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Approximate analytical solution of the Boussinesq equation with numerical validation

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Abstract. A new approximate solution of the one-dimensional Boussinesq equation is presented for a semi-infinite aquifer when the hydraulic head at the source is an arbitrary function of time. Estimates for recharge, discharge, and elevation of the water table are given. The simplicity and accuracy of the approximation are compared with "exact" numerical and analytical solutions. The method of solution is illustrated with several examples including the commonly treated case of a constant boundary head and a nonmonotonically varying boundary head.

1. Introduction

Groundwater flow in an unconfined aquifer may be approximately modeled by the nonlinear Boussinesq equation, assuming Dupuit's hypothesis of horizontal flow applies [*de Marsily*, 1986]. Solutions of the Boussinesq equation are applicable in catchment hydrology and base flow studies [e.g., *Troch et al.*, 1993] as well as in agricultural drainage problems [e.g., *Perrochet and Musy*, 1992] and constructed, subsurface wetlands. In the present paper the one-dimensional Boussinesq equation is used to describe lateral groundwater flow into an initially dry stream bank which is bounded from below by an impermeable layer when the stream level experiences time variation (Figure 1). The equation has the form

$$S \frac{\partial h}{\partial t} = K \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right), \tag{1}$$

where K and S are hydraulic conductivity and specific yield of the aquifer, respectively; h is the water depth measured from the impervious substratum; x is the horizontal distance from the origin; and t is time.

The boundary conditions are given by

$$h(0, t) = H(t) \qquad t > 0 \tag{2}$$

$$h(\infty, t) = h_i \qquad t > 0. \tag{3}$$

The initial water level is

$$h(x, 0) = h_i \qquad x > 0. \tag{4}$$

In this note we extend the analytical solution of *Parlange et al*. [1992] on the *Bruce and Klute* [1956] equation to solve (1) subject to (2), (3), and (4). The method is based on the work of *Heaslet and Alksne* [1961]. The new analytical solution is compared with the numerical solution for a range of cases.

Several approximate analytical solutions already exist for the simple case of H constant [e.g., *Tolikas et al*., 1984; *Lockington*, 1997]. The solution of *Tolikas et al*. [1984] is limited by its requirement of the solution of a system of nonlinear algebraic equations. *Lockington*'s [1997] algebraic solution is found by the method of weighted residuals for recharging and dewatering aquifers and is compared with the present approximation for the special case of constant boundary head.

2. New Approximate Solution

From (4) of *Parlange et al*. [1992], where in the present notation θ is replaced by $h - h_i$, θ_s is replaced by $(H - h_i)$, and D is replaced by Kh/S, we write

$$\frac{K}{S}\int_{h-h_i}^{H-h_i} \frac{\bar{h}}{\bar{h} - h_i}\, d\bar{h} = \frac{q}{H - h_i}\, x + B(t)x^2, \tag{5}$$

where $B(t)$ is an unknown function of time and q is defined by

$$q = -\frac{K}{S} H\left(\frac{\partial h}{\partial x}\right)_{x=0}, \tag{6}$$

where qS is the surface flux entering the bank from the stream. Equation (5) represents the beginning of a simple expansion of the integral on the left-hand side of (5) in a Taylor series in powers of x, which was shown to hold by *Heaslet and Alksne* [1961]. However, *Parlange et al*. [1992] suggested that keeping the first two terms in the series only, greatly simplifies the analysis and can still be accurate. We follow the same approach here.

Equation (5) can be rewritten as

$$\frac{K}{S}(H - h_i)\left(H - h + h_i \ln\left(\frac{H - h_i}{h - h_i}\right)\right) = qx + f(t)x^2, \tag{7}$$

where the unknown function $f(t)$ stands for $B(t)(H - h_i)$. Equation (2) is automatically satisfied by (7), and $q(t)$ and $f(t)$ must be found to complete the determination of $h(x, t)$ in (7) when $H(t)$ is given.

We note here that, in principle, $H(t)$ is arbitrary. Integrating (1) twice gives

$$-\frac{d}{dt}\int_0^\infty x^2 \frac{\partial h}{\partial x}\, dx = \frac{K}{S}(H^2 - h_i^2). \tag{8}$$

If we know $H(t)$, then (8) can be integrated in time to give

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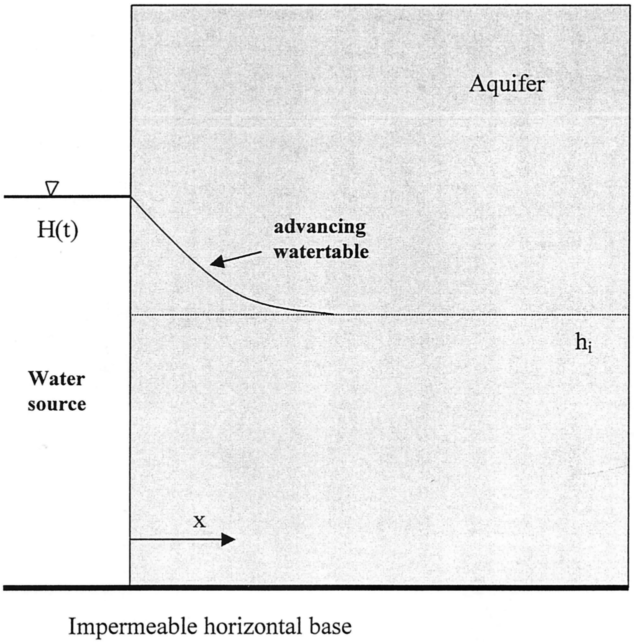

Figure 1. Schematic diagram of the physical problem.

$$-\int_0^\infty x^2 \frac{\partial h}{\partial x}\, dx.$$

We now write

$$I = \int_0^\infty (h - h_i)\, dx = -\int_0^\infty x \frac{\partial h}{\partial x}\, dx. \tag{9}$$

Note that $dI/dt = q$, so integrating (7) with respect to x from 0 to ∞ and multiplying it first by $\partial h/\partial x$ yields

$$\frac{K}{2S}(H^2 - h_i^2)(H - h_i) = qI - f(t) \int_0^\infty x^2 \frac{\partial h}{\partial x}\, dx. \tag{10}$$

Since (2) is a condition at $x = 0$, we constrain the solution to be valid as $x \to 0$; to do this we first calculate $h\partial h/\partial x$ as a function of x from (7) or

$$\frac{K}{S}(H - h_i)h \frac{\partial h}{\partial x} = (q + 2xf)(h - h_i), \tag{11}$$

and as $x \to 0$, this yields

$$\frac{K}{S}(H - h_i) \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) = -2f(H - h_i) + \frac{q^2 S}{KH}. \tag{12}$$

Thus we impose

$$\frac{dH}{dt} = -2f + \frac{q^2 S}{HK(H - h_i)}. \tag{13}$$

Knowing $H(t)$ and eliminating f gives a differential equation with which to calculate $I(t)$, i.e.,

$$\frac{K(H - h_i)(H^2 - h_i^2)}{S} = 2qI + \frac{K}{S}\int_0^t (H^2 - h_i^2)\, d\bar{t}$$

$$\cdot \left[-\frac{dH}{dt} + \frac{q^2 S}{H(H - h_i)K}\right]. \tag{14}$$

By knowing $H(t)$, equation (14) provides an ordinary differential equation between I and q (equal to dI/dt), and then (13) yields $f(t)$. This completes the determination of the functions (q and f) needed to obtain $h(x, t)$ in (7). The presence of h_i tends to make the analytical results somewhat lengthy, and in the following two examples we take $h_i = 0$ for simplicity.

3. Examples

Two examples are considered below.

We first take a particular, but still fairly general, case to illustrate the accuracy of the approximate formulation by comparison with an "exact" numerical solution. This first example uses a similarity solution, which, however, limits us to $H(t)$ increasing or decreasing monotonically with time. For this reason, the second example corresponds to an $H(t)$ first increasing and then decreasing with time.

Take $H(t)$ of the form

$$H = \phi_0 \left(\frac{Kt}{S}\right)^{2n-1}, \tag{15}$$

where ϕ_0 is a constant.

The dependence of H on t in (15) reduces (1) to an ordinary differential equation, thus allowing simple solutions. For instance, with $n = 1$ we obtain the standard Boltzmann transformation, and with $n = 0.5$ we maintain a constant level at the stream. In general, allowing n to be arbitrary gives a rich family of solutions using similarity techniques. These techniques have been illustrated by *Imray* [1966] and *Hogarth et al.* [1989]. In addition, (15) will approximate H, at least for some time, but in reality it will eventually break down, i.e., when $n > 0.5$, since H cannot grow without limit. In (15), ϕ_0 is a constant necessary to scale height H to any value at a particular time; we now look for a solution of the form

$$h = \phi(\chi)\left(\frac{Kt}{S}\right)^{2n-1}, \tag{16}$$

with

$$\chi = x \Big/ \left(\frac{Kt}{S}\right)^n. \tag{17}$$

Hence once $\phi(\chi)$ is obtained, then $h(x, t)$ is given by using (16). Note that ϕ requires only one variable χ, instead of x and t, making the solution easier to describe, which, of course, is the reason why similarity solutions are used. Thus (1) is transformed into an ordinary differential equation as follows:

$$\frac{d}{d\chi}\left(\phi \frac{d\phi}{d\chi}\right) = (2n - 1)\phi - n\chi \frac{d\phi}{d\chi} \tag{18}$$

with boundary conditions

$$\phi(\chi = 0) = \phi_0 \qquad \phi(\chi \to \infty) = 0. \tag{19}$$

Equations (18) and (19) form a two-point boundary value problem which can be solved numerically using the method of *Shampine* [1973]. This method was illustrated by *Hogarth et al.* [1992]. Following their paper, (18) is rewritten as

$$\frac{d\phi}{d\chi} = u \tag{20}$$

$$\frac{du}{d\phi} = \frac{(-n\chi - u)}{\phi} + \frac{(2n-1)}{u}. \tag{21}$$

In order to start the numerical integrations the following expansions can be used:

$$\chi = \chi_0 + A\phi + B\phi^2 \tag{22}$$

$$u = -n\chi_0 + C\phi + D\phi^2, \tag{23}$$

where A, B, C, and D are to be determined. Substituting (22) and (23) into (20) and (21) gives A, B, C, D, or

$$\chi = \chi_0 - \frac{1}{n\chi_0}\phi - \frac{(1-n)}{4n^3\chi_0^3}\phi^2 \tag{24}$$

$$u = -n\chi_0 + \frac{(1-n)}{2n\chi_0}\phi + \frac{(1-n)(2-3n)}{12n^3\chi_0^3}\phi^2. \tag{25}$$

Now since Shampine's method integrates from the front $\phi = 0$ at an arbitrary χ_0, we can simply take $\chi_0 = 1$ without loss of generality. We now can essentially obtain an exact solution which can be compared with the new approximate solution in order to estimate the precision of the latter.

From (9),

$$I = \int_0^\infty \phi d\chi \left(\frac{Kt}{S}\right)^{3n-1}, \tag{26}$$

which shows that n must be greater than $\frac{1}{3}$, since $I(0) = 0$, and by differentiation, from eq. (10),

$$q = \int_0^\infty \phi d\chi (3n-1) t^{3n-2} \left(\frac{K}{S}\right)^{3n-1}. \tag{27}$$

In addition, (8) gives

$$-\int_0^\infty x^2 \frac{dh}{dx}\, dx = \left(\frac{K}{S}\right)^{4n-1} t^{4n-1}\phi_0^2/(4n-1). \tag{28}$$

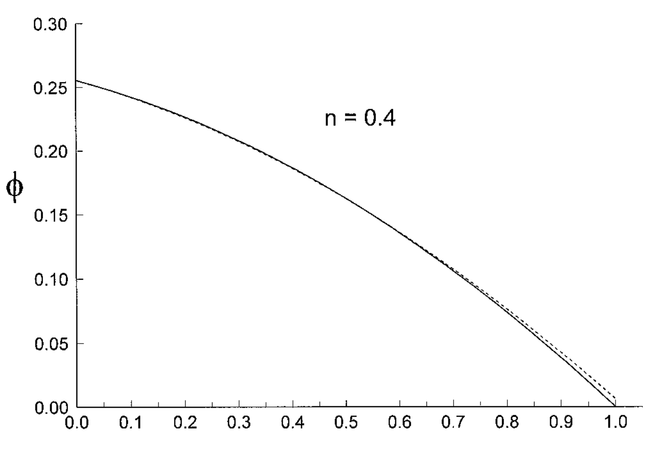

Figure 2. Solution of $\phi(\chi)$ with $n = 0.4$. Numerical solution is the solid line, and equation (30) is the dashed line.

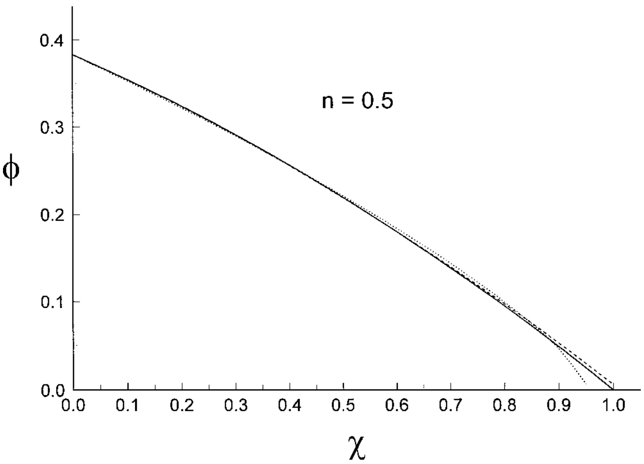

Figure 3. Solution of $\phi(\chi)$ with $n = 0.5$. Numerical solution is the solid line, and equation (30) is the dashed line. *Lockington*'s [1997] solution is the dotted line.

Using (26), (27), and (28), equation (14) gives the remarkably simple result

$$\frac{\left(\int_0^\infty \phi d\chi\right)^2}{\phi_0^3} = \frac{2}{11n-3}. \tag{29}$$

Finally, (7) gives

$$\phi_0 - \phi = \sqrt{\frac{2}{11n-3}}\,(3n-1)\phi_0^{1/2}\chi + \frac{(4n-1)}{2(11n-3)}(1-n)\chi^2 \tag{30}$$

with ϕ_0 given.

We thus find that the present approximation gives the exact results

$$\phi_0 - \phi = \phi_0^{1/2}\chi \qquad n = 1 \tag{31}$$

$$\phi_0 - \phi = \chi^2/6 \qquad n = \tfrac{1}{3}. \tag{32}$$

These are the only cases when it is exact [*Parlange et al.*, 1999; *Barenblatt et al.*, 1990].

Note that for $n = \frac{1}{3}$ the flux $q = 0$ (equation (27)), whereas I is a constant (equation (26)); that is, this case represents the redistribution of water initially present in the aquifer.

Now if we take $\chi_0 = 1$ at $\phi = 0$ without loss of generality, as previously indicated for the *Shampine* [1973] method, then the corresponding ϕ_0 is given by the simple quadratic equation

$$\phi_0 = \sqrt{\frac{2}{11n-3}}\,(3n-1)\phi_0^{1/2} + \frac{(4n-1)(1-n)}{2(11n-3)}. \tag{33}$$

It is sufficient to check the accuracy of the approximation for this value of ϕ_0 with no loss in generality, since, as shown by (30), χ_0 and ϕ_0 scale such that $\phi_0^{-1/2}\chi_0$ remains constant, and this property forms the basis of Shampine's method.

The analytical solution, equation (30), is plotted against the numerical solution for a range of values of n in Figures 2–5. Of course, only $\phi(\chi)$ is given, and $h(x, t)$ can easily be deduced from (16) for any position and time. In each case, there is very little difference between the profiles except near the toe where some slight overprediction or underprediction occurs. The case

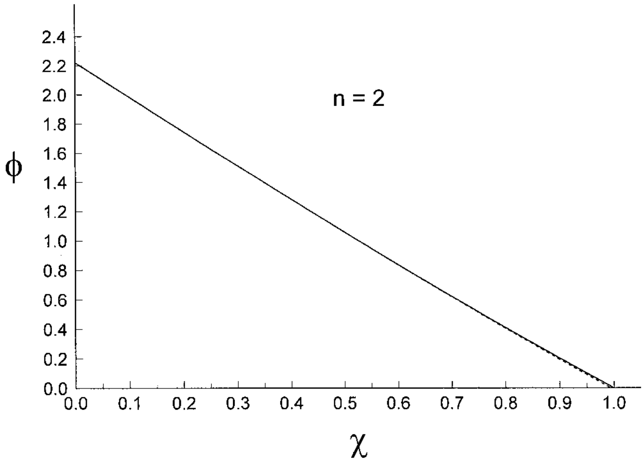

Figure 4. Solution of $\phi(\chi)$ with $n = 2.0$. Numerical solution is the solid line, and equation (30) is the dashed line.

of a constant boundary head corresponds to $n = 0.5$, and in this case the analytical approximation of *Lockington* [1997] is also graphed for comparison. The latter significantly underestimates the position of the advancing front, in this case where the aquifer is initially dry.

For the second example we consider the case where H increases with time from zero to a maximum and then decreases back to zero, thus providing a realistic behavior for all times. We take

$$H(t) = \tfrac{3}{2}\left[(t + 1)^{2/3} - 1\right]/(t + 1), \tag{34}$$

shown in Figure 6. H reaches a maximum equal to $1/\sqrt{3}$ at $t_n = 3^{3/2} - 1$. Note that we take $K/S = 1$; otherwise t would stand for Kt/S, as in the previous example. The interest of (34) is that an exact solution of the problem with this specific H can be found [*Parlange et al.*, 1999], generalizing an earlier solution of *Barenblatt et al.* [1990].

By knowing $H(t)$, equation (14) provides a differential equation for $I(t)$, with the initial condition $I(0) = 0$. Note that (14) is quadratic in q and can be solved easily to provide an explicit q as a function of t and I, which can then be solved numerically, for instance, using a Runge-Kutta technique. For

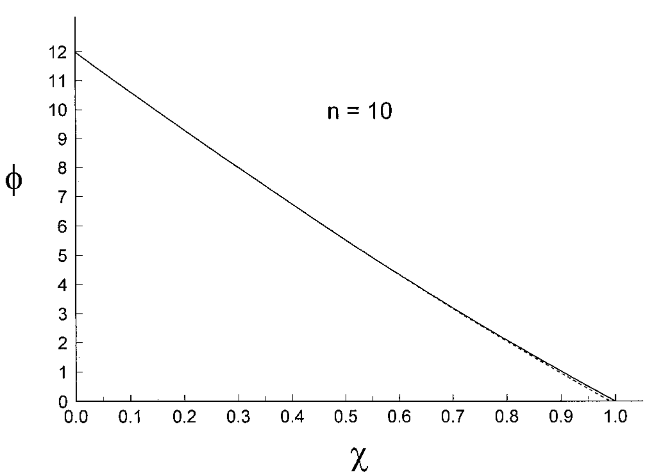

Figure 5. Solution of $\phi(\chi)$ with $n = 10.0$. Numerical solution is the solid line, and equation (30) is the dashed line.

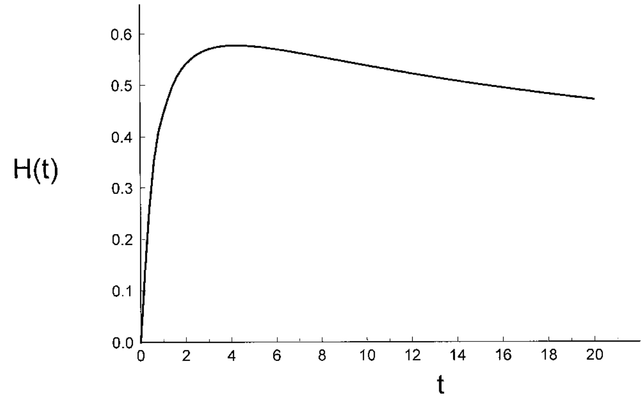

Figure 6. $H(t)$ as given in equation (34).

the case of (34), we find that the solution of (14) is written exactly as

$$I = \tfrac{3}{2}\left[2 + (t + 1)^{-1} - 3(t + 1)^{-1/3}\right], \tag{35}$$

yielding also

$$q = H/(t + 1). \tag{36}$$

Then (13) gives f as

$$f = H/6(t + 1). \tag{37}$$

Finally, (7) gives

$$h = H - x/(t + 1) - x^2/(6(t + 1)). \tag{38}$$

Remarkably, this turns out to be the exact solution of (1), as can be found by direct substitution. Thus once again the approximate approach yields an exact solution, this time with $H(t)$ first increasing and then decreasing with time [*Parlange et al.*, 1999].

4. Conclusion

The method of *Parlange et al.* [1992] has been extended to obtain an approximate analytical solution of the Boussinesq equation for water depth at some position which is an arbitrary function of time. This new approximation gives a simple and accurate solution by comparison with the numerical solution of the Boussinesq equation when the boundary condition is a power to time. Interestingly, for this particular example the approximation technique yields two exact solutions. Another example is also given where the level of the water at the boundary first increases and then decreases with time. In this case as well, the approximation technique yields an exact solution.

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