Shape optimization of conical hoppers to increase mass discharging rate

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Abstract

Mass discharging rate (MDR) is a critical aspect of hopper’s performance in bulk solids handling. A shape optimization method is established in this study to increase the MDR of cohesionless granular materials from hoppers. This method is based on a continuum model of granular matter and the Eulerian Finite Element Method (FEM) which can efficiently simulate the discharging process and predict the MDR. In this work with the focus on conical hoppers, the widths of silo and hopper outlet as well as the vertical height of hopper are fixed. The meridian of the hopper, however, evolves from a straight line to some optimal curve, guided by a combined genetic algorithm (GA) and gradient descent method (GDM). Cubic spline function is employed to parametrize the hopper shape. The effectiveness of the shape optimization is examined by comparing the MDRs of the optimal hopper and conventional conical hopper, obtained by both FEM and discrete element method (DEM) respectively. It is shown that this shape optimization method can automatically search the optimal shape of the hopper in a given range of constraints, and increase the MDR substantially. In a typical hopper with an initial half angle of 45°, the MDR is increased by over 130% after the shape optimization. Notably, the optimal shape depends mainly on the geometrical factors, i.e. the allowed width and height for the hopper, whilst insensitive to the material properties, which favors its general use for different particles. Such curved hoppers are particularly useful for increasing the discharge rate of hoppers ranging from 30° to 50°, which, facilitated with advanced manufacturing technology, will find wide potential applications in bulk solids handling.

Keywords: hopper design; shape optimization; granular materials; mass discharging rate; finite element method
1. Introduction

Hoppers are widely used in process engineering to handle granular materials such as food grains in agricultural industry, mineral powders in mining industry and bulk chemicals in pharmaceutical industry. Mass discharge rate (MDR) of the bulk solids is one of the critical concerns when evaluating the performance of a hopper [1]. How to achieve a desired value of MDR has the top priority in hopper design. For instance, in additive manufacturing where the metal powders need to be spread evenly via a hopper, a steady and precisely controlled MDR is indispensable for preparing the powder bed [2].

Numerous researches have been conducted in the past several decades to determine the MDR in various hoppers [3-7]. The Beverloo equation is widely accepted for evaluating the discharging rate of flat-bottomed cylindrical silos, where the MDR varies with the diameter of the outlet raised to 5/2 [8]. This equation was later extended to consider the hoppers with various slope angles [9, 10]. In addition to the hopper geometry, the material properties such as frictional coefficient are also an important influential factor of MDR. The so-called hourglass theories, developed based on the plastic limit analyses of granular materials, can account for the effects of material properties, especially when the hoppers are steep [11, 12]. Some silo standards have been established based on past research and experiments to give a general guide on hopper design, including the British Standard EN 1991-4:2006, Australian standard AS 3774-1996 and American Concrete Institute ACI 313-97, which are most commonly used worldwide. Unfortunately, the state-of-art design codes only cover a few hopper geometries such as symmetric single cone, square pyramid, wedge with vertical end walls, etc. [13]. Many types of hoppers in practical use are not included in the standard, for example, the transition, chisel, non-symmetric pyramid, and the expanded-flow hoppers [14, 15].

Most of the existing hoppers have fixed slope angles or joint of several slopes. To improve the MDR, two ways are generally most effective: enlarging the size of opening or steepening the hopper angle. In practices, the operational environment sometimes restricts the freedom of adjusting hopper outlet. For example, in a bulk materials port, the width of the silo should be sufficiently large to store bulk materials, while the outlet should be appropriately small to fit the size of a truck during unloading [16]. With these constraints, how to obtain the optimal hopper design to fulfil the practical need should be answered. Modulating the slope angle seems to be one of the options at hand. For example, by combing two different slopes –
typically a steep slope near the outlet and a gentle slope elsewhere – the hopper can create a much wider mass-flow zone within a tight geometrical constraint [14, 15]. Apparently, a transiting shape of hopper would be favorable for the discharge process. But what exactly is the optimal shape with the maximum discharge rate has no answer yet.

In aerospace and automotive engineering, shape optimization is widely applied in the design of aerodynamic shapes such as aircraft wing design and vehicle design [17, 18]. It is prosperous with the development of numerical models such as computational fluid dynamics (CFD). With such models at hand, we can then use the intelligent optimization methods such as Gradient Descent Method (GDM), Simulate Anneal Arithmetic (SAA), Genetic Algorithm (GA), and Particle Swarm Optimization (PSO), et al., to find the maximum or minimum value of an objective variable under given conditions.

In granular systems, such studies of intelligent optimization are by far very limited, mainly because of the obstacles related to the physical model of granular matter. At present, there exist two types of models for granular matter: discrete models and continuum models. The discrete element method (DEM) proposed by Cundall and Strack [19] simulated the movement and interacting forces of microscopic particles based on the Newtonian law. It is overall a credible first-principle method, but its computational efficiency is too low to be used in shape optimization which could involve tens to hundreds of iterations. Some parametric studies have been conducted previously in silos/hoppers using DEM [6, 20-22], which are valuable to find a relatively better hopper design. Nonetheless, such studies only examine the values of MDR in part of the parameter space, e.g. considering the variation of one parameter at a time while others are fixed. It remains challenging to find the optimal across a continuous space of parameters using DEM. The continuum model uses a constitutive relationship to describe the bulk materials without going to details of individual particles, and is therefore computationally efficient. Recently, it is found that some continuum models are able to capture the salient characteristic of hopper discharge, such as the Beverloo scaling, the invariance of MDR with hopper height, and the variation with hopper slope angle [23-25]. Therefore, it is expected that such continuum models can be applied intelligently to find some interesting hopper shapes in a wide space of parameters that has not be explored before.

In this study, a shape optimization is performed on an initially conical hopper with a focus on achieving the maximum MDR. Other concerns of hopper design such as the flow patterns
and wall pressures are temporarily left aside in this work. A combined GA and GDM optimization method is employed to efficiently search the optimal shape. The Eulerian FEM approach is selected to calculate the MDR of hopper during each iteration of shape optimization. The final outcome of optimization is assessed by comparing with the DEM simulation results. The sensitivity of the optimal shape to different geometrical conditions and material properties is also investigated.

2. Numerical models

In this work, granular materials are modelled by both FEM and DEM. The DEM model is generally considered credible in physics but it is inappropriate for the current optimization due to the huge demands of computational resources. The FEM model is sufficiently efficient for the iterative optimization, whose results however require careful examinations. Consequently, the continuum model is mainly used for optimization while the DEM model serves as a benchmark for assessing the results. The details of the two models have been elaborated in literature. Thus only a brief introduction of the models is presented below.

2.1 FEM model

The FEM model is based on the continuum assumption of granular materials. This assumption is usually acceptable in situations where the size of particles is much smaller than hopper outlet and the subjects of interest are macroscopic quantities, e.g. MDR, that not much sensitive to microscale particle characteristics. Recent studies demonstrate that the methods built on this continuum assumption such as the Arbitrary-Lagrangian-Eulerian method (ALE), Eulerian FEM and hydrodynamic method, SPH, et al. can give a satisfactory prediction of the hopper discharge. The Eulerian FEM is adopted here in consistency with the previous work [26]. The equations of conservation of mass, momentum, and energy, as written below:

Mass conservation,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]  

(1)

Momentum conservation,
\[
\frac{\partial \rho \textbf{v}}{\partial t} + \nabla \cdot (\rho \textbf{v} \otimes \textbf{v}) = \nabla \cdot \sigma + \rho \textbf{b}
\] (2)

and energy conservation,
\[
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \textbf{v}) = \sigma : \dot{\varepsilon}
\] (3)

Where \( \rho \) is the bulk density of granular material, \( \textbf{v} \) is the velocity vector, \( \sigma \) refers to the Cauchy stress tensor, \( \textbf{b} \) is the body force, and \( \varepsilon \) is the internal energy per unit volume. \( \dot{\varepsilon} = (1/2)(\nabla \textbf{v} + (\nabla \textbf{v})^T) \), which represents the strain rate.

In continuum models, the constitutive equation of a granular material is required to describe the behaviors. Although much work needs to be done in this regard, it was found that the elastoplastic models can describe the behaviors of granular material in sandpile [27] and hoppers [26]. The Mohr-Coulomb elastoplastic model is employed in this work, which consists of a linear isotropic elastic law, a yield criterion and a plastic potential \( G \) to determine the granular flow after yielding.

The linear elastic law depends on Young’s modulus \( E \) and Poisson’s ratio \( \nu \), written as
\[
\sigma_{ij} = D_{ijkl}^{el} \varepsilon_{kl}^{el}
\] (4)

where \( \sigma_{ij} \) represents the components of Cauchy stress; \( \varepsilon_{kl}^{el} \) is elastic strain; and \( D_{ijkl}^{el} \) refers to the fourth-order tensor of elasticity.

The yield condition is:
\[
R_{mc} q - p \tan \varphi - c = 0
\] (5)

where
\[
R_{mc} = \frac{1}{\sqrt{3} \cos \varphi} \sin \left( \theta + \frac{\pi}{3} \right) + \frac{1}{3} \cos(\theta + \frac{\pi}{3}) \tan \varphi
\] (6)

Here, \( q = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \), which is the Mises stress; \( p = -\frac{1}{3} \text{trace}(\sigma_{ij}) \), which represents the isotropic pressure; and \( S_{ij} \) is the deviatoric stress; \( \varphi \) is the internal friction angle, which indicates the slope of the yield surface in \( p - R_{mc} q \) stress plane; \( c \) is the cohesion of the
granular material, which is set zero in this study as we focus on cohesionless granular materials; \( \theta \) is the deviatoric polar angle defined as \( \cos(3\theta) = (r/q)^3 \), where \( r = \left( \frac{9}{2} S_{ji} S_{ik} S_{kl} \right)^{\frac{1}{3}} \) is an invariant measure of deviatoric stress.

The elastic law that determines the flow potential \( G \) is defined as:

\[
G = \sqrt{(\xi \tan \psi)^2 + (R_{mw}q)^2} - ptan\psi
\]  

\[
R_{mw} = \frac{4(1-e^2)\cos^2 \theta + (2e-1)^2}{2(1-e^2)\cos \theta + (2e-1)\sqrt{4(1-e^2)\cos^2 \theta + 5e^2 - 4e}} \times \frac{3 - \sin \varphi}{6\cos \varphi}
\]  

Where \( \xi \) is a parameter that characterizes the eccentricity of the flow potential; \( \psi \) is the dilatancy angle of the bulk material; and \( e \) refers to the deviatoric eccentricity, defined as \( e = (3 - \sin \varphi)/(3 + \sin \varphi) \). Table 2 shows the dimensions of the hopper and the elastic properties of the granular material as a case study. The elasticity is not so relevant to the granular flow, and can take some typical values without fear of losing accuracy [28]. The plastic properties are determined from a shear test and related to the microscopic particle properties used in DEM, as discussed in next subsection.

Fig. 1 (a) shows the implementation of the FEM model for simulating hopper flow, which is similar to previous studies [25, 28, 29]. To reduce the computational cost, only a 90° portion of the revolving silo is considered, and the symmetric boundary conditions are defined on the corresponding symmetric planes. The hopper structure is defined as a rigid shell in FEM. The granular material is modelled by using Eulerian elements. Their interaction is handled using an immersed boundary method. On the top surface of the computational domain, a void inflow constraint is defined to avoid re-filling. A free out-flow constraint is defined at the bottom outlet of the hopper. The granular material is discharged under gravity from the hopper whose position is fixed. The discharging process lasts for a physical time of 1 s to reach a steady MDR. Doing so can dramatically reduce the computational cost from 10.5h (if the hopper needs to be emptied) to 20 minutes.

2.2 DEM model

The DEM developed by Cundall and Strack [30] is a credible first-principle approach to simulate granular materials. The advantage of this method lies in the realistic consideration of the mobilization of microscopic particles based on Newtonian Law and the interactions...
between particles or between particles and surrounding media. Unlike continuum models, it requires no assumptions on the constitutive relationship. Briefly, the governing equations of DEM are expressed as

\[
m_i \frac{dV_i}{dt} = \sum_j F_{ij} + m_i g
\]

\[
l_i \frac{d\omega_i}{dt} = \sum_j M_{ij}
\]

where \(m_i, l_i\) are respectively the mass and rotational inertia of particle \(i\). \(V_i\) and \(\omega_i\) are the translational and rotational velocities. \(g\) is the gravity acceleration. \(F_{ij}\) and \(M_{ij}\) denote respectively the force and torque on particle \(i\) resulting from neighbouring particle \(j\).

In DEM, the interaction between particles is often calculated by some established contact force models, which include an elastic part derived from the classic Hertzian-Mindlin theory [31], and a viscous part to account for the energy dissipation during contacts/collisions. The damping coefficient used is a dimensionless fraction of the critical damping, and has the same value in normal and tangential directions. The DEM code provided in Abaqus is used in this work, whose details have been described elsewhere [32].

As noted, the DEM is adopted here mainly for examining the outcome of FEM optimization, so it is important to carefully correlate the DEM and FEM parameters. This is often done through a shear test, where the particles are packed in a cell and sheared to examine its macroscopic response, from which the internal friction angle and wall friction can be identified. In this work, the DEM and FEM parameters are taken from [33], which were shown to generate consistent results of MDR in conical hoppers [33]. Table 1 shows the parameters of macro internal friction angle and wall friction used in FEM. The corresponding micro parameters used in DEM include particle-particle friction 0.3, particle wall friction 0.5, and particle diameter 4 mm. Note that the parameter of cohesion \(c=0\) in FEM, and thus no adhesive forces like liquid bridge and electrostatic force were considered in DEM. The discharge of cohesive particles is much more complicated, possibly leading to intermittent avalanche, cohesive arching and the so-called piping flow. In these situations, the definition of MDR is uncertain since the flow is unstable. Therefore, to avoid such uncertainty in research, only the cohesionless granular materials will be focused on in this study.
3. Optimization process and algorithm

3.1 Parametrization of hopper shape

To facilitate the optimization, the hopper shape needs to be represented by a mathematical equation of parameters. Fig. 1 (b) shows the geometrical parameter of a typical conical hopper. \( D \) is the diameter of the inlet, \( D_0 \) is the diameter of the outlet and \( H \) is the height of hopper measured from the inlet to the outlet. These three parameters also determine the slope angle of hopper, i.e. \( \alpha \). If the hopper is conical, this angle can represent the hopper’s half-angle.

The hopper shape in optimization is generated by revolving a cubic spline. The cubic spline is defined by only three points, i.e. A, B, C as shown in Fig. 2, in order to save computational resources. Point A and B are fixed to constrain the size of in-flow and out-flow surface. Point C, initially the mid-point between A and B, moves freely in the rectangular regime bounded by A and B. The coordinate position \((x, y)\) of point C are the two design variables mathematically. In addition to the boundary condition imposed on C \((x, y)\), we also limit the slope in each point of the spline to be positive to avoid any odd shape of the spline [34].

In principle, the cubic spline function to represent the hopper shape is bounded by three boundary conditions. Using the coordinates A \((a, F_1(a))\), B \((b, F_2(b))\), C \((x, y)\), we have:

i) The curve passes all the three points A, B and C, which means \( F(a) = F_1(a) \); \( F(b) = F_2(b) \); \( F(x) = F_2(x) = y \).

ii) The curve is continuous and smooth in each point, which means \( F(x-0) = F(x+0) \); \( F'(x-0) = F'(x+0) \); \( F''(x-0) = F''(x+0) \).

iii) The natural boundary conditions at two endpoints A and B, which means \( F''(a) = c \); \( F''(b) = d \), where \( c \) and \( d \) are constant, they can be the same or different.

3.2 Genetic Algorithm (GA)

In this study, a genetic algorithm was employed at first to sweep the space of parameters so as to get an overall trend of MDR with the hopper shape parameterized as \((x, y)\). Genetic algorithm is a typical heuristic algorithm widely used in the aerodynamic shape optimization of the airplane, high-speed train and ship hull, etc. [35]. It plays an important role in those situations where the optimal design is unknown and associated with high uncertainty since
the heuristic algorithm has a better variability to avoid local optima [36].

The process to implement the genetic algorithm is elaborated in Fig. 3. This iterative algorithm is analogous with the rule of natural selection. Firstly, the inputted design variables are coded into binary numbers as genetic information. This genetic information is carried by the samples of the first generation which is evaluated according to the result of objective MDR. Secondly, following the principle of evolution, the best results are selected as the survivals. Then, two operations, crossover and mutation, are conducted to exchange genetic information and generate children samples with increased gene diversity. A brief illustration of these two operations is shown in Fig. 4. The crossover and mutation, which normally happen together, contribute to a selective diversity of design variables - the key feature of GA to avoid local optima and approximately reach the global optimum. Subsequently, the children samples, as a new parent, will form the next generation and repeat this iteration until the total number of iterations reaches 50, which is set as the stop criterion in the current GA, since we will refine the optimal results with GDM later.

3.3 Gradient Descent Method (GDM)

GDM is a gradient-based iterative optimization algorithm. Unlike the genetic algorithm which is heuristic and computational expansive, GDM normally converges to optima more efficiently [37]. This feature offers the feasibility of the large-scale optimization. Consequently, after obtaining an approximate distribution of optima using GA, GDM is performed to accurately find the optimal shape for the hopper with several iterations. Each iteration follows the steps below [38]:

i) Find the negative gradient $d_k$ in the initial point, $k$ indicates the number of iterations.

$$d_k = - \nabla F(X_k)$$  \hspace{1cm} (11)

ii) Backtracking line search in the negative gradient direction,

$$X_{k+1} = X_k + \alpha_k \cdot d_k$$  \hspace{1cm} (12)

Where $\alpha_k$ is the searching step length, determined through the iteration loop for $i=0$

$$\alpha_{k,i} = r \cdot a^i \hspace{0.5cm} (0 < a < 1)$$  \hspace{1cm} (13)

$r$ is the initial value of the step length, which is $\alpha_{k,0}$ when $i=0$; $a$ is the penalty factor.
If the following Armijo-Goldstein rule is satisfied [39]

\[ F(X_k) + (1 - \zeta)\alpha_{k,i}d_k \leq F(X_k + \alpha_{k,i}d_k) \]  
(14)

\[ F(X_k) + \zeta\alpha_{k,i}d_k \geq F(X_k + \alpha_{k,i}d_k) \], where \( 0 < \zeta < 0.5 \)  
(15)

Then \( \alpha_k = \alpha_{k,i} \) and exit loop

Else \( i = i + 1 \) \((i \leq 20)\)

iii) Cycle to next iteration and repeat the process above start with \( X_{k+1} \)

The searching step length and the range of Armijo-Goldstein rule, represented by \( r, a \) and \( \zeta \) in Eqs. (13), (14) and (15), are critical to the speed of optimization and the avoidance of local optima. Their values are evaluated according to the heuristic results of GA, i.e. Fig. 5. The detailed process are described in Section 4.1. The whole optimization process will be ceased if no better design is found in an iteration after 20 searching trials have been conducted, in which case the searching step length would decrease to \( \alpha_{k,20} = r \ast a^{20} \) with a change position of point C in a distance less than \( 10^{-6} \) m. The design from the previous iteration is the final optimal design and the previous iteration is the final iteration.

4. Results and discussion

4.1 A case study of the optimal hopper and its MDR increase

We first discuss the approximate distribution of MDR optima obtained by GA. As mentioned earlier, the design variables in the present optimization are the coordinates of point C (x, y) in the cubic spline. Starting from the mid-point of A and B, the coordinates (x, y) are inputted to construct the shape of hopper in Eulerian FEM via an interface coded with Python. Based on the FEM results of MDR, the smart GA will provide the likely better values of (x, y) for the next attempt. Note that as a heuristic optimization method, GA may not give literally the best result at the final attempt (50th iteration). We therefore plot the MDR obtained in each iteration and the corresponding coordinates (x, y) to observe the general trend, as shown in Fig. 5. From this figure, it can be seen that the flow rate increases as point C moves towards the top left of the design area, which physically indicates that the initial conical shape of hopper gradually evolves to an inward concave surface. The red color suggests the possible locations of the best point C, which corresponds to the globally optimal shape of hopper. There also exist some local optima in this region which will be discussed as well in this work.
Note that the red area in Fig. 5 remains too coarse to determine the best shape of hopper. The heuristic GA method is costive to further refine the result. By reference to Fig. 5, a GDM is then constructed to overcome this drawback, whose details are described in Eqs. (11)-(15). Therein, the key issues to be determined are the searching step length and the range of Armijo-Goldstein rule, represented by $r$, $a$ and $\zeta$ in Eqs. (13), (14) and (15), since they are critical to the speed of optimization and the avoidance of local optima. Their values are determined wisely by resorting to GA again. Differ to the previous GA application, $r$, $a$ and $\zeta$ are the new optimization variables and the objective is to construct a GDM with these three parameters, which emerge to the red area in Fig. 5 with the least number of iteration. This operation is done only once for the case study, yielding that $r = 0.43D_o$, $a = 0.68$ and $\zeta = 0.21$, which are found to apply in all other cases as well. The performances of the GDM and GA are compared in Table 2. Judged from the optimal location of $C$, both methods indicate that the concave shape is the globally best design for hopper MDR. Nonetheless, there still exist noticeable differences between the two locations, and between the two MDRs, which are hard to overcome by GA alone. As seen from the computation time, the cost of GA is almost ten times of that of GDM in the current problem. The difficulty could be further increased if we pursue the accuracy of optima by using GA. The combination of GA and GDM would be a good choice for the studied problem, which has complicated dependences on boundary and materials but involves no randomness since the granular materials are described by mathematical governing equations and constitutive relationships. Similar methods have been used in previous structural optimizations [40].

Fig. 6 shows the step-by-step evolution from the initial conical shape to the final optimal shape from GDM algorithm. It can be seen that the increase of MDR is faster during the first few iterations, then slows down in the subsequent stages and reaches a plateau after about 10 iterations, which clearly demonstrates the efficiency of GDM. With the optimal shape, as shown in Fig. 7, the MDR increases from 63.7 kg/s to 147.7 kg/s, yielding a remarkable 131.9% increase.

Interestingly, apart from the global optimum, it is witnessed in Fig. 5 that there is a local optimum on the right bottom side of the design area, which corresponds to a convex shape hopper. As shown in Fig. 8, the MDR of this convex shape is also higher than that of the
original conical shape, from 63.7 kg/s to about 69.76 kg/s, although the extent of increase is only 9.51%, which is much lower than that of the concave design.

4.2 Comparison with DEM prediction

DEM is implemented in this work to cross-check the reliability of the optimization design. It is used to determine the MDRs in the same initial conical hopper and the optimal hoppers as used in FEM modelling. Table 3 compares the results of MDR obtained by FEM and DEM using comparable parameters. We can see that despite some quantitative differences, the trends in FEM and DEM are quite consistent. The concave design of hopper can indeed double the MDR. The convex design is less effective but still can bring about an around 10% increase in MDR. This result suggests that although conical hoppers are widely used and easy to manufacture, they are in fact not the best choice for maximizing discharge rate. There is still room for improvement in this regard.

4.3 Sensitivity study

In this sensitivity study, unless being stated as the varied parameter to be studied, other geometry and material properties are kept the same as the case study. In terms of the variation of the geometry, diameter of inlet, diameter of outlet, hopper height and the initial hopper half-angle are the four parameters to be considered. Among these parameters, diameter of outlet is fixed in this study. The effects of the initial half-angle and the hopper height are studied in subsection 4.3.1 and 4.3.2, respectively, in which case the fourth parameter – diameter of inlet is a dependent variable and is determined by the other three parameters. Subsequently, the detailed data of the inlet diameter will not be presented.

4.3.1 Initial hopper half-angle

The initial hopper half-angle $\alpha$ is the critical geometric parameter that affects the MDR in a conventional conical hopper. In our optimization, the initial hopper half-angle remains important to the final optimal shape of the hopper. Fig. 9 (a) shows the MDRs of conical and optimized hoppers when the initial half-angle varies from $25^\circ$ to $60^\circ$. With the increases in $\alpha$, MDR of conical hopper decrease and basically plateaus after $45^\circ$, in good consistency with the literature. After being optimized, however, MDR peaks at around $40^\circ -45^\circ$ and declines moderately with a further increase of $\alpha$. From Fig. 9 (a), it can be seen further that if the half-angle of the hopper is simply steepened from $60^\circ$ to around $27^\circ$, the MDR can be increased from 64.4 kg/s to 76.5 kg/s. However, with the optimally designed hopper
from our study, the MDR can be increased by 68% from 64.4 kg/s to 108.2 kg/s. This case demonstrates that the new approach to optimize the shape is much more effective than the traditional design method to adjust the hopper angle.

We define the increase of the optimized MDR to the original MDR as a measure to evaluate the effectiveness of the optimization. Fig. 9 (b) shows the increase in percentage for different initial half-angles. Generally, the optimization method can double the MDR for 25° - 60° of $\alpha$. It is most effective for hoppers in the range from 38° to 45°, where MDR can be increased by more than 130%. While for hoppers with gentle angles, the efficiency of optimization declines to some extent. The optimized shape also varies with the initial hopper angle. The optimized splines of different $\alpha$ look similar in the vicinity of hopper outlet, but in other areas, the larger $\alpha$ can leads to much gentler curves, as can be seen from Fig. 10. Dark grey indicates the optimised shape; Light grey indicates the initial conical shape.

4.3.2 Hopper height

In normal conical hoppers, the hopper height is basically irrelevant to MDR when the hopper half angle and outlet diameter are fixed [5]. But here we see that the optimized MDR is dependent on the hopper’s original height. Fig. 11 (a) shows the MDRs before and after optimization in hoppers of different initial heights. The original MDRs are constant for all the hopper heights, just as stated in the literature [12]. The optimized MDRs, however, keep increasing with the hopper height, from the 96% for the height of 0.2 m to around 132% for the height of 0.4 m. The corresponding optimal shapes are illustrated in Fig. 12. It looks that the optimal shapes are similar for different initial heights, especially for the segment close to hopper outlet. The moderate effect of height may come from the outer segment of the spline which, as seen, indeed becomes steeper with increasing height. To show this trend quantitatively, Table 4 shows the coordinates of the joint point C in each height. Although the coordinates are quite different in the various hopper with different height, it can be noted that the ratios of $x/(D/2)$ and $y/H$ are similar in each case. The ratio of $x/(D/2)$ varied around 0.45 and the ratio of $y/H$ fluctuated around 0.8. Physically, these two ratios represent the relative position of the joint point C corresponding to the width and height of the initial hopper. In summary, the hopper height has a moderate effect on the optimal shape of hopper. A higher hopper height gives more
space to evolve to an optimal design that has a steeper curved surface of the wall, which can explain that the increase of hopper height will make the MDR increase larger after optimization.

4.3.3 Internal friction angle and wall friction coefficient

In practices, the hopper may handle different materials, or the material properties may change over time due to reasons of temperature, moistness and pressure, et al. It is important to know whether the optimized hopper is sensitive to material properties. To clarify this issue, we test nine different material properties in the same conical and concave hoppers discussed above. In these cases, the friction coefficient ranges from 0.2 to 0.6 and internal friction angle ranges from 20° to 40°.

Fig. 13 (a) shows the MDRs obtained for the nine different combinations of material properties. In the original conical hopper (α=45°), the largest MDR occurs for low internal friction angle (20°) and large wall friction (0.5). After optimization, the largest MDR happens for low internal friction angle (20°) and low wall friction (0.1). The latter seems to be more intuitive. It somehow suggests that the conical hopper still presents some mechanisms to discourage particle flow, such as stress arch, which is related to the frictional traction at walls. The optimized hopper provides a conforming surface to minimize such hindering mechanisms. Indeed, from Fig. 13. (b), we can see that the effectiveness of optimization is mainly related to the wall friction, possibly as high as 170.7% for wall friction of 0.1 and as low as 84.5% for wall friction of 0.5. It also varies with the internal friction angle but the trend is fluctuating.

In principle, the effectiveness of optimization does depend the material properties, but this does not mean that we should have different hopper shapes for different materials. Fig. 14 plotted the optimal shapes corresponding to the nine material properties considered above. All these shapes are almost overlapping with each other, despite the large difference in the absolute value of MDR. Therefore, the optimal shape of hopper is unchanged for a given geometrical constraint as discussed in Figs. 10-12. The material properties only affect the actual increase that can be achieved. This should be a positive evidence to support the general use of such optimized hopper in practices.

4.3.4 Comparison of flow pattern
In addition to the discharge rate, the flow pattern of particles will also be influenced by the change in hopper shape. In silo practices, two flow patterns are commonly identified, i.e. mass flow and funnel flow. In mass flow, all particles can move in the silo in the discharging process satisfying a ‘first in first out’ principle. In funnel flow, however, there exists a stagnant zone of particles, which will not flow out of the silo. In practical operations, the mass flow is preferred while the stagnant zone should be avoided. Therefore, it is necessary to check the flow pattern of particles in this new optimized hopper.

Fig.15 compares the steady speed field obtained using FEM for conical and optimal hoppers under 3 different initial conditions, respectively. The scale of legend in these 6 graphs are linear and consistent, but to better observe the flow and stagnant zones, we adjust the maximum speed in the legend to be 0.5 m/s. Generally, both the conical and optimized hopper lead to funnel flows for initial hopper half-angles of 32°, 45° and 53°. Moreover, the area of flow regime decreases when the initial hopper half-angle is increased in the two kinds of hoppers. However, when examined carefully, the performances of the two kinds of hoppers are different. It might be expected that the optimized hopper would create a worse flow patterns than the ordinary conical hoppers in view of its upward convex shape. However, we see from Fig. 15 that the flow pattern in the optimized hopper is actually not bad. It may result in a smaller flowing zone than the conical hopper, but of great importance, it also reduces the size of stagnant zone remarkably. Because of the upward convex shape, a large portion of the hopper, which is supposed to be a stagnant zone in conical shape, is cut away in the optimized shape. This is a key advantage of the optimized hopper in practice. In fact, a similar ‘expanded-flow hoppers’ comprised by two straight hopper slopes has been proposed in the past, and shown to be effective in improving the hopper flow pattern[14, 15]. Therefore, there is reason to believe that the optimized hopper not only maximize the discharge rate but also is beneficial to enhancing the flow patterns of particles.

5. Conclusions

The shape optimization is performed on conical hoppers with the objective to increase the mass discharge rate. The optimization is based on the FEM model and a coupled genetic algorithm and gradient descent method. The effectiveness of this optimization is examined
in various cases of geometry and material properties. The following conclusions can be drawn from this work:

1. The optimization method established on FEM model is effective to improve the hopper discharge rate. Although the FEM model may still need improvement in terms of quantitative accuracy, it captures the salient features of hopper discharge rate such as the Beverloo scaling and the dependency on hopper slope angle. Combining this model with the optimization methods, the optimal shape can be automatically searched within hours. The performances of these optimized hoppers are tested by using DEM simulations, where similar increases in discharge rate are indeed yielded.

2. The optimization across a wide space of parameters illuminates both a globally and a locally optimal shape for hopper. The globally optimal shape has a steep slope near the outlet and gentle slope in the outer area, i.e. concave to the outside. It can typically double the discharge rate as compared to the conventional conical hoppers, almost triple in certain cases. Note that the optimization also reveals a less effective but interesting local optimum, i.e. the convex hopper (to outside), which can increase the discharge rate by about 10%. The shape effect is indeed complicated as shown in Fig. 5, which also indicates the room for improvement in practices.

3. The optimal shape depends on the initial hopper half-angle of conical hopper, but it is insensitive to granular material properties. The initial conical hopper sets up the bounds for the optimization, and thus the originally steeper hopper also has a steeper optimized shape. The highest increase rate in discharge rate is observed for the initial hopper half-angle between 30° to 50°, lying in the usual range of practical hoppers. The material properties like the internal friction angle can affect the absolute value of discharge rate, but only has a minor influence on the optimal shape. That is, the same optimized hopper can work for a variety of granular materials, which is encouraging from the perspective of practical applications.

In principle, this work introduces a smoothly curved shape into hopper design, which can increase the discharge rate by about 2-3 times. It is worth mentioning that only two design variables, i.e., the coordinates of the original central point C, were used in our optimization design. Nevertheless, the MDRs of the optimal designs have been dramatically increased. In theory, introducing more design variables, for example, by considering multi-segment spline, can achieve better optimal designs. However, the optimization problem will be
more complicated and the computing time to obtain the optimal results will also be increased significantly, which can be considered in the future. To our knowledge, the most comparable peer to this shape is the expanded-flow hopper comprised of multiple slopes. Such expanded-flow hopper was introduced mainly to improve the mass-flow pattern in a limited allowable space, while the present hopper is focused on maximizing the discharge rate. Nonetheless, the expanded-flow hopper uses a steep slope near the outlet and is expected to benefit the discharge rate as well, although it is not the best for this purpose. In this regard, it might be deemed as an approximation to the current optimal transiting shape, with the virtue of being easy to manufacture. With the development of manufacturing technologies such as 3D printing, this optimal shape may also be directly implemented in practices depending on the need and economic considerations.

Acknowledgement

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References


Fig. 1. Illustration of (a) numerical model and (b) geometry of a conical hopper.
Fig. 2. Illustration of the spline function of the hopper shape with co-ordinate.

Fig. 3. Flow chart of a genetic algorithm.
Fig. 4. Illustrations of (a) crossover and (b) mutation in the GA method (retrieved from www.oodlestechnologies.com).

Fig. 5. Contour map of optimal flow rate in terms of x and y of point C obtained from GA with the legend showing the magnitude of MDR (the scale of the contour legend is not linear so as to present a clear optimal location)
Fig. 6. Evolution of the hopper shape from the initial conical shape to the final optimal shape.

<table>
<thead>
<tr>
<th></th>
<th>Optimal MDR</th>
<th>147.7 kg/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>MDR</td>
<td></td>
</tr>
<tr>
<td>Increase</td>
<td></td>
<td>131.9%</td>
</tr>
</tbody>
</table>

Fig. 7. (a) The section view of the globally optimal hopper. (b) Performance of this optimal shape compared to the initial conical hopper.
Fig. 8. (a) Section view of the convex hopper (a locally optimal shape). (b) Performance of this convex hopper compared to the conical one.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Location of point C</td>
<td>(0.422, 0.184)</td>
</tr>
<tr>
<td>MDR in convex shape</td>
<td>69.76 kg/s</td>
</tr>
<tr>
<td>Initial MDR</td>
<td>63.7 kg/s</td>
</tr>
<tr>
<td>MDR increase</td>
<td>9.51%</td>
</tr>
</tbody>
</table>
Fig. 9 (a) influence of the hopper angle on initial MDR and optimal MDR. (b) The...
increase of MDR after optimization for different initial hopper half-angles.

Fig. 10 Optimal shapes when the initial hopper half-angle is (a) 32°, (b) 45° and (c) 53°. Dark grey indicates the optimised shape; Light grey indicates the initial conical shape.
Fig. 11 (a) The influence of the hopper height on initial MDR and optimal MDR when
the initial hopper half-angle is fixed at 45°. (b) The increase of MDR after optimization in
different height while the initial hopper half-angle is 45°.

Fig. 12 Optimal hopper shapes with different hopper heights when the initial hopper
half-angle is fixed at 45° with the length in unit of m.
Fig. 13 (a) Initial MDR and optimal MDR in nine cases with different friction coefficient and internal friction angle when the initial hopper half-angle is fixed at 45°. (b)
MDR increases in nine cases with different friction coefficient and internal friction angle when the initial hopper half-angle is fixed at 45°.

Fig. 14 Nine optimal hopper shape drawings in the same model with the different friction coefficient and internal friction angle when the initial hopper half-angle is fixed at 45°.
Fig. 15 Comparison of velocity field before and after optimization on (a) 32°, (b) 45° and (c) 53°

Table 2. Dimensions of the hopper and the elastic parameters of the granular material in the case study.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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</thead>
<tbody>
<tr>
<td>Hopper height</td>
<td>0.4</td>
<td>m</td>
</tr>
<tr>
<td>Diameter of inlet</td>
<td>1</td>
<td>m</td>
</tr>
<tr>
<td>Diameter of outlet</td>
<td>0.2</td>
<td>m</td>
</tr>
<tr>
<td>Initial hopper half-angle</td>
<td>45</td>
<td>degree</td>
</tr>
</tbody>
</table>
Bulk density 2000 kg/m³
Elastic modulus 10 MPa
Poisson’s ratio 0.3
Internal friction angle * 28 degree
Wall friction * 0.27

* These parameters are taken from shear cell test as discussed in Section 2.2

Table 2 Optimization results from the genetic algorithm and gradient descent method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Genetic Algorithm</th>
<th>Gradient Descent Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal location of C</td>
<td>(0.201, 0.298)</td>
<td>(0.195, 0.322)</td>
</tr>
<tr>
<td>Initial MDR</td>
<td>63.7 kg/s</td>
<td>63.7 kg/s</td>
</tr>
<tr>
<td>Optimal MDR</td>
<td>126.4 kg/s</td>
<td>147.7 kg/s</td>
</tr>
<tr>
<td>MDR increase</td>
<td>98.4%</td>
<td>131.9%</td>
</tr>
<tr>
<td>Computing time</td>
<td>102 hrs.</td>
<td>11 hrs.</td>
</tr>
</tbody>
</table>

Table 3 Comparison of MDR increase between FEA simulation and DEM simulation for two different proposed designs.

<table>
<thead>
<tr>
<th>Method</th>
<th>FEA</th>
<th>DEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>MDR increase in optimal design</td>
<td>131.9%</td>
<td>114.72%</td>
</tr>
<tr>
<td>MDR increase in convex design</td>
<td>9.51%</td>
<td>9.09%</td>
</tr>
</tbody>
</table>
Table 4 Optimal location of the joint point C in different hopper height H, with its relative coordinates to corresponding hopper width D/2 and height H.

<table>
<thead>
<tr>
<th>H (m)</th>
<th>D (m)</th>
<th>x (m)</th>
<th>y (m)</th>
<th>x/(D/2)</th>
<th>y/H</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>0.195</td>
<td>0.323</td>
<td>0.390</td>
<td>0.808</td>
</tr>
<tr>
<td>0.35</td>
<td>0.45</td>
<td>0.205</td>
<td>0.307</td>
<td>0.455</td>
<td>0.877</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>0.168</td>
<td>0.250</td>
<td>0.420</td>
<td>0.833</td>
</tr>
<tr>
<td>0.25</td>
<td>0.35</td>
<td>0.176</td>
<td>0.220</td>
<td>0.502</td>
<td>0.880</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>0.129</td>
<td>0.147</td>
<td>0.430</td>
<td>0.735</td>
</tr>
</tbody>
</table>