Pricing Vulnerable Options with Jump Clustering

Yong Ma, Keshab Shrestha, and Weidong Xu

This paper presents a valuation of vulnerable European options using a model with self-exciting Hawkes processes that allow for clustered jumps rather than independent jumps. Many existing valuation models can be regarded as special cases of the model proposed here. Using numerical analyses, this study also performs sensitivity analyses and compares the results to those of existing models for European call options. The results show that jump clustering has a significant impact on the option value.


1. INTRODUCTION

When an option buyer is exposed to the option writer’s default risk, the option is referred to as a vulnerable option. In this case, the option writer’s default risk is an example of a counterparty risk. Most over-the-counter options and other derivative securities are associated with counterparty risks. Traditional research has, to a large extent, neglected this type of risk. However, the recent financial crisis has brought the issue of counterparty risk to the forefront of the discussion among policymakers, practitioners, and academic researchers. Counterparty risks are also an important risk factor recognized by Basel III from the Bank of International Settlements (BIS) and the Dodd–Frank Act in the United States, which was signed into federal law by President Obama on July 21, 2010.

In order to price vulnerable options, we need to model the underlying asset price dynamics and the counterparty risk. Based on the default model of corporate bonds proposed by Merton (1974), Johnson and Stulz (1987) present several vulnerable option pricing models, in which the option itself is assumed to be the only liability of the option writer. Under the assumption of independence between the underlying asset and the default risk of the option writer, Hull and White (1995) and Jarrow and Turnbull (1995) derive analytical solutions for vulnerable option values. However, they assume that the payout ratio is exogenous and

Yong Ma is an Assistant Professor at the College of Finance and Statistics, Hunan University, Changsha, Hunan, China. Keshab Shrestha is a Professor of Banking and Finance at the School of Business, Monash University Malaysia, Bandar Sunway, Selangor, Malaysia. Weidong Xu is an Associated Professor at the School of Management, Zhejiang University, Hangzhou, Zhejiang, China. We are very grateful to the Editor Robert I. Webb and the anonymous referee for their valuable comments and suggestions. This research was supported by National Natural Science Foundation of China (71601075), Humanity and Social Science Youth Foundation of the Ministry of Education of China (15YJC790071), and Zhejiang and Hunan Provincial Natural Science Foundation of China (LY15G010002, 2016JJ3047). All errors are our own.

*Correspondence author, School of Management, Zhejiang University, Hangzhou 310085, China. Tel: +86-571-88206867, Fax: +86-571-88206867, e-mail: swd1981@163.com; weidxu@zju.edu.cn

Received October 2015; Accepted December 2016

© 2017 Wiley Periodicals, Inc.
Published online 23 February 2017 in Wiley Online Library (wileyonlinelibrary.com).
DOI: 10.1002/fut.21843
the default event is modeled using a reduced-form approach.\textsuperscript{1} Klein (1996) extends these models by allowing the option writer to have other liabilities, where the values of the assets underlying the option and the option writer's assets are allowed to be correlated. He also allows the payout ratio or recovery rate to be endogenous. Subsequently, many extensions and variants of the models used by Johnson and Stulz (1987) and Klein (1996) have been proposed. For instance, Klein and Inglis (1999) extend the model proposed by Klein (1996) by incorporating interest rate risk. In another study, Klein and Inglis (2001) allow the default boundary to depend on the value of the option itself. Hung and Liu (2005) present a model to value vulnerable options in an incomplete market. Chang and Hung (2006) investigate vulnerable American options based on the two-point Geske and Johnson method. Finally, Klein and Yang (2010) extend the models of Johnson and Stulz (1987) and Klein (1996) to price vulnerable American options.

All the works mentioned above use Brownian-based diffusion processes to model the stock price and firm-value dynamics. Even though diffusion models are used extensively in the valuation of derivative securities, owing to their tractability, these models have some limitations because they do not allow jumps. First, theoretically speaking, in an efficient market, asset prices fully reflect all available relevant information. In addition, news arrives randomly and, when it does so, we expect the stock price to exhibit jumps. Second, these models are considered to be inconsistent with empirical observations in financial markets, where jumps are clearly visible (Bakshi, Cao, & Chen, 1997; Bates, 1996, 2000; Eraker, 2004; Eraker, Johannes, & Polson, 2003; Pan, 2002). Finally, in such models, a firm will not default unexpectedly because of the impossibility of sudden big drops in the firm's value (Jones, Mason, & Rosenfeld, 1984). In order to overcome these limitations, Xu, Xu, Li, & Xiao (2012) and Tian, Wang, Wang, & Wang (2014) propose improved models in which the underlying stock price and the firm value follow Poisson jump-diffusion processes. Specifically, Tina et al. divide the jumps into an idiosyncratic component, which affects only one particular asset price, and a systematic component, which affects all the asset prices.

However, Poisson jump-diffusion models do not allow for clustered jumps, where a single jump increases the probability of future jumps occurring. When news initially arrives, we expect further news to arrive that would clarify the initial news, or that would reveal the extent of its impact. In this case, security prices would respond not only to the initial news, but also to the ways in which market participants and firms react to the news. Therefore, theoretically, we expect to observe jump clustering. The clustering of jumps is especially expected during a crisis. As Ait-Sahalia, Cacho-Diaz, & Laeven (2015) point out, “what makes a crisis worthy of that name is typically not the initial jump, but the amplification that takes place subsequently over hours or days, and the fact that other markets become

\textsuperscript{1}In the existing literature, there are two main approaches to modeling credit risk: (1) the structural approach; and (2) the reduced-form approach. In the structural approach, the default event is modeled based on the fundamental characteristics of the obligor (e.g., the values of the obligor's assets and debt). The default event is assumed to take place once the value of the asset falls below a threshold value (e.g., Black & Cox, 1976; Longstaff & Schwartz, 1995; Merton, 1974). The structural approach is intuitive because it links the default risk to the firm's economic fundamentals. The main shortcoming of this approach lies in the assumption that the value the obligor's assets can be observed directly. In the reduced-form approach, instead of modeling the value of the obligor's assets and its capital structure, a default time is modeled as a stopping time by exogenously specifying a hazard rate or intensity of default (Duffie & Singleton, 1999; Jarrow, Lando, & Turnbull, 1997; Madan & Unal, 1998. Owing to its tractability, the reduced-form approach to modeling credit risk is popular. Though there exist differences between the two approaches, Gündüz and Uhrig-Homburg (2014) have recently shown that their predictive powers are quite close, on average. For more details about the two approaches, please refer to Bielecki and Rutkowski (2013), and the references therein.
affected as well.” The standard jump-diffusion models are unable to replicate jump clusters, either in time or across markets. At the same time, in a typical jump model with arrival rates calibrated to historical data, jumps are infrequent events, occurring perhaps once a year, on average (Eraker et al., 2003). Hence, multiple jumps in close succession over days and across different markets would be extremely unlikely in a standard jump-diffusion model. On the other hand, the phenomenon in which large moves in an asset price tend to be followed by additional large moves is observed extensively in financial markets (e.g., Lux & Marchesi, 2000; Mandelbrot, 1963). This implies that jumps in financial assets tend to be clustered. Therefore, a Poisson jump-diffusion model is not ideal for describing the clustered jumps observed in financial markets.

The main objective of this study is to propose a univariate, self-exciting jump-diffusion model, called a Hawkes jump-diffusion model, that can be used to price vulnerable options. In a Hawkes process, the occurrence of a jump will likely accelerate the arrival of future jumps. Therefore, the Hawkes process is suitable for modeling clustered jumps. Specifically, we use the Hawkes process to model both the underlying asset value dynamics and the dynamics of the value of the option writer’s assets. The jumps in each asset value consists of two Hawkes processes: (1) an individual Hawkes process, representing idiosyncratic shocks, such as continuous bad news or frequent investment losses; and (2) a common Hawkes process, representing systemic shocks, such as financial crises or macroeconomic events.

We use a mixture of analytic and numerical techniques to establish some important relationships for vulnerable European call options. For example, we show that among the models with counterparty risks, the model with jump-clustering leads to the highest value for the vulnerable European call option. We also show that the value of the vulnerable European call option decreases with the level of the option writer’s debt, decreases with the bankruptcy costs, increases with the jump-clustering risk specific to the underlying asset, and decreases with the jump-clustering risks specific to the option writer’s assets. With regard to the correlation between the two Brownian motions, where one is driving the underlying asset value and the other is driving the value of the option writer’s assets, the call option value increases with the correlation.

We contribute to the existing literature in several ways. First, the proposed model is a general model, of which the models considered by Black and Scholes (1973), Klein (1996), Tian et al. (2014), and Xu et al. (2012) can be considered special cases. Therefore, the valuation presented here can be considered an extension of all these models. Second, our valuation can also be considered an extension of the Merton (1974) model for pricing regular options without counterparty risk, where we allow for the clustering of jumps. Third, we analyze the effect of the so-called right-way risk and wrong-way risk on vulnerable call options using numerical analyses. Here, we present a situation in which the value of a vulnerable call option can be higher than a non-vulnerable call option value derived using the Black & Scholes model for an extreme right-way risk. This result is shown by analyzing the impact of the correlation on the call option value. Finally, we present several numerical results that show the sensitivity of the European call option value to various other factors, in addition to the correlation. Lastly, we show that clustering has a significant impact on European call option values.

The remainder of the paper is structured as follows. In Section 2, we present our model, together with closed-form solutions to the valuation of vulnerable European options. In Section 3, we demonstrate the fact that many existing models can be considered special cases of the proposed model. Section 4 presents the results of the model comparisons and

---

2Jump clusters in a univariate time series setting and across markets are discussed by Eraker et al. (2003), Ehrmann, Fratzscher, & Rigobon (2011), Lee and Mykland (2008), Maheu and McCurdy (2004), and Yu (2004).
the sensitivity analyses obtained using numerical analyses. Finally, Section 5 concludes the paper.

2. THE MODEL

2.1. Model Specification

Let \((\Omega, \mathcal{F}, \mathbb{Q})\) be a complete probability space, and \(\mathbb{E}\) be the expectation under a risk-neutral measure \(\mathbb{Q}\). Under the risk-neutral measure, the dynamics of the underlying asset price, \(S_t\), are assumed to be given by the following Hawkes jump-diffusion process:

\[
dS_t = (r - k_S(\lambda_t + \lambda_t^{(1)}))dt + \sigma_S dW_t^{(1)} + d \sum_{i=1}^{N_t+N_t^{(1)}} Y_i, \tag{1}
\]

where \(r\) is the risk-free interest rate, \(\sigma_S\) is the volatility of the underlying asset price, and \(W_t^{(1)}\) is the standard Brownian motion. Then, \(Y_i\) are the independently and identically distributed magnitudes of jumps, with mean \(k_S\), and are assumed to have support on \((-1, \infty)\) guaranteeing the positivity of \(S_t\). Finally, \(\lambda_t\) and \(\lambda_t^{(1)}\) represent the intensity processes of the two independent Hawkes processes, \(N_t\) and \(N_t^{(1)}\), respectively. More specifically, the two independent Hawkes processes are

\[
N_t = \sum_{i=1}^{\infty} \mathbb{I}_{\{T_i \leq t\}}, \quad N_t^{(1)} = \sum_{i=1}^{\infty} \mathbb{I}_{\{T_i^{(1)} \leq t\}}. \tag{2}
\]

where \(\{T_i\}_{i=1}^{\infty} \subset [0, \infty)\) represents a sequence of jump times arising from a systematic shock, such as a financial crisis, which influence all entities in the financial system. On the other hand, \(\{T_i^{(1)}\}_{i=1}^{\infty} \subset [0, \infty)\) represents a sequence of jump times due to idiosyncratic shocks, which influence the underlying asset price only. Finally, \(\mathbb{I}_{\{\cdot\}}\) is the indicator function.

The jump intensity processes follow the dynamics

\[
\lambda_t = \lambda_0 + \theta \int_0^t e^{-\delta(t-s)} dN_s = \lambda_0 + \theta \sum_{T_i < t} e^{-\delta(t-T_i)}, \tag{3}
\]

\[
\lambda_t^{(1)} = \lambda_0^{(1)} + \theta_1 \int_0^t e^{-\delta_1(t-s)} dN_s^{(1)} = \lambda_0^{(1)} + \theta_1 \sum_{T_i^{(1)} < t} e^{-\delta_1(t-T_i^{(1)})}, \tag{4}
\]

where all the parameters \(\lambda_0, \theta, \delta, \lambda^{(1)}_0, \theta_1,\) and \(\delta_1\) are assumed to be positive, and \(\lambda_0\) and \(\lambda^{(1)}_0\) are referred to as base intensities. The model implies that an occurrence of a systematic jump, represented by the Hawkes process \(N_t\), would lead to an instantaneous increase in the intensity of the systematic jumps by \(\theta\), where the increment decays exponentially at a rate of \(\delta\). In other words, the occurrence of a systematic shock will stimulate occurrences of future systematic shocks, which leads to jump clustering. Furthermore, the larger the value of \(\theta\) and the smaller the value of \(\delta\), the larger the extent of the jump clustering will

\(^3\)There may be concerns that using the Hawkes process can lead to more clusters or jumps than needed or observed in reality. However, we would like to point out that our formulation has sufficient parameters to control for this concern via careful choices of their values. We would like to thank the editor and the anonymous reviewer for pointing out this issue.
Pricing Vulnerable Options with Jump Clustering

be. Owing to the clustering of jumps, self-exciting Hawkes processes can be used to model the phenomenon of volatility clustering (Chavez-Demoulin & McGill, 2012). In models that use Poisson processes to represent jumps, an occurrence of a jump will not affect the probability of the next jump because of the assumed independent increments. The effect of an idiosyncratic jump, represented by the Hawkes process \( N_t^{(1)} \), is similar to that of \( N_t \).

Combining (1), (3), and (4) yields

\[
dS_t = \left[ r - k_S \left( \lambda_0 + \lambda_0^{(1)} + \theta \sum_{T_i < t} e^{-\delta(t-T_i)} + \theta_1 \sum_{T_i^{(1)} < t} e^{-\delta_1(t-T_i^{(1)})} \right) \right] dt \\
+ \sigma_S dW_t^{(1)} + d \sum_{i=1}^{N_t+N_t^{(1)}} Y_i. \tag{5}
\]

In order to model the counterparty risk associated with the option writer’s default, we adopt Merton’s structural model, where the option writer will default when the value of the writer’s assets, \( A_T \), falls below a threshold or boundary, \( D^* \), on the option’s expiration date, \( T \). The default boundary is assumed to be independent of the option value. However, it is related to the outstanding claims \( D \) at time \( T \). In the case of a default at \( T \), the recovery rate of the outstanding nominal claims is set to \( (1 - \alpha)A_T/D \), where \( \alpha \) is the percentage loss of the value of the option writer’s assets that represents the deadweight costs of bankruptcy.

Before bankruptcy, the value of the option writer’s assets, \( A_t \), is assumed to follow the Hawkes jump-diffusion process:

\[
dA_t = \left( r - k_A(\lambda_0 + \lambda_0^{(2)}) \right) dt + \sigma_A dW_t^{(2)} + d \sum_{i=1}^{N_t+N_t^{(2)}} Z_i, \tag{6}
\]

where \( \sigma_A \) is the volatility of the assets, \( W_t^{(2)} \) is the standard Brownian motion, with \( dW_t^{(1)} dW_t^{(2)} = \rho dt \), and \( Z_i \) are independently and identically distributed jump sizes, with mean \( k_A \) and support on \((-1, \infty)\). The assumption that \( Z_i > -1 \) for \( i = 1, 2, \ldots \) ensures that the value of the assets of the writer is always positive. The third Hawkes process is represented by \( N_t^{(2)} = \sum_{i=1}^{\infty} 1_{\left\{ T_i^{(2)} < t \right\}} \), where the dynamics of its intensity process are given by

\[
\lambda_t^{(2)} = \lambda_0^{(2)} + \theta_2 \int_0^t e^{-\delta_2(t-s)} dN_t^{(2)} = \lambda_0^{(2)} + \theta_2 \sum_{T_i^{(2)} < t} e^{-\delta_2(t-T_i^{(2)})}, \tag{7}
\]

with positive parameters \( \lambda_0^{(2)}, \theta_2, \) and \( \delta_2 \). It follows from (3),(6), and (7) that the dynamics of \( A_t \) can be expressed as

\[
dA_t = \left[ r - k_A \left( \lambda_0 + \lambda_0^{(2)} + \theta \sum_{T_i < t} e^{-\delta(t-T_i)} + \theta_1 \sum_{T_i^{(1)} < t} e^{-\delta_1(t-T_i^{(1)})} \right) \right] dt \\
+ \sigma_A dW_t^{(2)} + d \sum_{i=1}^{N_t+N_t^{(2)}} Z_i. \tag{8}
\]

Finally, suppose that \( (W_t^{(1)}, W_t^{(2)}), N_t, N_t^{(1)}, N_t^{(2)}, \{Y_i\}, \text{ and } \{Z_i\} \) are mutually independent. Thus, in the proposed model, the underlying asset price and the value of the option
writer’s assets are linked by the correlated Brownian motions and the common Hawkes process \( N_t \), which characterizes the jumps resulting from the systematic shocks. For model comparison purposes, suppose \( 0 < \theta < \delta \) and \( 0 < \delta_j < \delta \). Then, it follows from Hawkes (1971) and Da Fonseca and Zaatour (2014) that these Hawkes processes are stationary, and the expected numbers of jumps are 

\[
E[N_T] = \frac{\lambda_0 T}{1-\theta/\delta} \quad \text{and} \quad E[N^{(j)}_T] = \frac{\lambda^{(j)}_0 T}{1-\delta_j/\delta}, \quad j = 1, 2.
\]

### 2.2. Valuation of Vulnerable European Options

Since the value of a European option is the expected present (discounted) value of the payoff of the option on the expiration date under a risk-neutral measure, the values of the vulnerable European call and put at time zero can be expressed as

\[
C^* = e^{-rT}E[(S_T - K)^+I_{[AT \geq D^*]} + \frac{1-\alpha}{D}AT^I_{[AT < D^*]}]
\]

and

\[
P^* = e^{-rT}E[(K - S_T)^+I_{[AT \geq D^*]} + \frac{1-\alpha}{D}AT^I_{[AT < D^*]}]
\]

respectively.

In this study, we use Merton’s approach to modeling a default, whereby a default can only occur at \( T \). An alternative approach would be to use the first-passage-time approach, where the default time is denoted by

\[
\tau = \inf \{0 \leq t \leq T : A_t < D^*\}
\]

In this case, the default may occur at any time prior to \( T \). There are several reasons for using Merton’s approach. First, to make the first-passage-time models tractable, the payout ratios are usually assumed to be fixed and independent of the cause and timing of the default, as well as the value of the option writer’s assets. Second, the average length of time typically involved in resolving the financial distress is around two years (e.g., Branch, 2002; Bris, Welch, & Zhu, 2006; Wruck, 1990). During that time, the assets of the counterparty may recover, in which case the claims can be paid out in full. Third, the time to expiration of most over-the-counter options will be within this period.\(^4\)

In order to proceed further, we first derive the analytic expressions for the underlying asset price, \( S_T \), and the value of the option writer’s assets, \( A_T \). Let \( \lambda_S = \lambda_0 + \lambda_0^{(1)} \) and \( \lambda_A = \lambda_0 + \lambda_0^{(2)} \). Then, the following theorems can be established.

**Theorem 2.1.** Let

\[
X^{(1)}_t = S_0 \exp \left\{ \sigma_S W^{(1)}_t + \left( r - k_S \lambda_S - \frac{1}{2} \sigma_S^2 \right) t \right\}
\]

\[
\cdot \prod_{i=1}^{N_t} \exp \left\{ \frac{k_S \theta}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N^{(1)}_t} \exp \left\{ \frac{k_S \theta_1}{\delta_1} [e^{-\delta_1(t-T^{(1)}_i)} - 1] \right\},
\]

\[
X^{(2)}_t = A_0 \exp \left\{ \sigma_A W^{(2)}_t + \left( r - k_A \lambda_A - \frac{1}{2} \sigma_A^2 \right) t \right\}
\]

\[
\cdot \prod_{i=1}^{N_t} \exp \left\{ \frac{k_A \theta}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N^{(2)}_t} \exp \left\{ \frac{k_A \theta_2}{\delta_2} [e^{-\delta_2(t-T^{(2)}_i)} - 1] \right\},
\]

\[
j^{(1)}_t = \prod_{i=1}^{N + N^{(1)}_t} (1 + Y_i) \quad \text{and} \quad j^{(2)}_t = \prod_{i=1}^{N + N^{(2)}_t} (1 + Z_i)
\]

\(^4\)We would like to thank the anonymous reviewer for pointing out this issue.
where the product over an empty set is taken to be one. Then,

\[ S_t = X_t^{(1)} f_t^{(1)} \]  

and

\[ A_t = X_t^{(2)} f_t^{(2)} \]  

**Theorem 2.2.** Let

\[
J_i^1 = \prod_{i=1}^{N_t+N_t^{(1)}} (1 + Y_i) \prod_{i=1}^{N_t} \exp \left\{ \frac{kS\theta_i}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N_t^{(1)}} \exp \left\{ \frac{kS\theta_i}{\delta_1} [e^{-\delta_1(t-T_i^{(1)})} - 1] \right\},
\]

\[
J_i^2 = \prod_{i=1}^{N_t+N_t^{(2)}} (1 + Z_i) \prod_{i=1}^{N_t} \exp \left\{ \frac{kA\theta_i}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N_t^{(2)}} \exp \left\{ \frac{kA\theta_i}{\delta_2} [e^{-\delta_2(t-T_i^{(2)})} - 1] \right\}.
\]

The values of the vulnerable European call and put options at time zero are given by

\[
C^* = S_0 e^{-kS^2 T} \mathbb{E}[J_1^1 N_2(a_2(J_1^1), b_2(J_2^1); \rho)] - K e^{-r T} \mathbb{E}[N_2(a_1(J_1^1), b_1(J_2^1); \rho)]
\]

\[
+ \frac{S_0 A_0(1 - \alpha)}{D} e^{(r+\rho \sigma_\lambda - kS\lambda - kA\lambda)T} \mathbb{E}[J_1^2 J_2^1 N_2(a_3(J_1^1), b_3(J_2^1); -\rho)]
\]

\[
- \frac{K A_0(1 - \alpha)}{D} e^{-kA\lambda T} \mathbb{E}[J_1^2 N_2(a_4(J_1^1), b_4(J_2^1); -\rho)]
\]

and

\[
P^* = K e^{-r T} \mathbb{E}[N_2(-a_1(J_1^1), b_1(J_2^1); -\rho)] - S_0 e^{-kS^2 T} \mathbb{E}[J_1^1 N_2(-a_2(J_1^1), b_2(J_2^1); -\rho)]
\]

\[
- \frac{S_0 A_0(1 - \alpha)}{D} e^{(r+\rho \sigma_\lambda - kS\lambda - kA\lambda)T} \mathbb{E}[J_1^2 J_2^1 N_2(-a_3(J_1^1), b_3(J_2^1); \rho)]
\]

\[
+ \frac{(1 - \alpha) K A_0}{D} e^{-kA\lambda T} \mathbb{E}[J_1^2 N_2(-a_4(J_1^1), b_4(J_2^1); \rho)],
\]

respectively, where \( N_2(\cdot, \cdot; \rho) \) is the bivariate normal distribution function with standard marginal distributions and correlation coefficient \( \rho \), and

\[
a_1(J_1^1) = \frac{\ln(S_0 J_1^1/K) + (r - kS\lambda S - \frac{1}{2} \sigma_S^2)T}{\sigma_S \sqrt{T}},
\]

\[
b_1(J_1^1) = \frac{\ln(A_0 J_1^1/D^*) + (r - kA\lambda A - \frac{1}{2} \sigma_A^2)T}{\sigma_A \sqrt{T}},
\]
Proof. See Appendix B.

In particular, if the magnitudes of the jumps follow log-normal distributions, we can simplify the pricing formulae in Theorem 2.2, as follows:

Theorem 2.3. Let

\[
Q^1 = \prod_{i=1}^{N_t} \exp \left\{ \frac{k_5 \theta}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N_t^{(1)}} \exp \left\{ \frac{k_5 \theta}{\delta_1} [e^{-\delta_1(t-T^{(1)}_i)} - 1] \right\},
\]

\[
Q^2 = \prod_{i=1}^{N_t} \exp \left\{ \frac{k_5 \theta}{\delta} [e^{-\delta(t-T_i)} - 1] \right\} \prod_{i=1}^{N_t^{(2)}} \exp \left\{ \frac{k_5 \theta}{\delta_2} [e^{-\delta_2(t-T^{(2)}_i)} - 1] \right\},
\]

and let \( \tilde{N}^{(1)}_t = N_t + N_t^{(1)} \) and \( \tilde{N}^{(2)}_t = N_t + N_t^{(2)} \). If \( \log(1 + Y_t) \sim N(\mu_1, \sigma_1) \), \( \log(1 + Z_t) \sim N(\mu_2, \sigma_2) \). Then, the values of the vulnerable European call and put options at time zero are given by

\[
C^a = S_0 e^{-k_s \lambda_s T} \mathbb{E}[Q^1 e^{(\mu_1 + \frac{1}{2} \sigma^2) \tilde{N}^{(1)}_t} N_2(a_1(Q^1_T, \tilde{N}^{(1)}_T), b_1(Q^1_T, \tilde{N}^{(2)}_T); \rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))] - Ke^{-rT} \mathbb{E}[N_2(a_2(Q^1_T, \tilde{N}^{(1)}_T), b_2(Q^1_T, \tilde{N}^{(2)}_T); \rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))]
\]

\[
+ \frac{1 - \alpha}{D} S_0 a_0 \mathbb{E}[Q^1 T^2 e^{T - k_s \lambda_s T - k_s \lambda_s T + m_1(\mu_1 + \frac{1}{2} \sigma^2) + m_2(\mu_2 + \frac{1}{2} \sigma^2) + \rho \sigma_s \sigma_T}]\]

\[
\cdot N_2(a_3(Q^1_T, \tilde{N}^{(1)}_T), b_3(Q^1_T, \tilde{N}^{(2)}_T); -\rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))] - \frac{1 - \alpha}{D} K a_0
\]

\[
\cdot \mathbb{E}[Q^2 e^{-k_s \lambda_s T + m_2 \mu_2 + \frac{1}{2} m_2 \sigma^2} N_2(a_4(Q^2_T, \tilde{N}^{(1)}_T), b_4(Q^2_T, \tilde{N}^{(2)}_T); -\rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))]
\]

and

\[
P^a = Ke^{-rT} \mathbb{E}[N_2(-a_2(Q^2_T, \tilde{N}^{(1)}_T), b_2(Q^2_T, \tilde{N}^{(2)}_T); -\rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T)) - S_0 e^{-k_s \lambda_s T}]
\]

\[
\cdot \mathbb{E}[Q^1 e^{(\mu_1 + \frac{1}{2} \sigma^2) \tilde{N}^{(1)}_t} N_2(-a_1(Q^1_T, \tilde{N}^{(1)}_T), b_1(Q^1_T, \tilde{N}^{(2)}_T); -\rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))] - \frac{1 - \alpha}{D} S_0 a_0 \mathbb{E}[Q^1 T^2 e^{T - k_s \lambda_s T - k_s \lambda_s T + m_1(\mu_1 + \frac{1}{2} \sigma^2) + m_2(\mu_2 + \frac{1}{2} \sigma^2) + \rho \sigma_s \sigma_T}]
\]

\[
\cdot N_2(-a_3(Q^1_T, \tilde{N}^{(1)}_T), b_3(Q^1_T, \tilde{N}^{(2)}_T); \rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T)] + \frac{1 - \alpha}{D} K a_0
\]

\[
\cdot \mathbb{E}[Q^2 e^{-k_s \lambda_s T + m_2 \mu_2 + \frac{1}{2} m_2 \sigma^2} N_2(-a_4(Q^2_T, \tilde{N}^{(1)}_T), b_4(Q^2_T, \tilde{N}^{(2)}_T); \rho(\tilde{N}^{(1)}_T, \tilde{N}^{(2)}_T))]
\]
respectively, where

\[
k_S = \mathbb{E}[Y_i] = e^{\mu_1 + \sigma_2^2/2} - 1, \quad k_A = \mathbb{E}[Z_i] = e^{\mu_2 + \sigma_2^2/2} - 1,
\]

\[
a_1(x, m) = \frac{\log(xS_0/K) + (r + \sigma_2^2/2 - k_S\lambda)T + m\mu_1 + m\sigma_1^2}{\sqrt{\sigma_2^2T + m\sigma_1^2}},
\]

\[
b_1(y, n) = \frac{\log(yA_0/D) + (r + \sigma_2^2/2 - k_A\lambda)T + m\mu_2 + \rho\sigma_S\sigma_A T}{\sqrt{\sigma_2^2T + n\sigma_2^2}},
\]

\[
a_2(x, m) = \frac{\log(xS_0/K) + (r - \sigma_2^2/2 - k_S\lambda)T + m\mu_1}{\sqrt{\sigma_2^2T + m\sigma_1^2}},
\]

\[
b_2(y, n) = \frac{\log(yA_0/D) + (r - \sigma_2^2/2 - k_A\lambda)T + n\mu_2}{\sqrt{\sigma_2^2T + n\sigma_2^2}},
\]

\[
a_3(x, m) = \frac{\log(xS_0/K) + (r + \sigma_2^2/2 - k_S\lambda)T + m\mu_1 + m\sigma_2^2 + \rho\sigma_S\sigma_A T}{\sqrt{\sigma_2^2T + m\sigma_1^2}},
\]

\[
b_3(y, n) = \frac{-\log(yA_0/D) + (r + \sigma_2^2/2 - k_A\lambda)T + n\mu_2 + \rho\sigma_S\sigma_A T}{\sqrt{\sigma_2^2T + n\sigma_2^2}},
\]

\[
a_4(x, m) = \frac{\log(xS_0/K) + (r - \sigma_2^2/2 - k_S\lambda)T + m\mu_1 + \rho\sigma_S\sigma_A T}{\sqrt{\sigma_2^2T + m\sigma_1^2}},
\]

\[
b_4(y, n) = \frac{-\log(yA_0/D) + (r - \sigma_2^2/2 - k_A\lambda)T + n\mu_2 + \rho\sigma_S\sigma_A T}{\sqrt{\sigma_2^2T + n\sigma_2^2}},
\]

\[
\rho(m, n) = \frac{\rho\sigma_S\sigma_A T}{\sqrt{\frac{\sigma_2^2}{T} + m\sigma_1^2}\sqrt{\frac{\sigma_2^2}{T} + n\sigma_2^2}}.
\]

Proof. Conditional on \(Q_{1T}^1\) and \(Q_{2T}^2\), the option prices can be derived by replacing \(S_0\) and \(V_0\) in Tian et al. (2014) with \(S_0Q_{1T}^1\) and \(V_0Q_{2T}^2\), respectively. Then, taking the expectations
with respect to $Q_1^1$ and $Q_2^2$ yields the unconditional values of the vulnerable European options.

Theorems 2.2 and 2.3 indicate that analytic solutions to the option prices are unavailable, irrespective of whether the amplitudes of the jumps follow log-normal distributions. To estimate the option prices, we simulate the occurrence times of jumps characterized by Hawkes processes and then calculate the expectation based on the simulated paths. In this study, we apply the thinning algorithm proposed by Ogata (1981) to simulate the Hawkes processes. Below, we briefly describe this algorithm for the Hawkes processes $N_t$ with intensity process $\lambda^*_t = \lambda_0 + \theta \sum_{i: T_i < t} e^{-\delta(t-T_i)}$.

**Thinning Algorithm**

1. Set $\lambda^* \leftarrow \lambda_0$, $n \leftarrow 1$.
2. Generate a standard uniformly distributed variate $U$ (i.e., $U \sim U(0,1)$). Let $s \leftarrow -\frac{1}{\lambda^*} \log U$. If $s \leq T$, then $t_1 \leftarrow s$; else, go to the last step.
3. Set $n \leftarrow n + 1$, $\lambda^* \leftarrow \lambda^*_{n-1} + \theta$.
4. Generate $U \sim U(0,1)$, and let $s \leftarrow s - \frac{1}{\lambda^*} \log U$. If $s \leq T$, then go to the next step; else, go to the last step.
5. Generate $V \sim U(0,1)$. If $V \leq \frac{1}{\lambda^*}$, a new event occurs at time $s$ (i.e., $t_n \leftarrow s$), in which case return to step 3; else, update $\lambda^* \leftarrow \lambda^*_n$ and return to step 4.
6. Stop the simulation. The occurrence times of the events on $[0, T]$ are $t_1, t_2, \ldots, t_n$.

### 3. The Relationships between the Proposed Model and Other Models

In this section, we demonstrate that many of the existing option and vulnerable option pricing models can be considered special cases of the model presented here. In the process, we also propose an extension of Merton (1974) model used for pricing options without counterparty risk by allowing for jump clustering. The proposed valuation method, referred to as the Merton–Hawkes model, can be used to price European options traded on organized exchanges, where the counterparty risk is essentially zero. Our demonstration is based on call options. However, similar results can be shown to be true for put options.

#### 3.1. Black–Scholes (BS) Model

If we assume no counterparty risk (i.e., $D^* = 0$), no jump risk (i.e., $\lambda_0 = \lambda^{(1)}_0 = \lambda_0^{(2)} = 0$), and no jump clustering (i.e., $\theta = \theta_1 = \theta_2 = 0$), then our model leads to the classic Black–Scholes model. In this case, the following can be shown to be true:

$$ f_T^1 = f_T^2 = 1 $$

$$ \lim_{D^* \to 0^+} N((a_1(1), b_1(1), \rho) = N(a_1(1), \infty; \rho) = N(a_1(1)), $$

$$ \lim_{D^* \to 0^+} N((a_2(1), b_2(1), \rho) = N(a_2(1), \infty; \rho) = N(a_2(1)), $$

$$ \lim_{D^* \to 0^+} N((a_3(1), b_3(1), -\rho) = N(a_3(1), -\infty; -\rho) = 0, $$

$$ \lim_{D^* \to 0^+} N((a_4(1), b_4(1), -\rho) = N(a_4(1), -\infty; -\rho) = 0. $$
Therefore, Equation (13) reduces to

\[ C^* = S_0 N(a_1(1)) - Ke^{-rT} N(a_2(1)), \]

which is exactly the Black–Scholes’s pricing formula for the European call option.

### 3.2. Merton Model

Suppose there is no counterparty risk (i.e., \( D^* = 0 \)) and no jump clustering (i.e., \( \theta = \theta_1 = \theta_2 = 0 \)). Then, the jumps are Poisson jumps and, in this case, our model leads to the Merton (1974) model with jumps. Clearly, \( N_2(a_3(J^1_T), b_3(J^2_T); -\rho) = N_2(a_4(J^1_T), b_4(J^2_T); -\rho) = 0 \). Moreover, \( N_T \) and \( N^{(1)}_T \) become independent Poisson processes, and so \( N^{(1)}_T = N_T + N^{(1)}_T \) is also a Poisson process with intensity \( \lambda_S \). Let \( X_n = \prod_{i=1}^n (1 + Y_i) \). Thus, in this case, \( J^1_T = \prod_{i=1}^{N^{(1)}_T} (1 + Y_i) \) and

\[ C^* = S_0 e^{-k_S T} \mathbb{E}[N^{(1)}_T N(a_2(J^1_T))] - Ke^{-rT} \mathbb{E}[N(a_1(J^1_T))] \]

\[ = \sum_{n=0}^{\infty} e^{-\lambda_S T} \frac{(\lambda_S T)^n}{n!} \mathbb{E}[S_0 X_n e^{-k_S T} N(a_2(X_n)) - Ke^{-rT} N(a_1(X_n))], \]

which is exactly the pricing formula in Merton (1974).

### 3.3. Klein Model

When there is no jump risk, and obviously no jump clustering risk, \( \lambda_0 = \lambda_0^{(1)} = \lambda_0^{(2)} = 0 \) and \( \theta = \theta_1 = \theta_2 = 0 \). Then, \( J^1_T = J^2_T = 1 \) and Equation (13) can be rewritten as follows:

\[ C^* = S_0 N_2(a_2(1), b_2(1); \rho) - Ke^{-rT} N_2(a_1(1), b_1(1); \rho) \]

\[ + \frac{A_0 (1 - \alpha)}{D} (S_0 e^{(r + \rho_S \sigma_S) T} N_2(a_3(1), b_3(1); -\rho) - KN_2(a_4(1), b_4(1); -\rho)), \]

which is the pricing formula for vulnerable European call options in Klein (1996).

### 3.4. Tian et al. (2014) (TWWW) Model

In the TWWW model, the jump clustering risk is ignored. When there is no jump clustering risk, \( \theta = \theta_1 = \theta_2 = 0 \) and \( Q^1_T = Q^2_T = 1 \). In this case, \( N_T, N^{(1)}_T \), and \( N^{(2)}_T \) become independent.
Poisson processes. Consequently, Equation (15) becomes

\[
C* = S_0e^{-kS \lambda S T}E[(\mu + \frac{1}{2} \sigma^2)N_T(a_1(1, \tilde{N}_T^{(1)}), b_1(1, \tilde{N}_T^{(2)}); \rho(\tilde{N}_T^{(1)}, \tilde{N}_T^{(2)}))]
\]

\[-Ke^{-rT}E[N_2(a_2(1, \tilde{N}_T^{(1)}), b_2(1, \tilde{N}_T^{(2)}); \rho(\tilde{N}_T^{(1)}, \tilde{N}_T^{(2)}))]
\]

\[+ \frac{1 - \alpha}{D}S_0A_0E[e^{rT-kS \lambda S T-k \lambda A T+m_1(\mu + \frac{1}{2} \sigma^2)+m_2(\mu + \frac{1}{2} \sigma^2)\rho \sigma \lambda T}
\]

\[\cdot N_2(a_3(1, \tilde{N}_T^{(1)}), b_3(1, \tilde{N}_T^{(2)}); -\rho(\tilde{N}_T^{(1)}, \tilde{N}_T^{(2)})) - \frac{1 - \alpha}{D}K A_0
\]

\[\cdot \mathbb{E}[Q_T^2e^{-k \lambda A T+m_2(\mu + \frac{1}{2} \sigma^2)N_2(a_4(1, \tilde{N}_T^{(1)}), b_4(1, \tilde{N}_T^{(2)}); -\rho(\tilde{N}_T^{(1)}, \tilde{N}_T^{(2)}))}
\]

\[= \sum_{n=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(\lambda_0 T)^n (\lambda_0 T)^{m_1-n} (\lambda_0 T)^{m_2-n}}{n! (m_1-n)! (m_2-n)!} S_0e^{-k \lambda S T+m_1(\mu + \frac{1}{2} \sigma^2)\rho \sigma \lambda T}
\]

\[\cdot N_2(a_1(1, m_1), b_1(1, m_2); \rho(m_1, m_2))
\]

\[\cdot Ke^{-rT}E[N_2(a_2(1, m_1), b_2(1, m_2); \rho(m_1, m_2))]
\]

\[+ \frac{1 - \alpha}{D}S_0A_0E[e^{rT-kS \lambda S T-k \lambda A T+m_1(\mu + \frac{1}{2} \sigma^2)+m_2(\mu + \frac{1}{2} \sigma^2)\rho \sigma \lambda T}
\]

\[\cdot N_2(a_3(1, m_1), b_3(1, m_2); -\rho(m_1, m_2)) - \frac{1 - \alpha}{D}K A_0
\]

\[\cdot \mathbb{E}[Q_T^2e^{-k \lambda A T+m_2(\mu + \frac{1}{2} \sigma^2)N_2(a_4(1, m_1), b_4(1, m_2); -\rho(m_1, m_2))},
\]

which is the pricing formula for vulnerable European call options under the framework of Tian et al. (2014).

### 3.5. Merton–Hawkes (M–H) Model

Suppose that the writer’s default probability is zero. This will be the case for options traded on organized exchanges, where, owing to margins, marking-to-market, and other arrangements, the default risk is essentially zero. Then, \(D^* = 0\). This model is referred to as the Merton–Hawkes model, which can be considered an extension of the Merton (1974) model, where we allow jump clustering. In this case, \(b_1(1^T) = b_2(1^T) = \infty\) and \(b_3(1^T) = b_4(1^T) = -\infty\). Furthermore, Equation (13) degenerates to

\[
C^* = S_0e^{-kS \lambda ST}E[J_T^1N(a_2(1^T))] - Ke^{-rT}E[N_2(a_1(1^T))],
\]

and (15) reduces to

\[
C^* = S_0e^{-kS \lambda ST}E[Q_T^1e^{\tilde{N}_T^{(1)}(\mu + \frac{1}{2} \sigma^2)}N(a_1(1^T), \tilde{N}_T^{(1)})] - Ke^{-rT}E[N_2(a_2(Q_T^1, \tilde{N}_T^{(1)}))].
\]

Above, we discussed how some of the existing popular models can be considered special cases of the model proposed in this study. We also presented an extension of the Merton (1974) model by allowing jump clustering. The above discussion is illustrated in Figure 1.
4. NUMERICAL ANALYSIS

In this section, we perform a sensitivity analysis and compare our model against the five models mentioned above using numerical examples. As before, we use European call options. In the numerical examples, we assume that the magnitudes of jumps are log-normally distributed. Therefore, Theorem 2.3 will be used to price the calls. The parameter values for the base case are presented in Table I.

For the model comparison, most of the parameters take the same values as those in the TWWW model and represent a typical business situation. For instance, the option is at the money and is written by a firm with a high leverage ratio, and the default barrier is equal to the value of the option writer’s debt. The value of the vulnerable European call option at time zero is 1.65 in the base case. In the model comparison and the sensitivity analyses, we change the value of only one of the parameters at a time, keeping the values of the remaining parameters unchanged from the base case. Additionally, we change the values of the basic parameters, which include the time to maturity, strike price, correlation coefficient, debt value, and deadweight cost associated with bankruptcy. Finally, in evaluating the effect of jump clustering, we match the expected numbers of jump processes. Specifically, the intensity of the Poisson process in the Merton model is taken to be \( \lambda_0/(1 - \theta/\delta) + \lambda_0^{(1)}/(1 - \theta_1/\delta_1) \), and the intensities of the Poisson processes in the TWWW model are \( \lambda_0/(1 - \theta/\delta) \), \( \lambda_0^{(1)}/(1 - \theta_1/\delta_1) \), and \( \lambda_0^{(2)}/(1 - \theta_2/\delta_2) \), respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility</td>
<td>( \sigma_0 = 0.3 )</td>
<td>Volatility</td>
<td>( \sigma_A = 0.3 )</td>
</tr>
<tr>
<td>Initial price</td>
<td>( S_0 = 10 )</td>
<td>Initial price</td>
<td>( A_0 = 10 )</td>
</tr>
<tr>
<td>Mean jump size of ( S )</td>
<td>( \mu_1 = 0 )</td>
<td>Mean jump size of ( A )</td>
<td>( \mu_2 = 0 )</td>
</tr>
<tr>
<td>S.E. of jump size</td>
<td>( \sigma_1 = 0.1 )</td>
<td>S.E. of jump size</td>
<td>( \sigma_0 = 0.1 )</td>
</tr>
<tr>
<td>Default barrier</td>
<td>( D^* = 10 )</td>
<td>Outstanding claims</td>
<td>( D = 10 )</td>
</tr>
<tr>
<td>Maturity</td>
<td>( T = 2 )</td>
<td>Risk-free interest rate</td>
<td>( r = 0.02 )</td>
</tr>
<tr>
<td>Strike</td>
<td>( K = 10 )</td>
<td>Correlation</td>
<td>( \rho = 0.5 )</td>
</tr>
<tr>
<td>Deadweight cost</td>
<td>( \alpha = 0.5 )</td>
<td>Base intensity</td>
<td>( \lambda_0 = 1 )</td>
</tr>
<tr>
<td>Jump size of intensity</td>
<td>( \theta = 1 )</td>
<td>Decay rate</td>
<td>( \delta = 2 )</td>
</tr>
<tr>
<td>Base intensity</td>
<td>( \lambda_0^{(1)} = 1 )</td>
<td>Base intensity</td>
<td>( \lambda_0^{(2)} = 1 )</td>
</tr>
<tr>
<td>Jump size of intensity</td>
<td>( \theta_1 = 1 )</td>
<td>Jump size of intensity</td>
<td>( \theta_2 = 1 )</td>
</tr>
<tr>
<td>Decay rate</td>
<td>( \delta_1 = 2 )</td>
<td>Decay rate</td>
<td>( \delta_2 = 2 )</td>
</tr>
</tbody>
</table>
Figure 2 presents the sensitivity of the option price to the time to maturity. On the whole, the option price increases as the time to maturity increases, no matter which model is applied. There are some other interesting observations that apply to many of the remaining analyses. First, among the three models without the counterparty risk, the Merton model leads to the highest call option price, followed by the M–H model, and then the BS model with the lowest value. Clearly, the jump risk increases the value of the call option. However, it is interesting to note that the jump clustering lowers the value of the option compared to the model with jump risk, without clustering for options without counterparty risk. Second, among the three models with counterparty risks, the proposed model leads to the highest call option value. Third, the call option values without counterparty risk are higher than the call option values with counterparty risk. The first two findings apply to all the cases considered here. Therefore, this feature of the findings will not be repeated in the discussion that follows.

The effect of the strike price is shown in Figure 3. The figure shows that the strike price reduces the call option value. This can also be seen from Equation (9), where it is clear that the (vulnerable) European call option value decreases with the strike price.

The effect of the correlation, $\rho$, on the call option price is exhibited in Figure 4. It is interesting to analyze the effect of the correlation on the vulnerable option value because the correlation is related to the concepts of the wrong-way risk and the right-way risk discussed in the counterparty risk literature. A negative correlation implies a wrong-way risk, where an increase in the value of the call option is associated with a higher probability of the option writer's default. Similarly, a positive correlation implies a right-way risk, where an increase in the value of the call option is associated with a lower probability of default. The above argument is based on the fact that, ceteris paribus, a positive $\rho$ implies that it is more likely for $S_T$ and $A_T$ to move in the same direction. We can see from Equation (9) that the call option value monotonically increases with the correlation. However, in order to compare the option values based on other models, we rely on the numerical results presented in Figure 4. It is interesting to note that, for correlations near $+1$, the call option value under our model...
is higher than the value based on the BS model. This implies that the negative effect of the default risk is less than the positive effect of the right-way risk for such high correlations. Furthermore, the relative values of the call option under the TWWW and Klein models depend on the correlation. For correlations near $-1$, the option value under the TWWW model is higher than the option value under the Klein model. However, this relation is reversed for correlations near $+1$.

Next, we analyze the effect of the option writer’s debt, $D$, where the default threshold is assumed to be the same as the value of the debt (i.e., $D^* = D$). The values of the debt range from $8$ (20% less than the base case) to $12$ (20% more than the base case). It is clear from Equation (9) that the value of the call option decreases with the value of debt, because
the larger the value of the debt, the higher the is the probability of default. The numerical results on the effects of debt are presented in Figure 5. For example, when $D = 8$, the call option value based on the BS model and that based on our model are approximately the same, implying that at this level of debt, the effect of counterparty risk approximately cancels the effect of jump clustering. The effect of the deadweight cost, $\alpha$, is shown in Figure 6. Clearly, it follows from Equation (9) that the call option values based on all three models that allow counterparty risk decrease with the deadweight cost. However, it is interesting to note that
our model is the most sensitive to the deadweight cost. When the deadweight cost approaches 0, the option value based on our model tends to be higher than that of the BS model. On the other hand, as \( \alpha \) approaches 1, the option values based on our model and the Klein model are very close, which implies that if the recovery rate is zero, the combined effect of the jump risk and the jump clustering risk is negligible. Finally, the BS model leads to a higher call option price than in the case of the TWWW and Klein models, for all possible values of \( \alpha \).

Figure 7 displays the effects of the base intensities of the three Hawkes processes: \( \lambda_0 \), \( \lambda_0^{(1)} \), and \( \lambda_0^{(2)} \). The common base jump intensity, \( \lambda_0 \), and the base jump intensity specific to the underlying asset, \( \lambda_0^{(1)} \), both have a positive effect on the call option value, with the latter having the higher effect. The effect of the common base jump intensity, \( \lambda_0 \), is moderated because it affects both the underlying asset value, with a positive effect on the call option value, and the value of the writer’s assets, with a negative effect on the call option value owing to the higher probability of default. However, the base jump intensity specific to the value of the option writer’s assets, \( \lambda_0^{(2)} \), has a negative effect on the call option value because of the higher probability of default.5

Finally, we analyze the effects of the intensity-based jump-size parameters (\( \theta, \theta_1 \), and \( \theta_2 \)) and the decay-rate parameters (\( \delta, \delta_1 \), and \( \delta_2 \)). Figure 8 presents the effects of these parameters. The call option value increases as \( \theta \) and \( \theta_1 \) increase, but it decreases as \( \theta_2 \) increases, as shown in the plot on the left. On the other hand, the call option value decreases as \( \delta \) and \( \delta_1 \) increase; however, it increases as \( \delta_2 \) increases, as shown in the plot on the right. In other words, the jump-clustering risk specific to the underlying asset increases the value of a call option. However, the jump-clustering risk specific to the writer’s assets decreases the option’s value. Lastly, the common jump-clustering risk, common to both the underlying asset and the writer’s assets, affects the call option value positively.6

5Actually, the impact of \( \lambda_0^{(2)} \) on the call option is complex because of the jumps, the non-linear effect, and the default being a discrete event. This is true with the remaining parameters. Therefore, our conclusion is based on the numerical analyses presented in this paper.

6Note that jump-clustering risk increases with \( \theta \) and decreases with \( \delta \).
As discussed above, the effects of different parameters on the value of the vulnerable call option vary greatly, depending on the parameter considered. The effects of the parameters considered in the proposed model are summarized in Table II.

### 5. CONCLUSION

In this study, we propose a valuation of vulnerable European options using the self-exciting Hawkes process to model the dynamics of the underlying asset value, as well as the value of the option writer’s assets. The model used here allows jump-clustering, which is consistent with what is observed in financial markets. The flexibility and generality of the proposed model are shown by establishing that the models of Black and Scholes (1973), Klein (1996), Tian et al. (2014), and Xu et al. (2012) can be considered as special cases. Therefore, the valuation presented here can be considered an extension of these models.

Using a mixture of analytic and numerical techniques, we establish many interesting relationships for the vulnerable European call option. We show that, among the models with counterparty risks, that with jump-clustering leads to the highest value for the vulnerable European call option. We also show that the value of the vulnerable European call option

### TABLE II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Effect on Option Price</th>
<th>Parameter</th>
<th>Effect on Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>+</td>
<td>$K$</td>
<td>–</td>
</tr>
<tr>
<td>$D$</td>
<td>–</td>
<td>$\rho$</td>
<td>+</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>–</td>
<td>$\lambda_0$</td>
<td>+</td>
</tr>
<tr>
<td>$\phi$</td>
<td>+</td>
<td>$\phi_0$</td>
<td>–</td>
</tr>
<tr>
<td>$\delta$</td>
<td>–</td>
<td>$\delta_1$</td>
<td>–</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>–</td>
<td>$\phi_2$</td>
<td>–</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>+</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Pricing Vulnerable Options with Jump Clustering

decreases with the level of the option writer’s debt, decreases with the bankruptcy costs, and increases with maturity.

Furthermore, we establish that an increase in the jump-clustering risk specific to the underlying asset leads to a higher value for the vulnerable European call option. However, an increase in the jump-clustering risks specific to the option writer’s assets leads to a lower value for the call option. With regard to the effect of the correlation between the two Brownian motions, where one drives the underlying asset value and the other drives the value of the option writer’s assets, we find several interesting results. The call option value increases with the correlation. When the correlation approaches 1, representing an extreme case of right-way risk, the vulnerable European call option value is even higher than the value implied by the Black and Scholes (1973) model. Even though we do not present the corresponding results with regard to the vulnerable European put options, the results can be established in a similar way.

APPENDIX A

Proof of Theorem 2.1

Let \( M = \{T_1, T_2, \ldots, T_N\} \) be the jump times of \( N_u \) during \([0, t]\) and \( M^{(1)} = \{T_1^{(1)}, T_2^{(1)}, \ldots, T_{N_1^{(1)}}^{(1)}\} \) be the jump times of \( N_u^{(1)} \). Rearranging the set \( M \cup M^{(1)} \) gives an increasing sequence, say, \( T_1', T_2', \ldots, T_L' \). Let \( T_0 = T_0^{(1)} = T_0^{(1)} = 0 \). Thus, for \( i = 1, \ldots, L = N_t + N_1^{(1)},^7 \) we have

\[
\Delta S_{T_i'} = S_{T_i'} - S_{T_i'} = Y_i S_{T_i'} - Y_i S_{T_i'}
\]

and between two jump times, \( T_i' < u < T_i'^{+} \), \( S_t \) evolves continuously. Specifically, for \( T_i' < u < T_i'^{+} \), \( i = 0, 1, \ldots, L - 1, \)

\[
\frac{dS_u}{S_u} = \left[ r - kS \left( \lambda_S + \theta \sum_{T_j < u} e^{-\delta(u-T_j)} + \theta_1 \sum_{T_k^{(1)} < u} e^{-\delta_1(u-T_k^{(1)})} \right) \right] du + \sigma_S dW_u^{(1)},
\]

which is a one-dimensional linear stochastic differential equation. Hence, for \( T_i' < u < T_i'^{+} \),

\[
S_u = S_{T_i'} \exp \left\{ \sigma_S (W_u^{(1)} - W_{T_i'}^{(1)}) + \left( r - kS \lambda_S - \frac{1}{2} \sigma_S^2 \right) (u - T_i') \right. \\
+ \frac{kS \theta}{\delta} \sum_{j: T_j \leq T_i'} \left[ e^{-\delta(u-T_j)} - e^{-\delta(T_i'-T_j)} \right] + \frac{kS \theta_1}{\delta_1} \sum_{k: T_k^{(1)} \leq T_i'} \left[ e^{-\delta_1(u-T_k^{(1)})} - e^{-\delta_1(T_i'-T_k^{(1)})} \right] \left\}
\]

(A2)

7Because the point processes \( N_t \) and \( N_1^{(1)} \) are independent of each other, they will not jump simultaneously; that is, \( M \) and \( M^{(1)} \) are almost surely disjoint.
Thus, it follows from (A1) and (A2) that, for $i = 0, 1, \ldots, L - 1$,

$$S_{T_{i+1}'} = S_{T_i'} + \Delta S_{T_{i+1}'},$$

and

$$S(T_i')(1 + Y_{i+1}) \exp \left\{ \sigma_S(W_{i+1}^{(1)} - W_i^{(1)}) + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2)(T_{i+1}' - T_i') \right\}$$

$$+ \frac{k_S \theta}{\delta} \sum_{j:T_j \leq T_i'} \left[ e^{-\delta(T_{i+1}' - T_j)} - e^{-\delta(T_i' - T_j)} \right]$$

$$+ \frac{k_S \theta_1}{\delta_1} \sum_{k: T_k^{(1)} \leq T_i'} \left[ e^{-\delta_1(T_{i+1}' - T_k^{(1)})} - e^{-\delta_1(T_i' - T_k^{(1)})} \right].$$

In addition, it is clear that

$$S(T_i') = S(0)(1 + Y_1) \exp \left\{ \sigma S W_{i+1}^{(1)} + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2) T_i' \right\},$$

and

$$\frac{S(t)}{S(T_L)} = \exp \left\{ \sigma S (W_{i+1}^{(1)} - W_{i+1}^{(1)}) + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2) (t - T_L') \right\}$$

$$+ \frac{k_S \theta}{\delta} \sum_{j=1}^{N_T} \left[ e^{-\delta(t - T_j)} - e^{-\delta(T_L' - T_j)} \right] + \frac{k_S \theta_1}{\delta_1} \sum_{k=1}^{N_T^{(1)}} \left[ e^{-\delta_1(T_{i+1}' - T_k^{(1)})} - e^{-\delta_1(T_i' - T_k^{(1)})} \right].$$

Therefore, combining (A3), (A4), and (A5), we have

$$S(t) = S(T_1') \frac{S(T_2')}{S(T_1')} \cdots \frac{S(T_L)}{S(T_{L-1})} \frac{S(t)}{S(T_L)}$$

$$= S(0) \exp \left\{ \sigma S W_{i+1}^{(1)} + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2) t \right\} \prod_{i=1}^{L-1} (1 + Y_i)$$

$$\cdot \exp \left\{ \frac{k_S \theta}{\delta} \sum_{j=1}^{L-1} \sum_{j: T_j \leq T_i'} \left[ e^{-\delta(t - T_j)} - e^{-\delta(T_L' - T_j)} \right] + \sum_{j=1}^{N_T} \left[ e^{-\delta(t - T_j)} - e^{-\delta(T_L' - T_j)} \right] \right\}$$

$$\cdot \exp \left\{ \frac{k_S \theta_1}{\delta_1} \sum_{j=1}^{L-1} \sum_{k: T_k^{(1)} \leq T_i'} \left[ e^{-\delta_1(t - T_k^{(1)})} - e^{-\delta_1(T_L' - T_k^{(1)})} \right] + \sum_{k=1}^{N_T^{(1)}} \left[ e^{-\delta_1(t - T_k^{(1)})} - e^{-\delta_1(T_L' - T_k^{(1)})} \right] \right\}$$

$$= S_0 \exp \left\{ \sigma S W_{i+1}^{(1)} + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2) t \right\} \prod_{i=1}^{N_T^{(1)}} (1 + Y_i) \prod_{i=1}^{N_T^{(1)}} \exp \left\{ \frac{k_S \theta}{\delta} \left[ e^{-\delta(t - T_i)} - 1 \right] \right\}$$

$$\cdot \prod_{i=1}^{N_T^{(1)}} \exp \left\{ \frac{k_S \theta_1}{\delta_1} \left[ e^{-\delta_1(t - T_k^{(1)})} - 1 \right] \right\}$$

$$= S_0 \exp \left\{ \sigma S W_{i+1}^{(1)} + (r - k_S \lambda S - \frac{1}{2} \sigma_S^2) t \right\} f_{1(T_i')}. $$

Using the same method and steps, we can prove (12).
Appendix B

Proof of Theorem 2.2

Let

$$X = \frac{\ln(S_0 J_1^T) + (r - k_2 \lambda_2^{\frac{1}{2}} \sigma_2^2) T}{\sigma_2 \sqrt{T}},$$

$$Y = \frac{\ln(S_0 J_2^T) + (r - k_2 \lambda_2^{\frac{1}{2}} \sigma_2^2) T}{\sigma_2 \sqrt{T}}.$$ 

Given $J_1^T$ and $J_2^T$, $(X, Y)$ follows standard bivariate Gaussian distribution with correlation coefficient $\rho$. Since $(S_T - K)^+ = (S_T - K)\mathbb{1}_{[S_T \geq K]}$,

$$e^{\rho T} C^* = \mathbb{E}[(S_T - K)\mathbb{1}_{[S_T \geq K, T \geq D^*]}]$$

$$+ \frac{1 - \alpha}{D} \mathbb{E}[A_T (S_T - K)\mathbb{1}_{[S_T \geq K, T < D^*]}]$$

$$= \mathbb{E}[P_1(J_1^T, J_2^T)] - \mathbb{E}[P_2(J_1^T, J_2^T)]$$

$$+ \frac{1 - \alpha}{D} \mathbb{E}[P_3(J_1^T, J_2^T)] - \frac{K(1 - \alpha)}{D} \mathbb{E}[P_2(J_1^T, J_2^T)] \quad (B1)$$

where

$$P_1(J_1^T, J_2^T) = \mathbb{E}[S_T \mathbb{1}_{[S_T \geq K, T \geq D^*]} | J_1^T, J_2^T],$$

$$P_2(J_1^T, J_2^T) = \mathbb{E}[K \mathbb{1}_{[S_T \geq K, T \geq D^*]} | J_1^T, J_2^T],$$

$$P_3(J_1^T, J_2^T) = \mathbb{E}[S_T A_T \mathbb{1}_{[S_T \geq K, T \geq D^*]} | J_1^T, J_2^T],$$

$$P_4(J_1^T, J_2^T) = \mathbb{E}[A_T \mathbb{1}_{[S_T \geq K, T > D^*]} | J_1^T, J_2^T].$$

On the other hand,

$$P_1(J_1^T, J_2^T) = S_0 J_1^T e^{(r - \frac{1}{2} \sigma_2^2 - k_2 \lambda_2^{\frac{1}{2}}) T} \mathbb{E}[e^{-\sigma_2 \sqrt{T} X} \mathbb{1}_{[X < a_1(J_1^T), Y < b_1(J_2^T)]}]$$

$$= S_0 J_1^T e^{(r - \frac{1}{2} \sigma_2^2 - k_2 \lambda_2^{\frac{1}{2}}) T} \int_{-\infty}^{a_1(J_1^T)} \int_{-\infty}^{b_1(J_2^T)} \frac{e^{a_2(J_1^T)})}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ - \frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right\} dxdy$$

$$= S_0 J_1^T e^{(r - \frac{1}{2} \sigma_2^2 - k_2 \lambda_2^{\frac{1}{2}}) T} \int_{-\infty}^{a_1(J_1^T)} \int_{-\infty}^{b_1(J_2^T)} \frac{e^{a_2^2(J_1^T)}}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} (x + \sigma_2 \sqrt{T}y)^2 - 2\rho(x + \sigma_2 \sqrt{T})y + (y + \rho_2 \sigma_2 \sqrt{T})^2 \right\} dxdy$$

$$= S_0 J_1^T e^{(r - k_2 \lambda_2^{\frac{1}{2}}) T} N_2(a_2(J_1^T), b_2(J_2^T); \rho).$$

Similarly, we have

$$P_2(J_1^T, J_2^T) = \mathbb{E}[K \mathbb{1}_{[X < a_1(J_1^T), Y < b_1(J_2^T)]}]$$

$$= KN_2(a_1(J_1^T), b_1(J_2^T); \rho).$$
Since \((X, Y)\) follows a standard bivariate normal distribution with correlation \(\rho\), \((X, -Y)\) follows a standard bivariate normal distribution with correlation \(-\rho\). Let \(Z = -Y\), then

\[
P_3(J_1^T, J_2^T) = \prod_{i=1}^{2} S_{0} A_{0} J_{i} e^{(r - \frac{1}{2} \sigma_{i}^2 - \beta_{0} - \beta_{i} \lambda_{0})T_i} \cdot \mathbb{E}[e^{-\sqrt{T} \sigma_1 X + \sigma_2 Y}]_{[X < a_1 (J_1^T), -Y < -b_1 (J_1^T)]} \]

\[
= \prod_{i=1}^{2} S_{0} A_{0} J_{i} e^{(r - \frac{1}{2} \sigma_{i}^2 - \beta_{0} - \beta_{i} \lambda_{0})T_i} \cdot \int_{-\infty}^{a_1 (J_1^T)} \int_{-\infty}^{-b_1 (J_1^T)} e^{-\sqrt{T} \sigma_1 x + \sigma_2 y} \frac{e^{-\frac{x^2}{2(1 - \rho^2)}}}{\frac{2\pi \sqrt{1 - \rho^2}}} \exp \left\{ - \frac{x^2 + 2\rho xy + y^2}{2(1 - \rho^2)} \right\} \, dx \, dy \]

\[
= \prod_{i=1}^{2} S_{0} A_{0} J_{i} e^{(r - \frac{1}{2} \sigma_{i}^2 - \beta_{0} - \beta_{i} \lambda_{0})T_i} \cdot \int_{-\infty}^{a_1 (J_1^T)} \int_{-\infty}^{-b_1 (J_1^T)} e^{\frac{\sigma_1^2 + 2\rho \sigma_1 \sigma_2 + \sigma_2^2}{2\pi \sqrt{1 - \rho^2}}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} \left( x + \frac{\sigma_2}{\sigma_1} \sqrt{T} \right)^2 \right\} \, dx \, dy \]

\[
= S_{0} A_{0} J_{1} J_{2} e^{2(x + \rho y \sigma_1 \sqrt{T} + \rho \sigma_2 \sqrt{T})} N_2(a_1 (J_1^T), b_1 (J_1^T); -\rho).
\]

Lastly,

\[
P_4(J_1^T, J_2^T) = A_{0} J_{1} J_{2} e^{(r - \frac{1}{2} \sigma_{1}^2 - k_{1} \lambda_{1})T_1} \mathbb{E}[e^{-\sigma Y}]_{[X < a_1 (J_1^T), -Y < -b_1 (J_1^T)]} \]

\[
= A_{0} J_{1} J_{2} e^{(r - \frac{1}{2} \sigma_{1}^2 - k_{1} \lambda_{1})T_1} \int_{-\infty}^{a_1 (J_1^T)} \int_{-\infty}^{-b_1 (J_1^T)} e^{\frac{\sigma_1 Y}{\sigma_1}} \frac{e^{T \sigma_1 Y}}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ - \frac{x^2 + 2\rho xy + y^2}{2(1 - \rho^2)} \right\} \, dx \, dy \]

\[
= A_{0} J_{1} J_{2} e^{(r - \frac{1}{2} \sigma_{1}^2 - k_{1} \lambda_{1})T_1} \int_{-\infty}^{a_1 (J_1^T)} \int_{-\infty}^{-b_1 (J_1^T)} e^{\frac{\sigma_1^2 Y^2}{2\pi \sqrt{1 - \rho^2}}} \exp \left\{ - \frac{1}{2(1 - \rho^2)} \left( x + \frac{\sigma_2}{\sigma_1} \sqrt{T} \right)^2 \right\} \, dx \, dy \]

\[
= A_{0} J_{1} J_{2} e^{(r - k_{1} \lambda_{1})T_1} N_2(a_1 (J_1^T), b_1 (J_1^T); -\rho).
\]

Therefore,

\[
C^* = S_{0} e^{-k \lambda_{1}T} E[J_{1}^T N_2(a_2 (J_1^T), b_2 (J_1^T); \rho)] - K e^{-r T} E[N_2(a_1 (J_1^T), b_1 (J_1^T); \rho)]
\]

\[
+ \frac{S_{0} A_{0} (1 - \alpha)}{D} e^{(r + \rho \sigma_{1} \sigma_{2} - k \lambda_{1})T} E[J_{1}^T J_{2}^T N_2(a_3 (J_1^T), b_3 (J_1^T); -\rho)]
\]

\[
- \frac{K A_{0} (1 - \alpha)}{D} e^{-k \lambda_{1}T} E[J_{1}^T N_2(a_4 (J_1^T), b_4 (J_1^T); -\rho)].
\]

The formula for a put option can be proved in a similar way.
REFERENCES


