Analysis of Time Compression Approximations

W.L. Hogarth[1], D.A. Lockington[2], D.A. Barry[3], M.B. Parlange[3],

G.C. Sander[4], L. Li[2], R. Haverkamp[5], W. Brutsaert[6], J.-Y. Parlange[7]

[1] Faculty of Science and IT, The University of Newcastle, Callaghan, NSW, Australia, 2300

[2] School of Civil Engineering and National Centre for Groundwater Research and Training, The University of Queensland, St Lucia, Qld, Australia, 4072

[3] Ecole polytechnique fédérale de Lausanne (EPFL), Faculté de l'environnement naturel, architectural et construit, Institut d'ingénierie de l'environnement, Station no. 2, CH-1015 Lausanne, Switzerland.

[4] Loughborough University, Dept Civil & Bldg Engn, Loughborough, LE11 3TU, UK

[5] University Grenoble 1, CNRS, URA 1512, IMG, LTHE, F-38041, France

[6] Department of Civil and Environmental Engineering, Cornell University, Ithaca, NY, 14853, USA

[7] Department of Biological and Environmental Engineering, Cornell University, Ithaca, NY, 14853, USA

Abstract

TCA, time compression approximation, is a practical and often quite accurate tool to predict postponding infiltration for field applications. A modified approximation (MTCA) can be used just as easily and, in general, will reduce the error by about 50%. This is based on two results:

1. After ponding, TCA and MTCA predict very close infiltration rates; and
2. MTCA, but not TCA, uses the actual cumulative infiltration up to the ponding time.

   Thus, TCA has an additional error in its prediction of postponding infiltration.

Previously, those results, including the 50% reduction in error, were observed numerically for linear and Burger's soils. They are illustrated here numerically with an actual soil (a Grenoble sand). More importantly, we developed a general analytical approximation for this problem and showed that it can provide a very convenient predictive tool which can then be used for arbitrary soil properties.

Keywords: Time Compression, Infiltration, Constant flux, Constant surface water content

Running title: Time Compression Approximations

1. Introduction

Time Compression (sometimes “Condensation”) Approximation (sometimes “Analysis”) or TCA postulates that infiltration after ponding depends only on the total cumulative infiltration at ponding not on the details of the rainfall rate [Brutsaert, 2005]. Thus, when TCA applies, one can replace the true rainfall rate before ponding by its average value. It follows that if one knows the cumulative infiltration, I, as a function of flux, q, for saturated surface water content, then for the average rainfall rate, q_p, this relation will give the cumulative infiltration at ponding and thus provide an estimate of ponding time, t_p. After this ponding time estimate, the saturated solution is continued.

However, if the average value of rainfall rate is known until ponding, then the ponding time must be known fairly accurately as well as the cumulative rainfall amount, which is also the cumulative infiltration, at ponding time. Thus, MTCA assumes knowledge of ponding time, t_p, and cumulative infiltration at that time, I_p, and does not assume that the average flux before ponding is the flux at ponding.

To extend further our present understanding of TCA and MTCA, see Liu *et al*. [1998], Parlange *et al.* [2000], Basha [2002], Brutsaert [2005], Barry *et al*. [2007]. We will analyze numerically and analytically infiltration for constant flux and for constant surface water content for non-linear soils, revisiting earlier papers [Parlange *et al.* 1985; Hogarth *et al*. 1991; Parlange *et al.* 1997; Parlange *et al.* 1999] which compared numerical results with analytical results. The analytical approach was refined by Barry *et al.* [2007] and is used here to reanalyze the numerical results of Parlange *et al.* [1985] and Hogarth *et al.* [1991] obtained for a Grenoble sand. The sand’s hydraulic properties are fully reported in those two papers, Parlange *et al.*

[1985] and Hogarth *et al.* [1991], and will be used here to illustrate our results. The earlier numerical solutions have been reproduced using COMSOL numerical software. The converged COMSOL finite element solutions agreed with the original solutions presented in Hogarth *et al.* [1991].

2. Analysis

The method is based on a double integration of Richards' equation [Parlange and Haverkamp, 1989], yielding:

$$z(\theta, t) = \int_{\theta}^{\theta_s} \frac{D(\bar{\theta})d\bar{\theta}}{\partial \int_0^{\bar{\theta}} z d\tilde{\theta} / \partial t - k(\bar{\theta})}. \qquad (1)$$

In Eq. (1), D and k are the soil water diffusivity and hydraulic conductivity, respectively, and z is the distance from the surface (positive downwards), t the time, θ the water content at z, θ_s is θ for $z = 0$ (the surface). The expression $\partial \int_0^{\bar{\theta}} z d\tilde{\theta} / \partial t$ is the flux, which does not vary much, unlike k. Parlange [1972] suggested that a first approximation to solving Eq. (1) for z is to replace the flux term by $q\theta / \theta_s$, where q is the surface flux. That substitution has the desirable property that it gives the exact result, usually called the travelling wave solution, when q / θ_s is constant [Fleming *et al.* 1984]. In the long time limit, Eq. (1) reproduces the so-called "profile at infinity" for θ_s constant [Philip 1969]. A straightforward iterative scheme replacing the resulting value of z from Eq. (1) in the integrand has not proved convenient. Another approach is to generalize the method of Heaslet and Alksne [1969] and to expand instead the first approximation in terms of z or [Parlange *et al.* 1997, Barry *et al.* 2007]:

$$\int_{\theta}^{\theta_s} \frac{D d\bar{\theta}}{q\bar{\theta}/\theta_s - k\left(\bar{\theta}\right)} = z + Mz^2 + \cdots. \tag{2}$$

In practice excellent accuracy is obtained keeping only the first two terms on the right side of Eq. (2). $M(t)$ satisfies [Barry *et al*. 2007]:

$$2M = \frac{q}{\theta_s D_s} - \frac{1}{q - k_s}\frac{d\theta_s}{dt}, \tag{3}$$

where D_s and k_s are, respectively, the values of D and k at θ_s. Note that near saturation, D_s is basically undefined and, from the short time limit, could be estimated by [Parlange *et al*. 1999]:

$$\frac{1}{\theta_s D_s} = \frac{\int_0^{\theta_s} \left(\theta_s - \theta\right) D d\theta}{\int_0^{\theta_s} D d\theta \int_0^{\theta_s} \theta D d\theta}. \tag{4}$$

If the relationship between θ_s and q is known, then Eq. (3) yields M. Integrating Eq. (2) provides the additional equation:

$$\int_0^{\theta_s} \frac{D\theta d\theta}{q\dfrac{\theta}{\theta_s} - k\left(\theta\right)} = I + M\int_0^{\theta_s} z^2 d\theta, \tag{5}$$

where $I(t)$ is the cumulative infiltration:

$$I = \int_0^{\theta_s} z d\theta. \quad (6)$$

As it is only a small correction, the last term in Eq. (5), $\int z^2 d\theta$, can be evaluated roughly, assuming a Green and Ampt-type flow, or:

$$\int_0^{\theta_s} z^2 d\theta \approx I^2 / \theta_s, \quad (7)$$

in which case Eq. (5) becomes:

$$\int_0^{\theta_s} \frac{D\theta d\theta}{q\frac{\theta}{\theta_s} - k} = I + M\ I^2 / \theta_s. \quad (8)$$

Up to now, the analysis applies whether q or θ_s is imposed. However, Eq. (3) leads to very different results depending on whether q or θ_s is constant.

2.1 Constant Flux Analysis

Differentiation of Eq. (8) yields:

$$q + \frac{dMI^2 / \theta_s}{dt} = \frac{\theta_s D_s}{q - k_s}\frac{d\theta_s}{dt} - \int_0^{\theta_s} \frac{D\theta^2 d\theta}{\left(q\frac{\theta}{\theta_s} - k\right)^2} q \frac{d\, 1/\theta_s}{dt}, \quad (9)$$

and, combining with Eq. (3):

$$2M\theta_s D_s + \frac{dMI^2/\theta_s}{dt} = -\int_0^{\theta_s} \frac{D\theta^2 d\theta}{\left(q\frac{\theta}{\theta_s} - k\right)^2} q \frac{d\,1/\theta_s}{dt}, \tag{10}$$

we can estimate the order of magnitude of the second term as:

$$\frac{MI^2 d1/\theta_s}{dt} = \mathrm{O}\left[M\int_0^{\theta_s} \frac{D\theta^2 d\theta}{\left(q\frac{\theta}{\theta_s} - k\right)^2} \int_0^{\theta_s} Dd\theta \frac{d1/\theta_s}{dt} \right]. \tag{11}$$

Thus, if this second term were of the order of the third term in Eq. (10), we would have:

$$M\int_0^{\theta_s} Dd\theta = \mathrm{O}[q]. \tag{12}$$

However, in that case, the first term in Eq. (10), $2M\theta_s D_s$, would be an order of magnitude greater than all the other terms in that equation and it could not be balanced by any other term. Hence, the second term in Eq. (10) can be neglected giving:

$$2M\theta_s D_s = q\int_0^{\theta_s} \frac{D(\theta/\theta_s)^2 d\theta}{\left(q\frac{\theta}{\theta_s} - k\right)^2} \frac{d\theta_s}{dt}. \tag{13}$$

Given that *M* is an order of magnitude smaller than suggested by Eq. (12), then *M* can be obtained from Eq. (13), replacing $d\theta_s/dt$ by $q(q-k_s)/\theta_s D_s$ from Eq. (3), where *M* has been dropped, or:

$$2M\theta_s^2 D_s^2 = (q - k_s) q^2 \int_0^{\theta_s} \frac{D(\theta/\theta_s)^2 \, d\theta}{\left(q\frac{\theta}{\theta_s} - k\right)^2}. \tag{14}$$

As $q \to k_s$, the integral is singular since $q - k_s \to 0$. We remove the singularity by using a Gardner-type soil obeying [Barry *et al.*, 2007]:

$$D \approx \theta_s \int_0^{\theta_s} D d\bar{\theta} \frac{dk/\theta}{d\theta} / k_s. \tag{15}$$

Although not exact, such a *D* introduces only a small error on the value of *M* giving

$$2M\,\theta_s D_s = q \left(\frac{\int_0^{\theta_s} D\,d\theta}{\theta_s D_s} \right). \tag{16}$$

For a rapidly increasing *D*, the term in the parenthesis is much less than unity, as can be estimated from Eq. (4). This also shows that, in Eq. (3), the *M* term is much smaller than the other two terms, which basically balance each other. According to Eq. (16), *M* approaches a constant when $t \to \infty$. This, of course, means that the MI^2/θ_s correction in Eq. (8) becomes increasingly large if $t \to \infty$. For *q* sufficiently larger than k_s, ponding will occur for short times and the correction remains small. However, for *q* less than or close to k_{sat}, the contribution of dI^2/dt in Eq. (10) has to be considered, so that $2M\theta_s D_s$ in Eq. (10) is replaced by $2M(\theta_s D_s + 2Iq/\theta_s)$ and Eq. (14) is replaced by the more accurate:

$$2M\theta_s D_s (\theta_s D_s + Iq/\theta_s) = q\int_0^{\theta_s} D d\theta. \tag{17}$$

In Barry *et al*. [2007] this additional term was not kept as only $q > k_{sat}$ was considered and ponding occurred, so in that case this term is normally negligible.

2.2 Infiltration Analysis with Surface Saturation

This case is especially important for using the TCA technique as it serves as a reference. Of course, for $\theta_s = \theta_{sat}$ the $d\theta_s / dt$ term drops out of Eq. (3) and *M* is given by:

$$2M = q \ / \ \theta_{sat} D_{sat} \tag{18}$$

Under constant flux, the $d\theta_{sat} / dt$ term and $q / \theta_{sat} D_{sat}$ largely balanced each other giving $M << q / \theta_{sat} D_{sat}$. This cannot happen here, for $\theta_s = \theta_{sat}$, so that the *M*-term introduces an order of magnitude larger correction. With such an *M*, Eq. (8) holds and relates *I* and *q*.

As noted by Sivapalan and Milly [1985], TCA, to be exact, would require the same $I(q)$ relation for an arbitrary dependence of the flux *q* on time. Obviously, this is impossible. For instance, we have shown that for constant q the *M*-term has essentially no effect on ponding; here, on the other hand, the $I(q)$ relationship is affected as *M* is much larger.

As noted earlier, $1 \ / \ \theta_{sat} D_{sat}$ is not a very meaningful parameter, which means that our condition is unreliable but the estimate of Eq. (4) holds in the short time limit. If we use that estimate for all times in Eq. (8), there is an obvious difficulty for the long time case as the last term, no matter how small *M* is, will eventually dominate and cease to be a small correction. An alternative is to apply Eq. (8) in the short-time limit only so that Eq. (4) leads to:

$$MI^2 / \theta_{sat} = \int_0^{\theta_{sat}} (\theta_{sat} - \theta)\, D d\theta / 2q \quad (19)$$

Writing the correction in this form has the great advantage that if we apply it for long times (even though it was derived for short times), it remains finite in the long times when $q \to k_s$, and as a result, is negligible in that limit, when $I \to \infty$ in Eq. (8). Eq. (8) then becomes:

$$\int_0^{\theta_{sat}} \frac{D\theta d\theta}{\left(q\frac{\theta}{\theta_{sat}} - k\right)} = I + \int_0^{\theta_{sat}} (\theta_{sat} - \theta)\, D d\theta / 2q\,, \quad (20)$$

which, for a given q, gives I quite easily. Note that time not appear in Eq. (20), and I is only a function of q for given soil properties. Fig. 1 gives various $I(q)$ for the Grenoble sand. First, for q constant, I corresponds to its value at ponding obtained numerically and from Eq. (2) dropping the M term altogether, the agreement is obviously excellent. The figure also gives $I(q)$ when θ at the surface is saturated for all times and from Eq. (20). Again, the agreement is quite good, up to higher order terms neglected in Eq. (20).

In the figure, the numerical results for the case $q = 50 cm/hr$ until ponding, followed by θ_{sat} at the surface is also given. Of course, as q decreases, with increasing time this $I(q)$ approaches the results when θ_{sat} at the surface holds for all times. The figure also indicates the relationships assumed by TCA, (*BACF*) and MTCA (*BCF*); see also sketches of Fig. 2. In that sketch, point F represents the long time limit when all the $q(t)$ merge as $q \to k_{sat}$. The other points (ABCDEG) are close together, as $q_{2p} - q_p$ must be small for TCA, and MTCA, to apply, and points (DCG) are even closer to each other as discussed below. TCA assumes that, at ponding, point A in Figs. 1 and 2, q is continuous so that $I = I_{1P}$ given by Eq. (20) for $q = q_p$. Hence $I_{1p} = q_p t_{1p}$ is less than $I_p = q_p t_p$ with, from Eq. (20):

$$q_p(t_p - t_{1p}) = \int_0^{\theta_{sat}} (\theta_{sat} - \theta)\, D d\theta \,/\, 2q \tag{21}$$

MTCA rather assumes that $q = q_p$ until ponding time, point B, then q drops discontinuously to q_{2p}, to point C in Fig. 1 and point D in Fig. 2. Eq. (20) yields $q = q_{2p}$ taking $I = I_p = q_p t_p$.

The $I(q)$ curve when $q = 50 cm/hr$ at the surface until ponding followed by infiltration with the surface saturated obviously shows on the figure as an interpolation between the two cases of q constant and $\theta_s = \theta_{sat}$. An analytical interpolation is now guessed. The expression:

$$\int_0^{\theta_{sat}} \frac{D\theta d\theta}{\left(q\dfrac{\theta}{\theta_{sat}} - k\right)} = I + \frac{1}{2q}\int_0^{\theta_{sat}} (\theta_{sat} - \theta)\, D d\theta \left[\left(\frac{q}{q_p}\right)^{\alpha} - 1\right] \Big/ \left[\left(\frac{k_{sat}}{q_p}\right)^{\alpha} - 1\right] \tag{22}$$

is chosen because it goes to the right limits, i.e., $M = 0$ at $q = q_p$ and $M = \int_0^{\theta_{sat}} (\theta_{sat} - \theta) D d\theta \,/\, 2q$ when $q = k_{sat}$. In addition, we introduce a parameter, α, in Eq. [22] which allows us to satisfy another condition which is available in the transition. As $q_p \to \infty$ the transition is instantaneous so we impose the condition $dI / dq = 0$ in that limit, giving:

$$\alpha \approx \int_0^{\theta_{sat}} D d\theta \Big/ \int_0^{\theta_{sat}} D(\theta_{sat} - \theta) D d\theta \tag{23}$$

where we neglected $(k_{sat}/q_p)^\alpha$ compared to 1, since we assume that q_p is not too close to k_s and Eq. (23) shows that $\alpha >> 1$.

In our illustration, $\alpha \approx$ 18. Figure 1 also gives the transition curve based on Eqs. (22) and (23) – the agreement is obviously quite good.

We are now going to give analytical expressions to estimate $q(t)$. Differentiation of Eq. (20) gives:

$$dt = -\frac{dq}{q}\left[\int_0^{\theta_{sat}} \frac{D\theta^2 d\theta}{\theta_{sat}\left(q\theta/\theta_{sat}-k\right)^2} - \frac{1}{2q^2}\int_0^{\theta_{sat}}(\theta_{sat}-\theta)\, Dd\theta\right] \tag{24}$$

and by integration imposing the condition that $q \to \infty$ as $t \to 0$:

$$t = \int_0^{\theta_{sat}} \frac{D\theta^2}{k^2\theta_{sat}} \ln\left(\frac{q\theta/\theta_{sat}-k}{q\theta/\theta_{sat}}\right) d\theta + \int_0^{\theta_{sat}} \frac{D\theta^2 d\theta}{k\theta_{sat}\left(q\theta/\theta_{sat}-k\right)} - \frac{1}{4q^2}\int_0^{\theta_{sat}}(\theta_{sat}-\theta)\, Dd\theta\,. \tag{25}$$

In addition to giving a sketch of the fluxes as a function of time for TCA (curve *ACF*) and MTCA (curve *DF*) Fig. 2 also shows the interpolation case (curve *BEF*). The curves $t_1(q)$ for TCA and $t_2(q)$ for MTCA are based on a translation of $t(q)$ in Eq. (25) or:

$$t_1 - t_{1p} = t(q) - t(q_p), \tag{26}$$

and

$$t_2 - t_p = t(q) - t(q_{2p}). \tag{27}$$

Repeating the same procedure with Eq. (22) as with Eq. (20), differentiation and integration gives $t_3(q)$ for the interpolation curve (BEF) in Fig.2. or:

$$t_3 = t_1 + \frac{1}{2}\frac{(q/q_p)^\alpha}{q^2}\int_0^{\theta_{sat}} D(\theta_{sat} - \theta)Dd\theta \tag{28}$$

for α large.

We are now going to prove two general results observed previously for linear soils and for Burgers' soils. First, we are going to show that points *C*, *G* and *D* are practically the same (as observed by Basha [2002] for a Burgers' soil, see his Figure 4) so that in the $q(t)$ plane the MTCA curve and the TCA curve are effectively the same for $t > t_p$.

Second, we are going to show that the area *ABC* and the area *BDF* are of the same order of magnitude, which means that the error of TCA in predicting the cumulative infiltration is about twice the error of MTCA.

First we want to show that $t_1 - t_2$ for $q = q_{2p}$ is much smaller than $t_{1p} - t_p$ so that *DG* << *AB* in Fig.2 and the 3 points *CDG* are essentially indistinguishable. Equations (26) and (27) give:

$$t_1 - t_2 = t_{1p} - t_p + t(q_{2p}) - t(q_p), \tag{29}$$

but $q_p(t_{1p} - t_p) = I_{1p} - I_p$ and $I_{1p} - I_p = I(q_p) - I(q_{2p})$, with *I* given by Eq. (20). Since $dI = qdt$, thus $(I_{1p} - I_p)/q_p \approx [t(q_p) - t(q_{2p})]\overline{q}/q_p$ where $q_{2p} < \overline{q} < q_p$. Finally:

$$t_1 - t_2 = [t(q_{2p}) - t(q_p)]\left(1 - \overline{q}/q_p\right), \tag{30}$$

which is small compared to $t_{1p}-t_p$ since $\left(1-\bar{q}/q_p\right)$ is small for TCA, and MTCA, to be applicable.

Second, the area *BAF* is obtained as:

$$\int_{k_{sat}}^{q_p}(t_3-t_1)dq \approx q_p^3(t_p-t_{1p})^2 \Big/ \left(\theta_{sat}\int_0^{\theta_{sat}} D d\theta\right) \tag{31}$$

from Eq. (28), and the *ABC* area is given by $\frac{1}{2}(t_p-t_{1p})(q_p-q_{2p})$ or:

$$\text{ABC area} \approx \frac{1}{2}q_p^3(t_p-t_{1p})^2 \Big/ \left(\theta_{sat}\int_0^{\theta_{sat}} D d\theta\right) \tag{32}$$

using Eq. (24), as long as k_s is not close to q. Thus the area of *BAF* is roughly twice the area of *ABC*, or *BCF* has about half the area of *BAF*. Since those areas correspond to the errors in cumulative infiltration of MTCA and TCA, the latter has roughly twice the error of MTCA as already observed for linear and Burgers' soils. The same improvement of 50% was also observed by Parlange *et al.* [2000] for a power law diffusivity in the absence of gravity to allow analytical treatment with the tools available at that time. Parlange *et al.* [2000] obtained some analytical results with gravity; however, it was not possible to extend them to predict infiltration after ponding. Here we predict analytically that the reduction of the error in the cumulative infiltration with MTCA should apply to any soil. Figure 3 gives the various $q(t)$ obtained numerically for our example and analytically from Eqs. (26), (27) and (28). Not surprisingly, the agreement is quite good, and we cannot distinguish points D, C and G on the figure as expected.

3. Conclusion

One practical advantage of MTCA over TCA is that its application requires a knowledge of ponding time rather than rainfall rates. However, when TCA and MTCA are accurate tools, they both assume that using average rainfall rates, rather than the actual values, does not lead to large errors in predicting postponding infiltration. We assumed that this is the case here, i.e., we did not discuss those situations when the use of an average flux leads to large predictive errors. Rather, we showed that when TCA is a good predictor of postponding infiltration then, MTCA, which is as easy to apply, reduces the error of cumulative infiltration by about 50%

We derived analytically two results valid for any soil property. First, infiltration rates after ponding are the same for TCA and MTCA. Second, the error of the predicted cumulative infiltration for MTCA is about half of what it is for TCA. Both results were obtained previously for linear and Burger's soils and are checked here for a Grenoble sand. More importantly, we predict that they should hold for any soil.

We are able to model TCA, MTCA analytically and the transition from constant flux to constant surface water content for arbitrary soil properties. Small corrections to the cumulative infiltration in Eq. (5) had to be estimated. Being small, we could use rough, i.e., Green and Ampt or Gardner, approximations that affect the small corrections to a higher order which are negligible. We illustrated the accuracy of the analytical model by comparison with the numerical results for the Grenoble sand.

The analytical results presented here apply potentially to any soil, which is more general than previous analytical results that use specific forms of the soil-water properties. As a consequence, those results could be used as a predictive tool under field conditions, when the soil properties are known but do not follow specific forms.

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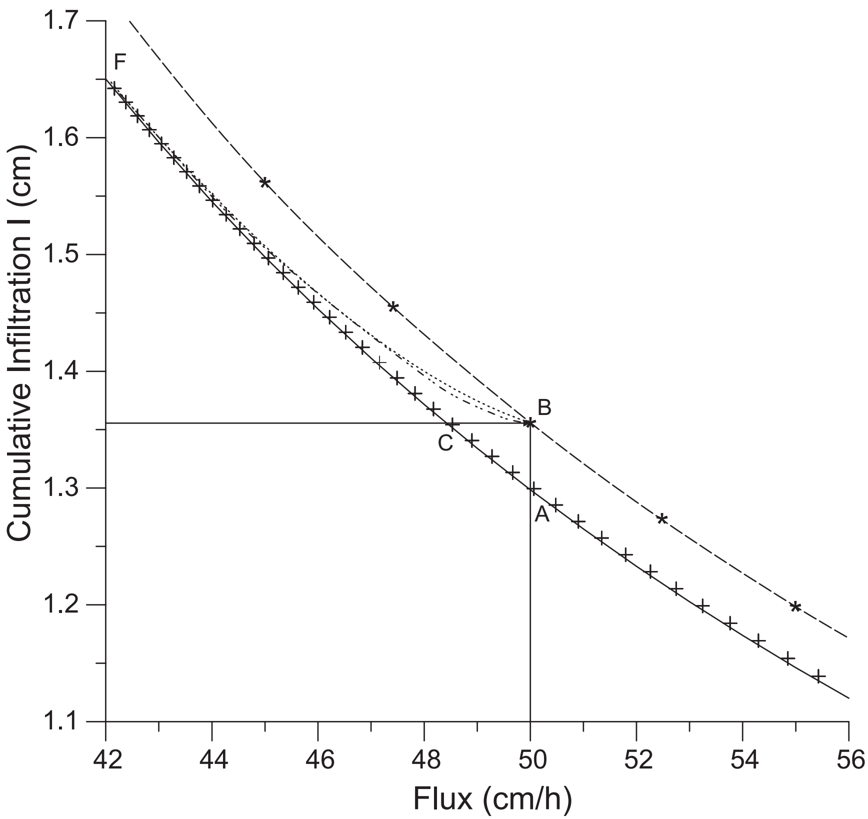
FIGURE CAPTIONS

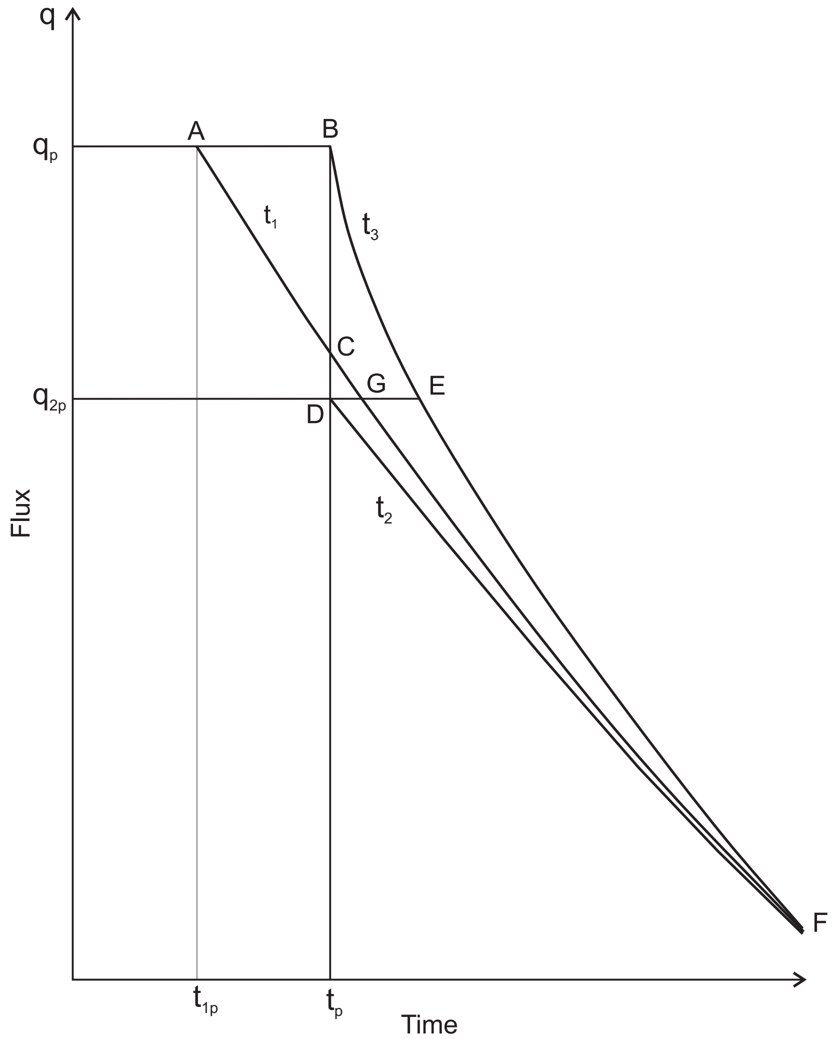
Fig. 1. Cumulative infiltration is given as a function of flux, showing the relationship between the cases of q constant and $\theta = \theta_{sat}$. The (---) is Eq. (2) with *M* neglected, * gives the numerical values, (——) is Eq. (20), + gives the numerical values, (- · -) is Eq. (21) with α = 18.105, and (····) gives the numerical values for the transition from q constant to $\theta = \theta_{sat}$. *BACF* is the TCA and *BCF* is the MTCA.

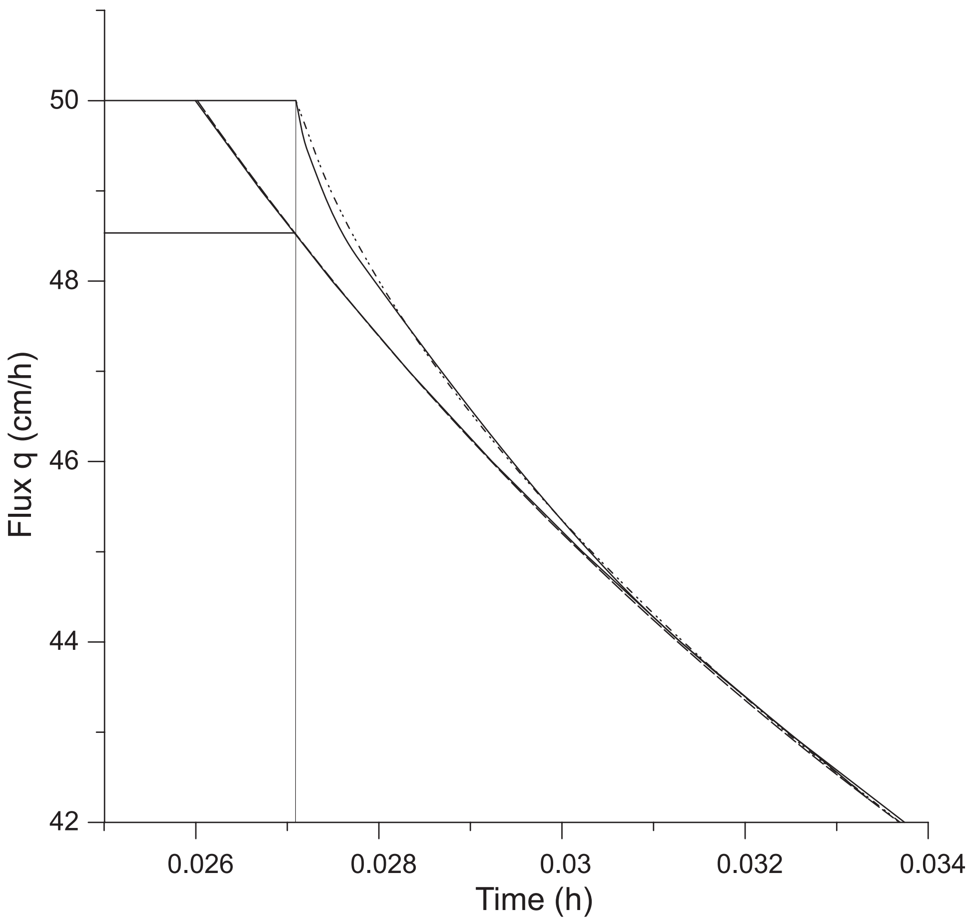
Fig. 2. Sketch of fluxes versus time illustrating the relationship between TCA and MTCA.

Fig. 3. Flux versus time showing the numerical results, (——) and the analytical results of Eqs. (26) and (27) (- - -) and Eq. (28) (- · -).



Cumulative Infiltration I (cm)
Flux (cm/h)
1.7
1.6
1.5
1.4
1.3
1.2
1.1
42
44
46
48
50
52
54
56
F
B
C
A


q
q_p
q_{2p}
Flux
A
B
C
D
G
E
F
t_1
t_2
t_3
t_{1p}
t_p
Time


50
48
46
44
42
Flux q (cm/h)
0.026
0.028
0.03
0.032
0.034
Time (h)