of Theorem 1 are satisfied, then any trajectory $\mathbf{z}(t)$ of system (28) starting from $\Omega_0(\mathbf{P}, \delta)$ will remain in $\Omega_1(\mathbf{P}, \delta + \eta(\alpha+\beta))$, and the saturation constraint $\Omega_1(\mathbf{P}, \delta + \eta(\alpha+\beta)) \subset \mathrm{L}(\mathbf{H})$ will be also satisfied. Furthermore, the finite $\mathbf{L}_2$-gain from $\tilde{\omega}$ to $\mathbf{y}$ will present a bias term and will become

$$\|\mathbf{y}\|_2^2 \le \gamma^2\left(\frac{\delta}{\eta} + \|\tilde{\omega}\|_2^2\right) \quad (37)$$

The matrix inequality in (34) is a BMI that is difficult to solve numerically. The following corollary shows how (34) and (35) can be converted to LMIs.

Corollary 1: If there exist symmetric positive semi-definite matrices $\mathbf{P}_0 \in \mathfrak{R}^{n\times n}$, $\mathbf{R} \in \mathfrak{R}^{n\times n}$ and matrices $\mathbf{H}, \mathbf{K}$ and scalars $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ such that the following LMIs are satisfied:

$$\begin{bmatrix} \mathbf{X}^i & \tilde{\mathbf{P}}\mathbf{C}_z^{\mathrm{T}} & \tilde{\mathbf{P}}\mathbf{P}\mathbf{B}_z & \mathbf{Y}^i \\ * & -\gamma^2\mathbf{I} & 0 & 0 \\ * & 0 & -\mathbf{I} & 0 \\ * & 0 & 0 & \mathbf{Z} \end{bmatrix} \le 0 \quad (38)$$

and

$$\begin{bmatrix} \Gamma\mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} - c_i^2/(\alpha+\beta) & 0 & \hat{\mathbf{H}}_{1i} & \hat{\mathbf{H}}_{1i} & 0 \\ * & -\mathbf{R} & 0 & 0 & \mathbf{I} \\ * & * & -\mathbf{Q} & 0 & 0 \\ * & * & * & -\mathbf{Q}/\varepsilon & 0 \\ * & * & * & * & -\varepsilon\mathbf{Q} \end{bmatrix} \le 0 \quad (39)$$

where

$$\mathbf{P} = \mathrm{diag}(\mathbf{P}_0, \mathbf{R}, \Gamma^{-1}), \quad \mathbf{P}^{-1} = \mathrm{diag}(\mathbf{Q}, \mathbf{Q}_r, \Gamma),$$
$$\tilde{\mathbf{P}} = \mathrm{diag}(\mathbf{Q}, \mathbf{I}, \mathbf{I}) \quad (40)$$

and

$$\mathbf{X}^i = \begin{bmatrix} \mathbf{X}_{11}^i & 0 & -\bar{\mathbf{B}}_2 \\ * & \mathbf{X}_{22}^i & \mathbf{X}_{23}^i \\ * & * & \mathbf{X}_{33}^i \end{bmatrix} \quad (41)$$

$$(\mathbf{Y}^i)^{\mathrm{T}} = [\mathbf{Y}_1; \mathbf{Y}_2; \mathbf{Y}_3] \quad (42)$$

$$\mathbf{Z} = \mathrm{diag}\{-\mathbf{Q}/\varepsilon_1, -\varepsilon_1\mathbf{Q}, -\varepsilon_2\mathbf{I}, -\mathbf{I}/\varepsilon_3 - \varepsilon_3\mathbf{I}, -\varepsilon_4\mathbf{I}, -\mathbf{R}/\varepsilon_0, -\varepsilon_0\mathbf{R}, -\mathbf{I}/\varepsilon_4\} \quad (43)$$

$$\mathbf{Y}_1 = \left[\bar{\mathbf{B}}_1\mathbf{Q} \ 0 \ \mathbf{Q}\mathbf{E}_A^{\mathrm{T}} \ \mathbf{Q}\mathbf{E}_A^{\mathrm{T}} \ 0 \ \mathbf{Q}\mathbf{E}_A^{\mathrm{T}} \ 0 \ 0 \ 0\right] \quad (44)$$

$$\mathbf{Y}_2 = \left[0 \ \ 0 \ \ 0 \ \ 0 \ \ \mathbf{RTG}_A \ \ 0 \ \ \hat{\mathbf{L}}\mathbf{C} \ \ 0 \ \ 0\right] \quad (45)$$

$$\mathbf{Y}_3 = \left[0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ \mathbf{F}_1\mathbf{C} \ \ \mathbf{F}_1\mathbf{CTG}_A\right] \quad (46)$$

$$\mathbf{X}_{11}^i = \mathrm{He}\left(\mathbf{AQ} + \bar{\mathbf{B}}_1\mathbf{Q}\right) + \varepsilon_2\mathbf{G}_A\mathbf{G}_A^{\mathrm{T}} \quad (47)$$

$$\mathbf{X}_{22}^i = \mathrm{He}\left(\mathbf{RTA} - \hat{\mathbf{L}}_1\mathbf{C}\right) \quad (48)$$

$$\mathbf{X}_{23}^i = \mathbf{RTF} + (-\mathbf{F}_1\mathbf{CTA} + \mathbf{F}_2\mathbf{C})^{\mathrm{T}} \quad (49)$$

$$\mathbf{X}_{33}^i = \mathrm{He}(-\mathbf{F}_1\mathbf{CTF}) \quad (50)$$

then all the conditions of Theorem 1 are satisfied. The state feedback gain and the observer gain are obtained as $\hat{\mathbf{K}}_1 = \mathbf{K}_1\mathbf{Q}$ and $\hat{\mathbf{L}} = \mathbf{RL}$.

Proof: Refer to Appendix 2. □

Remark 6: We define the uncertainty in the system as $\Delta\mathbf{A} = \mathbf{G}_A\mathbf{F}_A(t)\mathbf{E}_A$. The matrix $\mathbf{F}_A$ is unknown and time-varying, but its bound is known and constrained by $\mathbf{F}_A^{\mathrm{T}}\mathbf{F}_A \le \mathbf{I}$. Whilst $\mathbf{G}_A$ and $\mathbf{E}_A$ are known and represent the structure of these uncertainty. Design LMIs (38) and (39) do not need $\Delta\mathbf{A}$ in the design, where the uncertainty $\Delta\mathbf{A}$ is resolved by using Young's relation (Lemma 2), but only need the matrices $\mathbf{G}_\mathbf{A}$ and $\mathbf{E}_\mathbf{A}$ which are known.

Hence, to design the optimal controller and observer to achieve the minimum γ, the following optimisation algorithm is implemented:

$$\min_{\mathbf{Q},\mathbf{R},\mathbf{P}_0,\mathbf{F}_1,\mathbf{F}_2,\hat{\mathbf{L}},\hat{\mathbf{H}}_1,\hat{\mathbf{K}}_1} (\gamma) \quad \text{subject to: (38) and (39)} \quad (51)$$

Remark 7: : If disturbance decomposition is not needed, then a Luenberger observer can be used instead of the PDUIO (9). Thus, all theorems and remarks can be used for this case by setting matrices $\mathbf{T} = \mathbf{I}$, $\mathbf{G} = \mathbf{0}$ and decomposing matrices $\mathbf{B}_z$ and $\tilde{\omega}$ to be

$$[\mathbf{B}_z] = \begin{bmatrix} \mathbf{E} & 0 \\ \mathbf{E} & 0 \\ -\mathbf{\Gamma F}_1\mathbf{CE} & 1 \end{bmatrix}, \quad \tilde{\omega} = \begin{bmatrix} \omega \\ \dot{f} \end{bmatrix} \quad (52)$$

Remark 8: : LMI (38) has a solution, only if the top-left sub-block matrix $\mathbf{X}^i$ is negative semi-definite. From the feasibility analysis point of view, it can be confirmed that the necessary conditions for $\mathbf{X}^i \le 0$ are:

(1) The pair $(\mathbf{A}, \mathbf{B})$ must be controllable.
(2) The pair $(\mathbf{TA}, \mathbf{C})$ must be detectable.
(3) **CTF** must be a full rank matrix.

It is clear that these conditions are satisfied when Assumptions 1 and 5 are given.

Remark 9: The necessary condition for stability and performance is the controllability and observability of the system; these conditions are stated in Assumption 1. However, there is no necessary and sufficient condition for the existence of **K** and **H** to deal with saturation. In other words, all conditions in Theorem 1 and Corollary 1 are sufficient conditions for the existence of controller gains that can deal with saturation. Therefore, there may be a situation where it is impossible to design a stabilising controller for a system with input saturation limits, subject to faults, uncertainties and unknown inputs. In such a scenario, the linear set (23) (where the states of the augmented system remain within it) does not exist and consequently there is no feasible solution for (39). To obtain a feasible solution we can relax performance goal in the minimisation problem (51). In the worst case, the physical constraint of the system such as the saturation of the actuator must be removed by changing the actuator in the practical engineering problem.

Remark 10: For the augmented system (28), admissible initial conditions for the states must be selected based on the invariant set from Remark 5. Suppose $\hat{\mathbf{x}}(0) = 0$, $\hat{\mathbf{f}}(0) = 0$ and the system is initially fault free $\mathbf{f}(0) = 0$. It can be deduced that $\mathbf{e}_x(0) = \mathbf{x}(0)$ and $\mathbf{e}_f(0) = 0$. Then from Theorem 1, if the initial states $\mathbf{z}(0)$ start from set $\Omega(\mathbf{P}, \delta)$, then it is obtained that

$$[\mathbf{x}(0) \quad \mathbf{x}(0) \quad 0]\begin{bmatrix}\mathbf{P}_0 & 0 & 0\\ 0 & \mathbf{R} & 0\\ 0 & 0 & \Gamma^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}(0)\\ \mathbf{x}(0)\\ 0\end{bmatrix} \le \delta \tag{53}$$

Since $\lambda_{\min}(\mathbf{P})\| \mathbf{x} \|^2 \le \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$, $\forall \mathbf{x} \in R^n$, where $\lambda_{\min}$ denotes the minimum eigenvalue of a matrix, then it follows from (53) that

$$\| \mathbf{x}(0) \| \le \sqrt{\frac{\delta}{\lambda_{\min}(\mathbf{P}_0 + \mathbf{R})}} \tag{54}$$

Inequality (54) means that system states must be initialised to be within the circle of radius $\sqrt{\delta/(\lambda_{\min}(\mathbf{P}_0 + \mathbf{R}))}$.

4 Case study

In this section, the effectiveness of the proposed method is evaluated by utilising a flexible joint robot link actuated by a DC motor, whose non-linear model is [33]

$$\begin{aligned}\dot{\theta}_m &= \omega_m\\ \dot{\omega}_m &= \frac{k_t}{J_m}(\theta_l - \theta_m) - \frac{B_r}{J_m}\omega_m + \frac{k_\tau}{J_m}u\\ \dot{\theta}_l &= \omega_l\\ \dot{\omega}_l &= -\frac{k_t}{J_l}(\theta_l - \theta_m) - \frac{m_p g h}{J_l}\sin(\theta_l)\end{aligned} \tag{55}$$

where θ_m, ω_m are angular velocity and angular rotation of the motor, respectively, and θ_l, ω_l are angular position and angular velocity of the link, respectively. J_l is the link inertia, J_m is the motor inertia, K_t is the torsional spring constant, B_r is the viscous friction coefficient, K_τ is the amplifier gain, m_p and h are the pointer mass and the link length, respectively. In this case, parameters $\theta_m, \omega_m, \theta_l$ are measured by sensors. Thus, the corresponding state-space model can be described in form of (1) where the system matrices are

Table 1 Constant design parameters

$\eta = 1$	$\alpha = 1$	$\varepsilon_1 = 0.01$
$\varepsilon_3 = 0.5$	$\varepsilon = 0.01$	$\varepsilon_4 = 0.9$

$$\begin{aligned}&\mathbf{x} = \begin{bmatrix}\theta_m\\ \omega_m\\ \theta_l\\ \omega_l\end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix}0 & 1 & 0 & 0\\ -48.6 & -1.25 & 48.6 & 0\\ 0 & 0 & 0 & 1\\ 19.5 & 0 & -19.5 & 0\end{bmatrix},\\ &\mathbf{B} = \mathbf{F} = \begin{bmatrix}0\\ 21.6\\ 0\\ 0\end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix}1 & 0\\ 0 & 0\\ 1 & 0\\ 1 & -.333\end{bmatrix},\\ &\omega(t) = \begin{bmatrix}\xi(t)\\ \upsilon(t)\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\end{aligned} \tag{56}$$

In (56), $\upsilon = \sin(\theta_l)$ is the non-linear function and ξ represents the input disturbance.

It is evident that

$$\mathbf{CE} = \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}^{\mathrm{T}} \tag{57}$$

and hence $\mathrm{rank}(\mathbf{CE}) \ne \mathrm{rank}(\mathbf{E})$ and the unmatched disturbance is '$\sin(\theta_l(t))$'. Here it is assumed that $\mathbf{F} = \mathbf{B}$. For simulation purposes, the fault is $f(t) = 1.2 + 0.8\sin(\pi t)$ and the disturbance is $\xi(t) = 0.5\sin(6\pi t)$. The modelling uncertainty is $\Delta\mathbf{A}(t) = \mathbf{G}_A\mathbf{F}_A(t)\mathbf{E}_A$, where $\mathbf{G}_A = \mathbf{E}_A = \mathbf{0.01} \times \mathbf{I}_4$ and $\mathbf{F}_A = \sin(2\ t)$. Since the conditions of Assumption 1 are satisfied in this study, the optimisation problem in (51) is solved by MATLAB LMI toolbox. The constant scalars introduced in Theorem 1 are listed in Table 1. Based on $u_{\max} = 0.5$, the gains for the controller ($\mathbf{K}_1$), the state observer ($\mathbf{L}$) and adaptive law gains ($\mathbf{F}_1$ and $\mathbf{F}_2$) are obtained. To limit the magnitude of the observer gain $\hat{\mathbf{L}}$, an additional constraint was imposed to the optimisation problem as follows, $\hat{\mathbf{L}}\hat{\mathbf{L}}^{\mathrm{T}} \le \sigma^2\mathbf{I}_m$, $\sigma > 0$; which by using the Schur complement is equivalent to

$$\begin{bmatrix}\sigma^2\mathbf{I}_m & \hat{\mathbf{L}}\\ \hat{\mathbf{L}}^{\mathrm{T}} & \mathbf{I}_n\end{bmatrix} \ge 0, \tag{58}$$

where σ can be pre-set by the designer. Subjecting the optimisation problem to the LMI constraints and minimising γ, the following are attained: (see (59)) . Figs. 3 and 4 show the trajectories of the fault and states. The states have an initial condition of $\mathbf{x}(0) = [0.3 \quad 0 \quad 0.4 \quad 0]$ and the PDUIO has zero initial conditions $\hat{\mathbf{x}}(0) = \mathbf{0}_{1\times 4}$, $\hat{f}(0) = 0$. From Fig. 3, it can be seen that the system states are stabilised despite the presence of input saturation, fault, disturbance and uncertainty. Thus, based on the introduced on-line FE, the designed FE/FTC can rapidly recover the system performance in the presence of the fault. Due to the disturbance $\xi(t)$ throughout the simulation, the states do not tend to zero and fluctuate around it. In addition from Fig. 3, the disturbance causes the fault estimate to not converge to the fault. The performance index (γ) is calculated in the simulation interval as illustrated in Fig. 5, where it is shown that, the maximum performance index is below 0.5, which is lower than 0.98 obtained from the design procedure. From the invariant set theorem, the trajectory of system states that starts from non-zero initial conditions should remain in the ellipsoid set $\Omega(\mathbf{P}, \delta + \eta(\alpha + \beta))$ (Remark (5)). In this study, based on considered initial conditions

$$\mathbf{K}_1 = [-2.85 \quad -4.62 \quad -0.41 \quad -15.70], \quad \mathbf{L} = \begin{bmatrix} -0.25 & -0.03 & -0.24 \\ 0.55 & 1.43 & -0.55 \\ 0.25 & 0.03 & 0.24 \\ -0.28 & 0.11 & -0.21 \end{bmatrix},$$
$$\mathbf{F}_1 = [5.56 \quad 1.79 \quad -5.56], \quad \mathbf{F}_2 = [559.52 \quad 168.42 \quad -559.54] \tag{59}$$
with $\gamma = 0.98$.

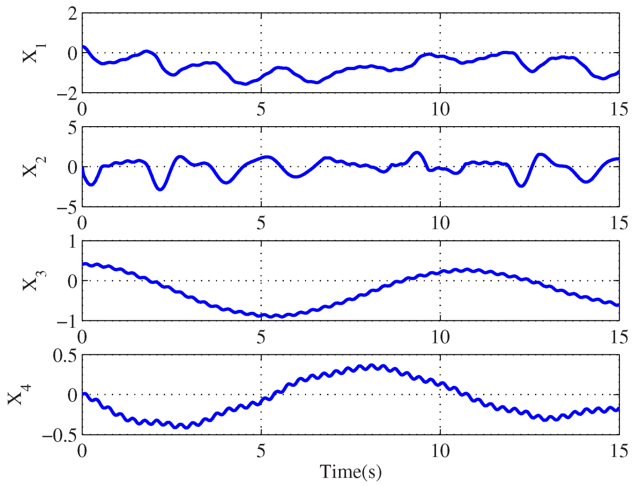

Fig. 3 *Time response of states*

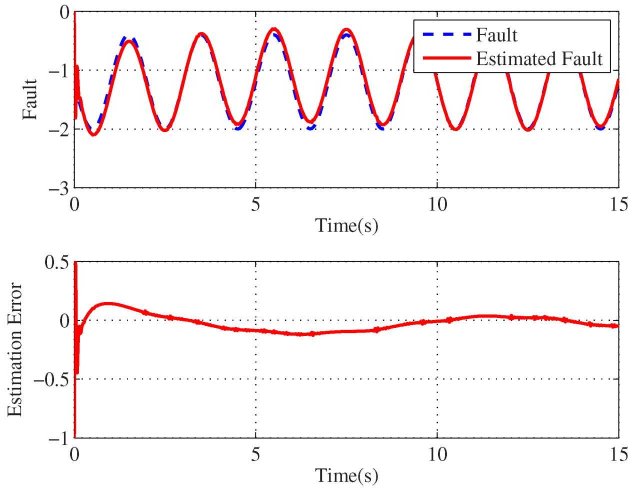

Fig. 4 *Trajectory of fault and FE error*

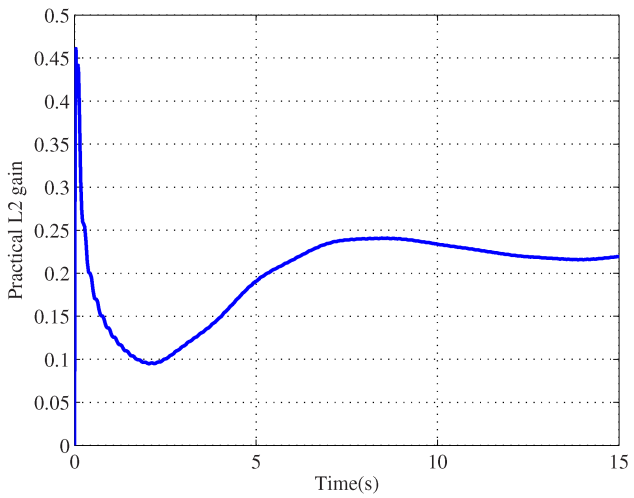

Fig. 5 *Performance index trajectory*

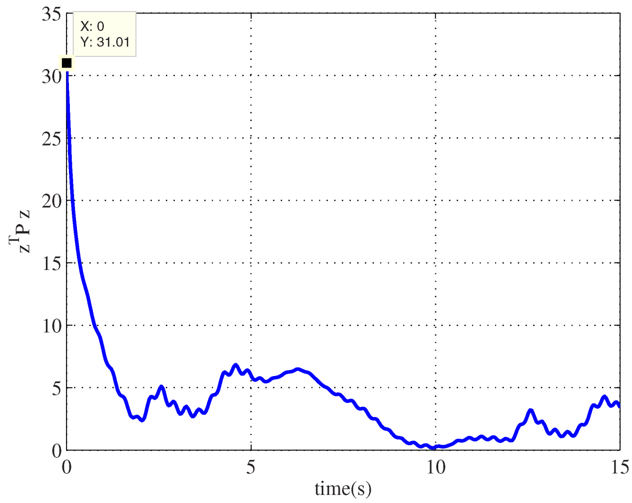

Fig. 6 *Invariant ellpsoid and states trajectory*

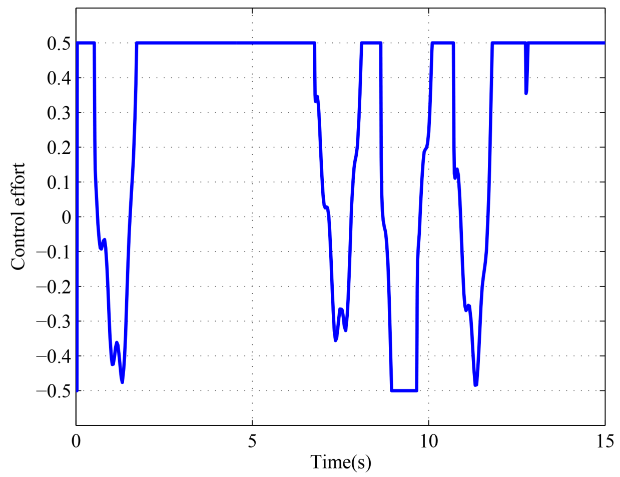

Fig. 7 *Control effort*

for augmented states ($\mathbf{x}(0)$, $\mathbf{e_x}(0)$ and $\mathbf{e_f}(0)$) we have $\delta = 31$. As illustrated in Table 1, $\eta = 1$ and the bound of disturbance energy is set to 1. Therefore, the state trajectory must stay within an ellipsoid with radius $\delta + \eta(\alpha + \beta) = 32$, and Fig. 6 shows that this holds. Also, we can see that the augmented states that start from the initial set, converge to a smaller set. This means that the disturbance rejection is effective.

While the control effort is saturated (Fig. 7) in more than 70% of simulation time, all the control objectives in the Problem Statement 1 are achieved based on results in previous figures. It can be concluded that the introduced method for the integrated FE/FTC in this paper is effective.

5 Conclusion

In this paper, we present a systematic method for the integrated design of FE and FTC for saturated systems with uncertainty and $\mathbf{L}_2$-bounded disturbance. Most earlier works did not consider the effect of actuator saturation in FTC-based FE. The full-state feedback controller with an additive fault compensation term guarantees stability and the $\mathbf{L}_2$-gain performance index. Due to uncertainty, the design of FE and FTC are combined to reduce the effect of the disturbance on the FE. Based on Young's inequality all BMIs conditions are cast into LMIs that can be solved in single step for FE/FTC design gains. All aforementioned conditions are derived by considering the QPDI model for saturation. This modelling approach eliminates saturation non-linearity in the design procedure. Simulation results show that LMIs conditions guarantee the stability and performance of the FE/FTC schemes. The method in this paper for saturated FTC based FE is introduced for LTI system, but in the future can be extended to non-linear systems.

6 References

[1] Lee, T.H., Lim, C.P., Nahavandi, S., *et al.*: 'Observer-based H_∞ fault-tolerant control for linear systems with sensor and actuator faults', *IEEE Syst. J.*, 2019, **13**, (2), pp. 1981–1990

[2] Zhang, Y., Jiang, J.: 'Bibliographical review on reconfigurable fault-tolerant control systems', *Annu. Rev. Control*, 2008, **32**, (2), pp. 229–252

[3] Zhang, Y., Jiang, J.: 'Fault tolerant control system design with explicit consideration of performance degradation', *IEEE Trans. Aerosp. Electron. Syst.*, 2003, **39**, (3), pp. 838–848

[4] Jiang, J., Zhang, Y.: 'Accepting performance degradation in fault-tolerant control system design', *IEEE Trans. Control Syst. Technol.*, 2006, **14**, (2), pp. 284–292

[5] Zhang, Y., Jiang, J., Theilliol, D.: 'Incorporating performance degradation in fault tolerant control system design with multiple actuator failures', *Int. J. Control Autom. Syst.*, 2008, **6**, (3), pp. 327–338

[6] Theilliol, D., Join, C., Zhang, Y.: 'Actuator fault tolerant control design based on a reconfigurable reference input', *Int. J. Appl. Math. Comput. Sci.*, 2008, **18**, (4), pp. 553–560

[7] Theilliol, D., Zhang, Y., Ponsart, J.C.: 'Fault tolerant control system against actuator failures based on re-configuring reference input'. 2009 Int. Conf. on Advances in Computational Tools for Engineering Applications (ACTEA'09), Zouk Mosbeh, Lebanon, 2009, pp. 480–485

[8] Benosman, M., Lum, K.Y.: 'Online references reshaping and control reallocation for nonlinear fault tolerant control', *IEEE Trans. Control Syst. Technol.*, 2009, **17**, (2), pp. 366–379

[9] Fan, J., Zhang, Y., Zheng, Z.: 'Integrated adaptive fault diagnosis and state-feedback control for systems with constant actuator faults and control input constraints'. ASME/IEEE Int. Conf. on Mechatronic and Embedded Systems and Applications (MESA'11), Washington, DC, 2011

[10] Huang, S., Lam, J., Chen, B.: 'Local reliable control for linear systems with saturating actuators'. 2002 41st IEEE Conf. on Proc. of the Decision and Control, Las Vegas, USA, 2002, vol. 4, pp. 4154–4159

[11] Mahmoud, M., Jiang, J., Zhang, Y.: 'Stability of fault tolerant control systems driven by actuators with saturation', *IFAC Proc. Vol.*, 2002, **35**, (1), pp. 473–478

[12] Li, X.J., Yang, G.H.: 'Robust adaptive fault-tolerant control for uncertain linear systems with actuator failures', *IET Control Theory Applic.*, 2012, **6**, (10), pp. 1544–1551

[13] Fan, J., Zheng, Z., Zhang, Y.: 'Fault-tolerant control for output tracking systems subject to actuator saturation and constant disturbances: an LMI approach'. Proc. AIAA Guidance, Navigation, and Control Conf., Portland, USA, August 2011

[14] Wenjie, D., Dayi, W., Chengrui, L.: 'Integral sliding mode fault-tolerant control for spacecraft with uncertainties and saturation', *Asian J. Control*, 2017, **19**, (1), pp. 372–381

[15] Luo, M., Gao, M., Cai, G.: 'Fault-tolerant control for flutter of airfoil subject to input saturation', *Asian J. Control*, 2016, **18**, (6), pp. 2256–2262

[16] Lan, J., Patton, R.J.: 'A new strategy for integration of fault estimation within fault-tolerant control', *Automatica*, 2016, **69**, pp. 48–59

[17] Lan, J., Patton, R.J.: 'Integrated fault estimation and fault-tolerant control for uncertain lipschitz nonlinear systems', *Int. J. Robust Nonlinear Control*, 2017, **27**, (5), pp. 761–780

[18] Jiang, B., Staroswiecki, M., Cocquempot, V.: 'Fault accommodation for nonlinear dynamic systems', *IEEE Trans. Autom. Control*, 2006, **51**, (9), pp. 1578–1583

[19] Zhang, K., Jiang, B., Shi, P.: 'Observer-based integrated robust fault estimation and accommodation design for discrete-time systems', *Int. J. Control*, 2010, **83**, (6), pp. 1167–1181

[20] Yan, X.G., Edwards, C.: 'Robust sliding mode observer-based actuator fault detection and isolation for a class of nonlinear systems', *Int. J. Syst. Sci.*, 2008, **39**, (4), pp. 349–359

[21] Gao, Z., Ding, S.X.: 'Actuator fault robust estimation and fault-tolerant control for a class of nonlinear descriptor systems', *Automatica*, 2007, **43**, (5), pp. 912–920

[22] Zhang, J., Swain, A.K., Nguang, S.K.: 'Robust sensor fault estimation and fault-tolerant control for uncertain lipschitz nonlinear systems'. 2014 American Control Conf. (ACC), Portland, USA, 2014, pp. 5515–5520

[23] Gao, Z., Liu, X., Chen, M.Z.: 'Unknown input observer-based robust fault estimation for systems corrupted by partially decoupled disturbances.', *IEEE Trans. Ind. Electron.*, 2016, **63**, (4), pp. 2537–2547

[24] Fan, J., Zhang, Y., Zheng, Z.: 'Adaptive observer-based integrated fault diagnosis and fault-tolerant control systems against actuator faults and saturation', *J. Dyn. Syst. Meas. Control*, 2013, **135**, (4), p. 041008

[25] Fan, J., Zhang, Y., Zheng, Z.: 'Observer-based reliable stabilization of uncertain linear systems subject to actuator faults, saturation, and bounded system disturbances', *ISA Trans.*, 2013, **52**, (6), pp. 730–737

[26] Lan, J., Patton, R.J., Zhu, X.: 'Integrated fault-tolerant control for a 3-dof helicopter with actuator faults and saturation', *IET Control Theory Applic.*, 2017, **11**, (14), pp. 2232–2241

[27] Corless, M., Tu, J.: 'State and input estimation for a class of uncertain systems', *Automatica*, 1998, **34**, (6), pp. 757–764

[28] Hu, T., Lin, Z., Chen, B.M.: 'An analysis and design method for linear systems subject to actuator saturation and disturbance', *Automatica*, 2002, **38**, (2), pp. 351–359

[29] Guo, L., Cao, S.: '*Anti-disturbance control for systems with multiple disturbances*' (CRC Press, Boca Raton, 2013)

[30] Cao, S., Zhao, Y., Qiao, J.: 'Adaptive fault tolerant attitude control based on a disturbance observer for satellites with multiple disturbances', *Trans. Inst. Meas. Control*, 2016, **38**, (6), pp. 722–731

[31] Cao, S., Zhao, Y.: 'Anti-disturbance fault-tolerant attitude control for satellites subject to multiple disturbances and actuator saturation', *Nonlinear Dyn.*, 2017, **89**, (4), pp. 2657–2667

[32] Zhang, K., Jiang, B., Cocquempot, V.: 'Adaptive observer-based fast fault estimation', *Int. J. Control Autom. Syst.*, 2008, **6**, (3), pp. 320–326

[33] Park, T.G.: 'Designing fault detection observers for linear systems with mismatched unknown inputs', *J. Process Control*, 2013, **23**, (8), pp. 1185–1196

[34] Lien, C.H.: 'Robust observer-based control of systems with state perturbations via LMI approach', *IEEE Trans. Autom. Control*, 2004, **49**, (8), pp. 1365–1370

[35] Kheloufi, H., Zemouche, A., Bedouhene, F., *et al.*: 'On LMI conditions to design observer-based controllers for linear systems with parameter uncertainties', *Automatica*, 2013, **49**, (12), pp. 3700–3704

[36] Zemouche, A., Rajamani, R., Kheloufi, H., *et al.*: 'Robust observer-based stabilization of lipschitz nonlinear uncertain systems via LMIS-discussions and new design procedure', *Int. J. Robust Nonlinear Control*, 2017, **27**, (11), pp. 1915–1939

7 Appendix

7.1 Appendix 1: proof of Theorem 1

Define a Lyapunov function $\mathrm{V}(t,\mathbf{z}) = \mathbf{z}^{\mathrm{T}}(t)\mathbf{P}\mathbf{z}(t)$. The following inequality

$$\dot{\mathbf{V}} \le \eta\tilde{\omega}^{\mathrm{T}}\tilde{\omega} - \frac{\eta}{\gamma^2}\mathbf{z}^{\mathrm{T}}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z\mathbf{z} \tag{60}$$

causes system (31) to satisfy the performance in (4). Owing to uncertainty and saturation, system (31) is actually a piecewise continuous time-varying system. Therefore, to calculate the time derivative of $\mathrm{V}(t,\mathbf{z})$, the time Dini derivative is used as follows:

$$\dot{\mathbf{V}}(\mathbf{z},t) = \lim_{h\to 0}\sup\frac{\mathbf{V}(t+h,\mathbf{z}+h\mathbf{A}_z^i\mathbf{z}+h\mathbf{B}_z\tilde{\omega}_z) - \mathrm{V}(t,\mathbf{z})}{h} \tag{61}$$

Then, by letting $h \to 0$, it follows that

$$\dot{\mathbf{V}}(\mathbf{z},t) = \mathbf{z}^{\mathrm{T}}\left[\mathbf{A}_z^{i\,\mathrm{T}}(t)\mathbf{P} + \mathbf{P}\mathbf{A}_z^i(t)\right]\mathbf{z} + 2\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{B}_z\tilde{\omega} \tag{62}$$

From Lemma 2, it follows that

$$2\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{B}_z\tilde{\omega} \le \frac{1}{\eta}\mathbf{z}^{\mathrm{T}}\mathbf{P}\mathbf{B}_z\mathbf{B}_z^{\mathrm{T}}\mathbf{P}\mathbf{z} + \eta\tilde{\omega}^{\mathrm{T}}\tilde{\tilde{\omega}} \tag{63}$$

and therefore (62) becomes

$$\dot{\mathbf{V}} \le \mathbf{z}^{\mathrm{T}}\left[\mathbf{A}_z^{i\,\mathrm{T}}(t)\mathbf{P} + \mathbf{P}\mathbf{A}_z^i(t) + \frac{1}{\eta}\mathbf{P}\mathbf{B}_z\mathbf{B}_z^{\mathrm{T}}\mathbf{P}\right]\mathbf{z} + \eta\tilde{\omega}^{\mathrm{T}}\tilde{\omega} \tag{64}$$

Using (64), it is clear that (60) holds if

$$\begin{aligned}&\mathbf{z}^{\mathrm{T}}\left[\mathbf{A}_z^{i\,\mathrm{T}}(t)\mathbf{P} + \mathbf{P}\mathbf{A}_z^i(t) + \frac{1}{\eta}\mathbf{P}\mathbf{B}_z\mathbf{B}_z^{\mathrm{T}}\mathbf{P}\right]\mathbf{z} + \eta\tilde{\omega}^{\mathrm{T}}\tilde{\omega}\\ &\le \eta\tilde{\omega}^{\mathrm{T}}\tilde{\omega} - \frac{\eta}{\gamma^2}\mathbf{z}^{\mathrm{T}}\mathbf{C}_z^{\mathrm{T}}\mathbf{C}_z\mathbf{z}\end{aligned} \tag{65}$$

It can be concluded that (34) implies that the inequalities (65) and consequently (4) hold true for system (31). It can also be concluded that if stability and performance condition (34) is satisfied for system (31) then the same conditions hold for system (28). To prove this proposition, multiply both sides of (34) $\forall i \in [1, 2^r]$ with θ_i and summing together, and the following inequality is obtained:

$$\sum_{j=1}^{2^r} (\theta_j \mathbf{A}_z^j)^{\mathrm{T}} \mathbf{P} + \mathbf{P} \sum_{j=1}^{2^r} \theta_j \mathbf{A}_z^j + \frac{\sum_{j=1}^{2^r} \theta_j}{\eta} \mathbf{P} \mathbf{B}_z \mathbf{B}_z^{\mathrm{T}} \mathbf{P} + \frac{\eta \sum_{j=1}^{2^r} \theta_j}{\gamma^2} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z \le 0 \tag{66}$$

Then, from $\sum_{j=1}^{2^r} \theta_j = 1$, following inequality holds:

$$\mathbf{A}_z^{\mathrm{T}} \mathbf{P} + \mathbf{P} \mathbf{A}_z + \frac{1}{\eta} \mathbf{P} \mathbf{B}_z \mathbf{B}_z^{\mathrm{T}} \mathbf{P} + \frac{\eta}{\gamma^2} \mathbf{C}_z^{\mathrm{T}} \mathbf{C}_z \le 0 \tag{67}$$

Therefore, the proposition is proved.

To use the saturation model (25) it is necessary to show that the second inequality (35) in Theorem 1 makes $\tilde{\mathbf{x}} \in \mathrm{L}(\mathbf{H})$. It is clear that the following equality holds:

$$\mathbf{H}_x \hat{\mathbf{x}}(t) + \mathbf{H}_f \hat{\mathbf{f}}(t) = \mathbf{H}_x \mathbf{x}(t) + \mathbf{H}_f \mathbf{f}(t) - \mathbf{H}\mathbf{x}_\mathbf{e} \tag{68}$$

Therefore, the inequality (23), which is defined for the linear set $\mathbf{L}$, is converted to

$$\left| (\mathbf{H}_x)_i \mathbf{x}(t) + (\mathbf{H}_f)_i \mathbf{f}(t) - (\mathbf{H})_i \mathbf{x}_\mathbf{e} \right| < u_{i,\max} \tag{69}$$

For any vector $\mathbf{X}$ and $\mathbf{Y}$, the following inequality can be established:

$$|\mathbf{X} + \mathbf{Y}| \le |\mathbf{X}| + |\mathbf{Y}| \quad \forall \mathbf{X}, \mathbf{Y} \tag{70}$$

By using (70), it can be seen that if the following inequality (71) holds, then inequality (69) also holds:

$$|(\mathbf{H}_x)_i \mathbf{x}(t) - (\mathbf{H})_i \mathbf{x}_\mathbf{e}| + \left| (\mathbf{H}_f)_i \mathbf{f}(t) \right| < u_{i,\max} \tag{71}$$

Since

$$\left| (\mathbf{H}_{f i} \mathbf{f}(t) \right| \le \sum_{j=1}^{l} \left| (\mathbf{H}_f)_{ij} \mathbf{f}_j(t) \right| \tag{72}$$

and notice that

$$\sum_{j=1}^{l} \left| (\mathbf{H}_f)_{ij} \right| \left| \mathbf{f}_j(t) \right| \le f_0 \sum_{j=1}^{l} \left| (\mathbf{H}_f)_{ij} \right| = f_0 \| \, (\mathbf{H}_f)_i \, \|_1 \tag{73}$$

then

$$|(\mathbf{H}_x)_i \mathbf{x}(t) - (\mathbf{H})_i \mathbf{x}_\mathbf{e}| < c_i, \quad i = 1, \ldots, m \tag{74}$$

where $c_i = u_{i,\max} - f_0 \| \, (\mathbf{H}_f)_i \, \|_1 > 0$ and hence the new linear region is defined as

$$\hat{L}(\mathbf{H}) = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_\mathbf{e} \end{bmatrix} \middle| \; |(\mathbf{H}_x)_i \mathbf{x}(t) - (\mathbf{H})_i \mathbf{x}_\mathbf{e}| < c_i, \quad i \in [1, m] \right\} \tag{75}$$

Obviously, it follows that

$$\hat{L}(\mathbf{H}) \subset L(\mathbf{H}) \tag{76}$$

It is known that the states of augmented system (31) remain in the set $\Omega(\mathbf{P}, \eta\alpha_{ag})$ at all times, hence it is adequate to show that $\Omega(\mathbf{P}, \eta\alpha_{ag}) \subset \hat{L}(\mathbf{H})$. Inequality (35) makes this condition hold. This is the end of proof.

7.2 Appendix 1: proof of Corollary 1

By using the Schur-complement, (34) is equivalent to

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{A}_z^{i\,\mathrm{T}} \mathbf{P} + \mathbf{P} \mathbf{A}_z^i & \mathbf{C}_z^{\mathrm{T}} & \mathbf{P} \mathbf{B}_z \\ \mathbf{C}_z & -\gamma^2 \mathbf{I} & 0 \\ \mathbf{B}_z^{\mathrm{T}} \mathbf{P} & 0 & -\mathbf{I} \end{bmatrix} \le 0 \tag{77}$$

where $\eta = 1$ has been set without loss of generality and $\mathbf{P}$ has a diagonal structure as in (40). If matrices $\mathbf{A}_z^i$ and $\mathbf{B}_z$ are replaced from (31), the coupling terms make (77) become BMIs. Notice that the top left block (i.e. $\mathbf{A}_z^{i\,\mathrm{T}} \mathbf{P} + \mathbf{P} \mathbf{A}_z^i$) has coupled terms ($\mathbf{P}_0 \mathbf{B} \Delta_j^- \mathbf{K}_1$, $\mathbf{P}_0 \mathbf{B} \Delta_j \mathbf{H}_1$, $\mathbf{R}\mathbf{L}_1\mathbf{C}$ and $\mathbf{F}_1 \mathbf{C}\mathbf{C}^{\mathrm{T}} \mathbf{L}_1$). The change of variable $\hat{\mathbf{L}}_\mathbf{1} = \mathbf{R}\mathbf{L}_\mathbf{1}$ resolves the one of them ($\mathbf{R}\mathbf{L}_1\mathbf{C}$). Also notice that the top-left block contains system uncertainty that can be overcome by re-writing it as the sum of two matrices, one containing the uncertainties $\tilde{\mathbf{B}}$ and the other one is defined without uncertainties $\tilde{\mathbf{A}}$ as $\tilde{\mathbf{A}} + \tilde{\mathbf{B}}$ where

$$\begin{aligned}
&\tilde{\mathbf{A}} = [\mathbf{a}_{ij}], \quad \tilde{\mathbf{B}} = [\mathbf{b}_{ij}] \\
&\text{such that} \\
&\mathbf{a}_{11} = He\left(\mathbf{P}_0 \left(\mathbf{A} + \bar{\mathbf{B}}_1\right)\right), \ \ \mathbf{a}_{12} = \bar{\mathbf{B}}_1, \ \ \mathbf{a}_{13} = -\bar{\mathbf{B}}_2 \\
&\mathbf{a}_{22} = He\left(\mathbf{RTA} - \hat{\mathbf{L}}_1 \mathbf{C}\right), \\
&\mathbf{a}_{23} = \mathbf{RTF} - (\mathbf{F}_1 \mathbf{C}(\mathbf{TA} - \mathbf{L}_1 \mathbf{C}) + \mathbf{F}_2 \mathbf{C})^{\mathrm{T}} \\
&\mathbf{a}_{33} = He(-\mathbf{F}_1 \mathbf{CTF}) \\
&\text{and} \\
&\mathbf{b}_{11} = He(\mathbf{P}_0 \, \Delta \, \mathbf{A}), \ \ \mathbf{b}_{12} = (\mathbf{RT} \, \Delta \, \mathbf{A})^{\mathrm{T}}, \ \ \mathbf{b}_{13} = -(\mathbf{F}_1 \mathbf{CT} \, \Delta \, \mathbf{A})^{\mathrm{T}} \\
&\mathbf{b}_{21} = \mathbf{b}_{23} = \mathbf{b}_{33} = 0
\end{aligned} \tag{78}$$

However, note that the coupling term $\mathbf{P}_0 \mathbf{B} \Delta_j^- \mathbf{K}_1$ cannot resolve with a change of variable or a congruence transformation. The same problem arises in the observer-based controller design. In that case, the similar term $\mathbf{PBK}$ exists that is usually resolved by the equality constraint $\mathbf{PB} = \mathbf{B}\hat{\mathbf{P}}$ and change of variable $\hat{\mathbf{K}} = \hat{\mathbf{P}}\mathbf{K}$ (refer to [34–36] for more details). The mentioned equality constraint is not used in this study because of Δ_j^- and to reduce conservatism. Pre- and post-multiply $\boldsymbol{\Sigma}$ by $\mathrm{diag}(\tilde{\mathbf{P}}, \mathbf{I}, \mathbf{I})$, makes the terms $\mathbf{P}_0 \mathbf{B} \Delta_j^- \mathbf{K}_1$ and $\mathbf{P}_0 \mathbf{B} \Delta_j \mathbf{H}_1$ in $\mathbf{a}_{11}$ become $\mathbf{B} \Delta_j^- \mathbf{K}_1 \mathbf{Q}$ and $\mathbf{B} \Delta_j \mathbf{H}_1 \mathbf{Q}$. Using the change of variables $\mathbf{K}_1 \mathbf{Q} = \hat{\mathbf{K}}$ and $\mathbf{H}_1 \mathbf{Q} = \hat{\mathbf{H}}_\mathbf{1}$ resolves another coupling term. Now $\mathbf{B} \Delta_j^- \mathbf{K}_1$ exists in $\mathbf{a}_{12}$ that will be resolved in following: Expand $\Delta \, \mathbf{A}(t) = \mathbf{G}_\mathrm{A} \mathbf{F}_\mathrm{A}(t) \mathbf{E}_A$ and applying Young's relation (Lemma 2), the following inequality is established for the top-left block Σ:

$$\begin{aligned}
\tilde{\mathbf{A}} + \tilde{\mathbf{B}} \le & \begin{bmatrix} \left(\mathbf{A} + \bar{\mathbf{B}}_1\right)\mathbf{Q} + \mathbf{Q}\left(\mathbf{A} + \bar{\mathbf{B}}_1\right)^{\mathrm{T}} & 0 & \bar{\mathbf{B}}_2 \\ * & \mathbf{a}_{22} & \mathbf{a}_{23} \\ * & * & \mathbf{a}_{33} \end{bmatrix} \\
& + \varepsilon_0 \begin{bmatrix} 0 \\ \mathbf{C}\mathbf{L}_1^{\mathrm{T}} \\ 0 \end{bmatrix} \mathbf{R} \begin{bmatrix} 0 \\ \mathbf{C}\mathbf{L}_1^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}} + \frac{1}{\varepsilon_0} \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_1 \mathbf{C} \end{bmatrix} \mathbf{R}^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_1 \mathbf{C} \end{bmatrix}^{\mathrm{T}} \\
& + \varepsilon_1 \begin{bmatrix} \bar{\mathbf{B}}_1 \\ 0 \\ 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \bar{\mathbf{B}}_1 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + \frac{1}{\varepsilon_1} \begin{bmatrix} 0 \\ -\mathbf{I} \\ 0 \end{bmatrix} \mathbf{Q}^{-1} \begin{bmatrix} 0 \\ -\mathbf{I} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon_2 \begin{bmatrix} \mathbf{G}_A \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{G}_A \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} \\
& + \frac{1}{\varepsilon_2} \begin{bmatrix} \mathbf{Q}\mathbf{E}_A{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{Q}\mathbf{E}_A{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon_3 \begin{bmatrix} \mathbf{Q}\mathbf{E}_A{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{Q}\mathbf{E}_A{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} \\
& + \frac{1}{\varepsilon_3} \begin{bmatrix} 0 \\ \mathbf{RTG}_\mathbf{A} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{RTG}_\mathbf{A} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon_4 \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_1 \mathbf{CTG}_\mathbf{A} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_1 \mathbf{CTG}_\mathbf{A} \end{bmatrix}^{\mathrm{T}} \\
& + \frac{1}{\varepsilon_4} \begin{bmatrix} \mathbf{Q}\mathbf{E}_\mathbf{A}{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{Q}\mathbf{E}_\mathbf{A}{}^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}
\end{aligned} \tag{79}$$

IET Control Theory Appl., 2019, Vol. 13 Iss. 13, pp. 2104-2113

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are positive constants. Using the introduced procedure makes the terms $\mathbf{F}_1\mathbf{C}$ and $\mathbf{C}^{\mathrm{T}}\mathbf{L}_1$ de-coupled from each other and thus the last coupling term is resolved. Also, the term $\mathbf{B}\Delta_j^{-}\mathbf{K}_1$ is now converted to $\mathbf{B}\Delta_j^{-}\mathbf{K}_1\mathbf{Q}$ which makes possible the change of variable $\mathbf{K}_1\mathbf{Q} = \hat{\mathbf{K}}$. The inequality (79) can be rewritten as the following form:

$$\tilde{\mathbf{A}} + \tilde{\mathbf{B}} \le \mathbf{X}^i - \mathbf{Y}^i\mathbf{Z}^{-1}\mathbf{Y}^{i\mathrm{T}} \tag{80}$$

where $\mathbf{X}^i$, $\mathbf{Y}^i$ and $\mathbf{Z}$ are given in (41)–(43). From (80) and (77) the following results:

$$\mathbf{\Sigma} \le \begin{bmatrix} \mathbf{X}^i - \mathbf{Y}^i\mathbf{Z}^{-1}\mathbf{Y}^{i\mathrm{T}} & \mathbf{C}_z^{\mathrm{T}} & \mathbf{P}\mathbf{B}_z \\ \mathbf{C}_z & -\gamma^2\mathbf{I} & 0 \\ \mathbf{B}_z^{\mathrm{T}}\mathbf{P} & 0 & -\mathbf{I} \end{bmatrix} \tag{81}$$

Finally, by using the Schur complement on (81), inequality (38) is derived.

In the following, the procedure of getting (39) from (35) is shown. From (35) it can be derived that (see (82)) and by simple manipulation, the following is obtained:

$$\begin{bmatrix} \hat{\mathbf{H}}_{1i} & -\mathbf{H}_{1i} \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 & 0 \\ 0 & \mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{H}}_{1i} & -\mathbf{H}_{1i} \end{bmatrix}^{\mathrm{T}} \le \frac{c_i^2}{\eta(\alpha+\beta)} - \Gamma \mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} \tag{83}$$

Applying the Schur complement to (83) yields

$$\begin{bmatrix} \Gamma\mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} - c_i^2/(\alpha+\beta) & \mathbf{H}_{1i} & \hat{\mathbf{H}}_{1i} \\ \mathbf{H}_{1i}^{\mathrm{T}} & -\mathbf{R} & 0 \\ \hat{\mathbf{H}}_{1i}^{\mathrm{T}} & 0 & -\mathbf{Q} \end{bmatrix} \le 0 \tag{84}$$

It is derived from (84) that

$$\begin{bmatrix} \Gamma\mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} - c_i^2/(\alpha+\beta) & 0 & \hat{\mathbf{H}}_{1i} \\ * & -\mathbf{R} & 0 \\ * & * & -\mathbf{Q} \end{bmatrix} + \begin{bmatrix} \mathbf{H}_{1i} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{I} \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{H}_{1i}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \le 0 \tag{85}$$

Now by applying Young's inequality (Lemma 2), the following is obtained:

$$\begin{bmatrix} \Gamma\mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} - c_i^2/\alpha & 0 & 0 \\ * & -\mathbf{R} & 0 \\ * & * & -\mathbf{P}_0^{-1} \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{H}_{1i} \\ 0 \\ 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \mathbf{H}_{1i} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + \frac{1}{\varepsilon} \begin{bmatrix} 0 \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1} \begin{bmatrix} 0 \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} \le 0 \tag{86}$$

It can be seen that (39) is the Schur complement of (86) and Corollary 1 is proven.

$$\begin{bmatrix} \mathbf{H}_{1i} & -\mathbf{H}_{1i} \end{bmatrix} \begin{bmatrix} \mathbf{P}_0^{-1} & 0 \\ 0 & \mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{1i} & -\mathbf{H}_{1i} \end{bmatrix}^{\mathrm{T}} \le \frac{c_i^2}{\eta(\alpha+\beta)} - \Gamma \mathbf{H}_{2i}\mathbf{H}_{2i}^{\mathrm{T}} \tag{82}$$